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Research Article

Property P in G-Metric Spaces

Renu Chugh,[1] Tamanna Kadian,[1] Anju Rani,[1] and B. E. Rhoades[2]

[1] *Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India*
[2] *Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA*

Correspondence should be addressed to Renu Chugh, chughrenu@yahoo.com

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We prove two general fixed theorems for maps in G-metric spaces and then show that these maps satisfy property P.

1. Introduction

Metric fixed point theory is an important mathematical discipline because of its applications in areas such as variational and linear inequalities, optimization, and approximation theory. Generalizations of metric spaces were proposed by Gahler [1, 2] (called 2-metric spaces) and Dhage [3, 4] (called D-metric spaces). Hsiao [5] showed that, for every contractive definition, with $x_n := T^n x_0$, every orbit is linearly dependent, thus rendering fixed point theorems in such spaces trivial. Unfortunately, it was shown that certain theorems involving Dhage's D-metric spaces are flawed, and most of the results claimed by Dhage and others are invalid. These errors were pointed out by Mustafa and Sims in [6], among others. They also introduced a valid generalized metric space structure, which they call G-metric spaces. Some other papers dealing with G-metric spaces are those in [7–11].

Let T be a self-map of a complete metric space (X, d) with a nonempty fixed point set $F(T)$. Then T is said to satisfy property P if $F(T) = F(T^n)$ for each $n \in N$. An interesting fact about maps satisfying property P is that they have no nontrivial periodic points. Papers dealing with property P are those in [12–14].

In this paper, we will prove two general fixed point theorems for maps in G-metric spaces and then show that these maps satisfy property P. Throughout this paper, we mean by N the set of all natural numbers.

Definition 1.1 (see [8]). Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2 (see [8]). Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$; if given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; $G(a, x, y) < \delta$ implies that $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 1.3 (see [8]). *Let (X, G), (X', G') be G-metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Definition 1.4 (see [8]). Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X; therefore, we say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\varepsilon > 0$, there exists $N \in N$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

Proposition 1.5 (see [8]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$,

(3) $G(x_n, x, x) \to 0$, *as* $n \to \infty$,

(4) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 1.6 (see [8]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there is $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all n, $m, l \geq N$; that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.7 (see [8]). *In a G-metric space (X, G) the following are equivalent*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*

(2) *For every $\varepsilon > 0$, there exists $N \in N$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all*

$n, m \geq N$.

Proposition 1.8 (see [8]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.9 (see [8]). A G-metric space (X, G) is called a symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \quad \forall x, y \in X. \tag{1.1}$$

Proposition 1.10 (see [8]). *Every G-metric space* (X, G) *defines a metric space* (X, d_G) *by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad \forall x, y \in X. \tag{1.2}$$

Note that, if (X, G) *is a symmetric G-metric space, then*

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{1.3}$$

However, if (X, G) *is not symmetric, then it holds by the G-metric properties that*

$$\frac{3}{2}G(x, y, y) \le d_G(x, y) \le 3G(x, y, y), \quad \forall x, y \in X. \tag{1.4}$$

In general, these inequalities cannot be improved.

Proposition 1.11 (see [8]). *A G-metric space* (X, G) *is G-complete if and only if* (X, d_G) *is a complete metric space.*

Proposition 1.12 (see [8]). *Let* (X, G) *be a G-metric space. Then, for any* $x, y, z, a \in X$, *it follows that*

(1) *if* $G(x, y, z) = 0$, *then* $x = y = z$,

(2) $G(x, y, z) \le G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \le 2G(y, x, x)$,

(4) $G(x, y, z) \le G(x, a, z) + G(a, y, z)$,

(5) $G(x, y, z) \le (2/3)\{G(x, a, a) + G(y, a, a) + G(z, a, a)\}$.

Theorem 1.13 (see [15]). *Let* T *be a self-map of a metric space* X *such that* X *is* T*-orbitally complete. Suppose that* T *satisfies*

$$d(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{1.5}$$

where k *is a real number satisfying* $0 \le k < 1$. *Then* T *has a unique fixed point* $u \in X$. *Moreover, for each* $x \in X$, $\lim T^n x = u$ *and*

$$d(T^n x, u) \le \frac{q^n}{1 - q} d(x, Tx). \tag{1.6}$$

2. Fixed Point Theorems

Theorem 2.1. *Let (X, G) be a complete G-metric space, and let T be a self-mapof X satisfying, for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \le k \max\left\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \frac{[G(x, Ty, Ty) + G(z, Tx, Tx)]}{2}, \frac{[G(x, Ty, Ty) + G(y, Tx, Tx)]}{2}, \frac{[G(y, Tz, Tz) + G(z, Ty, Ty)]}{2}, \frac{[G(x, Tz, Tz) + G(z, Tx, Tx)]}{2}\right\}, \tag{2.1}$$

where k is a constant satisfying $0 \le k < 1$. Then T has a unique fixed point (say p) and T is G-continuous at p.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n x_0$. We may assume that $x_n \ne x_{n+1}$ for each $n \in N \cup \{0\}$. For, if there exists an N such that $x_N = x_{N+1}$, then x_N is a fixed point of T.

From (2.1), with $x = x_{n-1}$, $y = z = x_n$,

$$G(x_n, x_{n+1}, x_{n+1}) \le k \max\left\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \frac{[G(x_{n-1}, x_{n+1}, x_{n+1}) + 0]}{2}, \frac{[G(x_{n-1}, x_{n+1}, x_{n+1}) + 0]}{2}, G(x_n, x_{n+1}, x_{n+1}), \frac{[G(x_{n-1}, x_{n+1}, x_{n+1}) + 0]}{2}\right\}, \tag{2.2}$$

$G(x_n, x_{n+1}, x_{n+1}) \le kM_n$, say.

Suppose that, for some $n \in N$, $M_n = G(x_n, x_{n+1}, x_{n+1})$. Then we have

$$G(x_n, x_{n+1}, x_{n+1}) \le kG(x_n, x_{n+1}, x_{n+1}), \tag{2.3}$$

which is a contradiction, since x_n's are distinct.

Suppose that there is an $n \in N$ for which $M_n = G(x_{n-1}, x_{n+1}, x_{n+1})/2$. Using property (G5),

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}), \tag{2.4}$$

and one obtains

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{k}{2}\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}, \tag{2.5}$$

which leads to

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{k}{2-k} G(x_{n-1}, x_n, x_n) < kG(x_{n-1}, x_n, x_n), \quad \text{since } k < 1. \tag{2.6}$$

Thus, we get

$$G(x_n, x_{n+1}, x_{n+1}) \le kG(x_{n-1}, x_n, x_n) \le \cdots \le k^n G(x_0, x_1, x_1). \tag{2.7}$$

For every $m, n \in N, m > n$, using (G5),

$$\begin{aligned} G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\le \left(k^n + \cdots + k^{m-1}\right) G(x_0, x_1, x_1) \le \frac{k^n}{1-k} G(x_0, x_1, x_1). \end{aligned} \tag{2.8}$$

Therefore $\{x_n\}$ is G-Cauchy, hence G-convergent, since X is G-complete. Call the limit p.

From (2.1) with $x = x_n$, $y = z = p$,

$$\begin{aligned} &G(x_{n+1}, Tp, Tp) \\ &\le k \max\Bigg\{ G(x_n, p, p), G(x_n, x_{n+1}, x_{n+1}), G(p, Tp, Tp), G(p, Tp, Tp), \\ &\qquad \frac{[G(x_n, Tp, Tp) + G(p, x_{n+1}, x_{n+1})]}{2}, \frac{[G(x_n, Tp, Tp) + G(p, x_{n+1}, x_{n+1})]}{2}, \\ &\qquad G(p, Tp, Tp), \frac{[G(x_n, Tp, Tp) + G(p, x_{n+1}, x_{n+1})]}{2} \Bigg\}. \end{aligned} \tag{2.9}$$

Taking the limit of both sides of (2.9) as $n \to \infty$ yields

$$G(p, Tp, Tp) \le kG(p, Tp, Tp), \tag{2.10}$$

which implies that $G(p, Tp, Tp) = 0$ and hence $p = Tp$.

Suppose that q is also a fixed point of T. Then, from (2.1) with $x = p, y = z = q$,

$$\begin{aligned} G(p, q, q) \le k \max\Bigg\{ & G(p, q, q), 0, 0, 0, \frac{[G(p, q, q) + G(q, p, p)]}{2}, \\ & \frac{[G(p, q, q) + G(q, p, p)]}{2}, 0, \frac{[G(p, q, q) + G(q, p, p)]}{2} \Bigg\}, \end{aligned} \tag{2.11}$$

which implies that

$$G(p, q, q) \le \frac{k}{2-k} G(q, p, p). \tag{2.12}$$

Using (2.1) again, this time with $x = q$, $y = z = p$, one obtains

$$G(q,p,p) \leq k \max\left\{G(q,p,p), 0, 0, 0, \frac{[G(q,p,p)+G(p,q,q)]}{2}, \frac{[G(q,p,p)+G(p,q,q)]}{2}, 0, \frac{[G(q,p,p)+G(p,q,q)]}{2}\right\}, \tag{2.13}$$

which implies that

$$G(q,p,p) \leq \frac{k}{2-k} G(p,q,q). \tag{2.14}$$

Combining (2.12) and (2.14) gives

$$G(p,q,q) \leq \left(\frac{k}{2-k}\right)^2 G(p,q,q). \tag{2.15}$$

Therefore, $p = q$, since $k/(2-k) < 1$.

Let $\{y_n\} \subset X$ be any sequence with limit p. Using (2.1) with $x = z = y_n$, $y = p$,

$$G(Ty_n, Tp, Ty_n) \leq k \max\left\{G(y_n,p,y_n), G(y_n,Ty_n,Ty_n), 0, G(y_n,Ty_n,Ty_n), \frac{[G(y_n,p,p)+G(y_n,Ty_n,Ty_n)]}{2}, \frac{[G(y_n,p,p)+G(p,Ty_n,Ty_n)]}{2}\right\}, \tag{2.16}$$

That is,

$$G(Ty_n, p, Ty_n) \leq k \max\left\{G(y_n,p,y_n), G(y_n,Ty_n,Ty_n), \frac{[G(y_n,p,p)+G(y_n,Ty_n,Ty_n)]}{2}, \frac{[G(y_n,p,p)+G(p,Ty_n,Ty_n)]}{2}\right\}. \tag{2.17}$$

Using the fact that, from (G5),

$$\begin{gathered} G(y_n,Ty_n,Ty_n) \leq G(y_n,p,p) + G(p,Ty_n,Ty_n), \\ G(Ty_n,p,Ty_n) \leq kL, \quad \text{say}. \end{gathered} \tag{2.18}$$

If, for some n, L is equal to $G(y_n, p, y_n)$, then we have

$$G(Ty_n, p, Ty_n) \leq kG(y_n, p, y_n). \tag{2.19}$$

If, for some n, L is equal to $G(y_n, Ty_n, Ty_n)$, then, using (G5),

$$G(Ty_n, p, Ty_n) \leq kG(y_n, Ty_n, Ty_n) \leq G(y_n, p, p) + G(p, Ty_n, Ty_n), \tag{2.20}$$

which implies that

$$G(Ty_n, p, Ty_n) \leq \frac{k}{1-k} G(y_n, p, p). \tag{2.21}$$

If, for some n, L is equal to $[G(y_n, p, p) + G(y_n, Ty_n, Ty_n)]/2$, then, using (G5),

$$G(Ty_n, p, Ty_n) \leq \frac{k}{2}[G(y_n, p, p) + G(y_n, p, p) + G(p, Ty_n, Ty_n)], \tag{2.22}$$

which implies that

$$G(Ty_n, p, Ty_n) \leq \frac{2k}{2-k} G(y_n, p, p). \tag{2.23}$$

If, for some n, L is equal to $[G(y_n, p, p) + G(p, Ty_n, Ty_n)]/2$, then, using (G5),

$$G(Ty_n, p, Ty_n) \leq \frac{k}{2}[G(y_n, p, p) + G(p, Ty_n, Ty_n)], \tag{2.24}$$

which implies that

$$G(Ty_n, p, Ty_n) \leq \frac{k}{2-k} G(y_n, p, p). \tag{2.25}$$

Therefore, for all n, $\lim G(p, Ty_n, Ty_n) = 0$ and T is G-continuous at p.

Special cases of Theorem 2.1 are Theorem 2.1 of [9] and Theorems 2.1, 2.4, 2.6, and 2.8 of [10]. □

Theorem 2.2. *Let (X, G) be a complete G-metric space, and let T be a self-map of X satisfying, for all $x, y, z \in X$,*

$$\begin{aligned} G(Tx, Ty, Tz) \leq k \max\{&G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), \\ &G(x, Ty, Ty), G(y, Tx, Tx), G(z, Tz, Tz)\}, \end{aligned} \tag{2.26}$$

or

$$\begin{aligned} G(Tx,Ty,Tz) \le k\max\{&G(x,y,z),G(x,x,Tx),G(y,y,Ty),\\ &G(x,x,Ty),G(y,y,Tx),G(z,z,Tx)\}, \end{aligned} \tag{2.27}$$

where k is a constant satisfying $0 \le k < 1$. Then T has a unique fixed point (call it p) and T is G-continuous at p.

Proof. Suppose that T satisfies (2.26). Using (2.26) with $z = y$, we have

$$G(Tx,Ty,Ty) \le k\max\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx)\}. \tag{2.28}$$

Suppose that (X,G) is symmetric.

From Proposition 1.10, d_G, defined by $d_G(x,y) = 2G(x,y,y)$ makes, (X,d_G) into a metric space. Substituting into (2.28) and then multiplying by 2 yield

$$d_G(Tx,Ty) \le k\max\{d_G(x,y),d_G(x,Tx),d_G(y,Ty),d_G(x,Ty),d_G(y,Tx)\}. \tag{2.29}$$

From Theorem 1.13, T has a unique fixed point.

Suppose that (X,G) is not symmetric. Define

$$\begin{aligned} A_n &= \left\{G\left(T^ix,T^jx,T^jx\right): \quad 0 \le i,j \le n\right\},\\ \delta_n &= \max_{i,j} A_n. \end{aligned} \tag{2.30}$$

Then $\delta_n = G(T^ix,T^mx,T^mx)$ for some i,m satisfying $0 \le i,m \le n$.

Suppose that $i > 0$. Then, from (2.26),

$$\begin{aligned} \delta_n &= G(x_i,x_m,x_m)\\ &\le k\max\{G(x_{i-1},x_{m-1},x_{m-1}),G(x_{i-1},x_i,x_i),G(x_{m-1},x_m,x_m),G(x_{i-1},x_m,x_m),G(x_{m-1},x_i,x_i)\}\\ &\le k\delta_n, \end{aligned} \tag{2.31}$$

a contradiction. Therefore, $i = 0$.

Thus, for some m satisfying $0 \le m \le n$, using property (G5) and (2.26),

$$\begin{aligned} \delta_n = G(x_0,x_m,x_m) &\le G(x_0,x_1,x_1) + G(x_1,x_m,x_m)\\ &\le G(x_0,x_1,x_1) + k\max\{G(x_0,x_{m-1},x_{m-1}),G(x_0,x_1,x_1),\\ &\qquad G(x_{m-1},x_m,x_m),G(x_0,x_m,x_m),G(x_{m-1},x_1,x_1)\}\\ &\le G(x_0,x_1,x_1) + k\delta_n, \end{aligned} \tag{2.32}$$

which implies that

$$\delta_n \leq \frac{1}{1-k} G(x_0, x_1, x_1), \tag{2.33}$$

and δ_n is bounded in n. Call this bound δ.

Define $x_n = Tx_{n-1}$. Without loss of generality, we may assume that $x_n \neq x_{n+1}$ for each n. For, if there exists an N for which $x_N = x_{N+1}$, then $x_{N+1} = Tx_N$ and x_N is a fixed point of T.

Again from (2.26),

$$\begin{aligned}
&G(x_n, x_{n+1}, x_{n+1}) \\
&\leq k \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), 0\} \\
&= k \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1})\} \\
&\leq k \max\{G(x_{n-1}, x_n, x_n), \delta\} \\
&\leq \cdots \leq k^n \max\{G(x_0, x_1, x_1), \delta\} \leq k^n \delta.
\end{aligned} \tag{2.34}$$

For any $m, n \in N$; $m > n$,

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + \cdots + G(x_{m-1}, x_m, x_m) \\
&\leq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right)\delta \leq \frac{k^n \delta}{1-k}.
\end{aligned} \tag{2.35}$$

Therefore, $\lim G(x_n, x_m, x_m) = 0$ as $m, n \to \infty$ and $\{x_n\}$ is G-Cauchy, hence G-convergent, since X is G-complete. Call the limit p.

From (2.26),

$$\begin{aligned}
G(x_n, Tp, Tp) \leq k \max\big\{&G(x_{n-1}, p, p), G(x_n, x_{n+1}, x_{n+1}), G(p, Tp, Tp), \\
&G(x_{n-1}, Tp, Tp), G(p, x_n, x_n)\big\}.
\end{aligned} \tag{2.36}$$

Taking the limit of both sides of (2.36) as $n \to \infty$ yields

$$G(p, Tp, Tp) \leq kG(p, Tp, Tp), \tag{2.37}$$

which implies that $p = Tp$.

Suppose that q is another fixed point of T with $p \neq q$. Then, from (2.26),

$$G(p, q, q) \leq k \max\big\{G(p, q, q), 0, 0, G(p, q, q), G(q, p, p)\big\} = kG(q, p, p). \tag{2.38}$$

Again using (2.26),

$$G(q,p,p) \le k \max\{G(q,p,p), 0, 0, G(q,p,p), G(p,q,q)\} = kG(p,q,q). \tag{2.39}$$

Combining (2.36) and (2.38) gives $G(p,q,q) \le k^2 G(p,q,q)$, a contradiction. Therefore $p = q$ and the fixed point is unique.

Now let $\{y_n\} \subset X$ with $\lim y_n = p$. Using (2.26),

$$\begin{aligned} G(Ty_n,p,Ty_n) \le k \max\{&G(y_n,p,y_n), G(y_n,Ty_n,Ty_n), 0, G(y_n,p,p),\\ &G(p,Ty_n,Ty_n), G(y_n,Ty_n,Ty_n)\} \end{aligned} \tag{2.40}$$

But from (G5), we have

$$G(y_n,Ty_n,Ty_n) \le G(y_n,p,p) + G(p,Ty_n,Ty_n). \tag{2.41}$$

Therefore, (2.40) reduces to

$$G(Ty_n,p,Ty_n) \le \max\left\{kG(y_n,p,y_n), \frac{k}{1-k}G(y_n,p,p)\right\}. \tag{2.42}$$

Taking the limit of both sides of the above equation as $n \to \infty$ gives $\lim G(Ty_n,p,Ty_n) = 0$, which implies that $\lim Ty_n = p$, and T is G-continuous at p. □

The proof using (2.27) is similar. Special cases of Theorem 2.2 are Theorems 2.5, 2.8, and 2.9 of [9].

3. Property P

In this section we shall show that maps satisfying (2.1) or (2.26) possess property P.

Theorem 3.1. *Under the conditions of Theorem 2.1, T has property P.*

Proof. From Theorem 2.1, T has a fixed point. Therefore $F(T^n) \neq \emptyset$ for each $n \in N$. Fix $n > 1$ and assume that $p \in F(T^n)$. We wish to show that $p \in F(T)$.

Suppose that $p \neq Tp$. Using (2.1),

$$
\begin{aligned}
G(p,Tp,Tp) &= G\left(T^n p, T^{n+1} p, T^{n+1} p\right) \\
&\leq k \max\Bigg\{ G\left(T^{n-1} p, T^n p, T^n p\right), G\left(T^{n-1} p, T^n p, T^n p\right), G\left(T^n p, T^{n+1} p, T^{n+1} p\right), \\
&\qquad G\left(T^n p, T^{n+1} p, T^{n+1} p\right), \frac{[G(T^{n-1} p, T^{n+1} p, T^{n+1} p) + 0]}{2}, \\
&\qquad \frac{[G(T^{n-1} p, T^{n+1} p, T^{n+1} p) + 0]}{2}, \\
&\qquad \frac{[G(T^n p, T^{n+1} p, T^{n+1} p) + G(T^n p, T^{n+1} p, T^{n+1} p)]}{2}, \\
&\qquad \frac{[G(T^{n-1} p, T^{n+1} p, T^{n+1} p) + 0]}{2} \Bigg\} \\
&= kG\left(T^{n-1} p, T^n p, T^n p\right) \leq k^2 G\left(T^{n-2} p, T^{n-1} p, T^{n-1} p\right) \\
&\leq \cdots \leq k^n G(p,Tp,Tp),
\end{aligned}
\tag{3.1}
$$

a contradiction.

Therefore $p \in F(T)$ and T has property P. □

Theorem 3.2. *Under the conditions of Theorem 2.2, T has property P.*

Proof. From Theorem 2.2, T has a fixed point. Therefore $F(T^n) \neq \emptyset$for each $n \in N$. Fix $n > 1$ and assume that $p \in F(T^n)$. Using (2.26) and assuming that $p \neq Tp$, we have

$$
\begin{aligned}
G(p,Tp,Tp) &= G\left(T^n p, T^{n+1} p, T^{n+1} p\right) \\
&\leq k \max\Big\{ G\left(T^{n-1} p, T^n p, T^n p\right), G\left(T^{n-1} p, T^n p, T^n p\right), \\
&\qquad G\left(T^n p, T^{n+1} p, T^{n+1} p\right), G\left(T^{n-1} p, T^{n+1} p, T^{n+1} p\right), 0, 0 \Big\}.
\end{aligned}
\tag{3.2}
$$

Define $B_n = \{G(T^i p, T^j p, T^j p) : 0 \leq i, j \leq n\}$. Then

$$
\delta_n = \max_{i,j} B_n. \tag{3.3}
$$

Then, $\delta_n = G(T^i p, T^m p, T^m p)$ for some $0 \leq i, m \leq n$.

Assume that $\delta_n > 0$. Then from (2.26),

$$\begin{aligned}
\delta_n &= G\left(T^i p, T^m p, T^m p\right) \\
&\leq k \max\Big\{ G\left(T^{i-1}p, T^{m-1}p, T^{m-1}p\right), G\left(T^{i-1}p, T^i p, T^i p\right), G\left(T^{m-1}p, T^m p, T^m p\right), \\
&\qquad G\left(T^{i-1}p, T^m p, T^m p\right), G\left(T^{m-1}p, T^i p, T^i p\right), G\left(T^{m-1}p, T^m p, T^m p\right) \Big\} \\
&\leq k\delta_n,
\end{aligned} \tag{3.4}$$

a contradiction. Therefore $\delta_n = 0$. In particular, $G(p, Tp, Tp) = 0$ and $p = Tp$. □

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