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Research Article

Existence of Fixed Points of Firmly Nonexpansive-Like Mappings in Banach Spaces

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The aim of this paper is to obtain some existence theorems related to a hybrid projection method and a hybrid shrinking projection method for firmly nonexpansive-like mappings (mappings of type (P)) in a Banach space. The class of mappings of type (P) contains the classes of resolvents of maximal monotone operators in Banach spaces and firmly nonexpansive mappings in Hilbert spaces.

1. Introduction

Many problems in optimization, such as convex minimization problems, variational inequality problems, minimax problems, and equilibrium problems, can be formulated as the problem of solving the inclusion

$$0 \in Au \tag{1.1}$$

for a maximal monotone operator $A : E \to 2^{E^*}$ defined in a Banach space E; see, for example, [1–5] for convex minimization problems, [3, 5, 6] for variational inequality problems, [3, 5, 7] for minimax problems, and [8] for equilibrium problems. It is also known that the problem can be regarded as a fixed point problem for a *firmly nonexpansive mapping* in the Hilbert space setting. In fact, if H is a Hilbert space and A: $H \to 2^H$ is a maximal monotone operator, then the resolvent $T = (I + A)^{-1}$ of A is a single-valued firmly nonexpansive mapping of H onto the domain $D(A)$ of A, that is, $T\colon H \to D(A)$ is onto and

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \tag{1.2}$$

for all $x, y \in H$. Further, the set of fixed points of T coincides with that of solutions to (1.1); see, for example, [5].

In 2000, Solodov and Svaiter [9] proved the following strong convergence theorem for maximal monotone operators in Hilbert spaces.

Theorem 1.1 (see [9]). *Let H be a Hilbert space, $A : H \to 2^H$ a maximal monotone operator such that $A^{-1}0$ is nonempty, and J_r the resolvent of A defined by $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{gathered}
x_1 = x \in H, \\
C_n = \{z \in H : \langle z - J_{r_n}x_n, x_n - J_{r_n}x_n\rangle \leq 0\}, \\
D_n = \{z \in H : \langle z - x_n, x - x_n\rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap D_n}(x)
\end{gathered} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence of positive real numbers such that $\inf_n r_n > 0$ and $P_{C_n \cap D_n}$ denotes the metric projection of H onto $C_n \cap D_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{A^{-1}0}(x)$.

This method is sometimes called a hybrid projection method; see also Bauschke and Combettes [10] on more general results for a class of nonlinear operators including that of resolvents of maximal monotone operators in Hilbert spaces. Ohsawa and Takahashi [11] obtained a generalization of Theorem 1.1 for maximal monotone operators in Banach spaces.

Many authors have investigated several types of hybrid projection methods since then; see, for example, [12–30] and references therein. In particular, Kamimura and Takahashi [17] obtained another generalization of Theorem 1.1 for maximal monotone operators in Banach spaces. Bauschke and Combettes [16] and Otero and Svaiter [25] also obtained generalizations of Theorem 1.1 with Bregman functions in Banach spaces. Matsushita and Takahashi [20] obtained a generalization of Ohsawa and Takahashi's theorem [11] and some existence theorems for their iterative method.

Recently, Aoyama et al. [31] discussed some properties of *mappings of type* (P), (Q), *and* (R) in Banach spaces. These are all generalizations of firmly nonexpansive mappings in Hilbert spaces. It is known that the classes of mappings of type (P), (Q), and (R) correspond to three types of resolvents of monotone operators in Banach spaces, respectively, [31, 32].

The aim of this paper is to investigate a hybrid projection method and a hybrid shrinking projection method introduced in [30] for a single mapping of type (P) in a Banach space; see (2.2) for the definition of mappings of type (P). Using the techniques in [12, 20, 21] we show that the sequences generated by these methods are well defined without assuming the existence of fixed points. We also show that the boundedness of the generated sequences is equivalent to the existence of fixed points of mappings of type (P).

2. Preliminaries

Throughout the present paper, every linear space is real. We denote the set of positive integers by $\mathbb{N}$. Let E be a Banach space with norm $\|\cdot\|$. Then the dual space of E is denoted by E^*. The norm of E^* is also denoted by $\|\cdot\|$. For $x \in E$ and $x^* \in E^*$, we denote $x^*(x)$ by $\langle x, x^* \rangle$. For a sequence $\{x_n\}$ of E and $x \in E$, strong convergence of $\{x_n\}$ and weak convergence of $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \tag{2.1}$$

for all $x \in E$. The space E is said to be *smooth* if $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in S(E)$, where $S(E)$ denotes the unit sphere of E. The space E is also said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is also said to be *uniformly convex* if for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$ imply $\|(x+y)/2\| \leq 1 - \delta$. The space E is said to have the *Kadec-Klee* property if $x_n \to x$ whenever $\{x_n\}$ is a sequence of E such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. We know the following (see, e.g., [4, 33, 34]).

(i) E is smooth if and only if J is single-valued. In this case, J is demicontinuous, that is, norm-to-weak* continuous.

(ii) If E is smooth, strictly convex, and reflexive, then J is single-valued, one-to-one, and onto.

(iii) If E is uniformly convex, then E is a strictly convex and reflexive Banach space which has the Kadec-Klee property.

Let E be a strictly convex and reflexive Banach space, C a nonempty closed convex subset of E, and $x \in E$. Then there exists a unique $z_x \in C$ such that $\|z_x - x\| = \min_{y\in C}\|y - x\|$. The mapping P_C defined by $P_C x = z_x$ for all $x \in E$ is called the *metric projection* of E onto C. We know that $z = P_C x$ if and only if $z \in C$ and $\langle y - z, J(x - z)\rangle \leq 0$ for all $y \in C$.

Let C be a nonempty subset of a Banach space E and $T : C \to E$ a mapping. Then the set of fixed points of T is denoted by $F(T)$. A point $u \in C$ is said to be an *asymptotic fixed point* of T [35] if there exists a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. The mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The *identity mapping* on E is denoted by I.

Let E be a smooth Banach space, C a nonempty subset of E, and $T : C \to E$ a mapping. Following [31], we say that T is *of type (P)* if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \geq 0 \tag{2.2}$$

for all $x, y \in C$. If E is a Hilbert space, then $J = I$ and hence T is of type (P) if and only if T is *firmly nonexpansive*, that is,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle \tag{2.3}$$

for all $x, y \in C$. We know that if C is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, then the metric projection P_C of E onto C is of type (P) and $F(P_C) = C$.

Let E be a smooth, strictly convex, and reflexive Banach space and $A : E \to 2^{E^*}$ a mapping. The graph of A, the domain of A, and the range of A are defined by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}, D(A) = \{x \in E : Ax \neq \emptyset\}$, and $R(A) = \bigcup_{x \in D(A)} Ax$, respectively. The mapping A is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x, x^*) \in G(A)$ and $(y, y^*) \in G(A)$. It is known that if A is monotone, then the resolvent $T = (I + J^{-1}A)^{-1}$ of A is a single-valued mapping of $R(I + J^{-1}A)$ onto $D(A)$ and of type (P), and moreover, $F(T) = A^{-1}0$; see [31]. A monotone operator A is said to be *maximal monotone* if $A = B$ whenever $B : E \to 2^{E^*}$ is a monotone operator and $G(A) \subset G(B)$. By Rockafellar's result [6], if $A : E \to 2^{E^*}$ is maximal monotone, then the resolvent $T = (I + J^{-1}A)^{-1}$ of A is a mapping of E onto $D(A)$; see [4, 36] for more details.

We know the following.

Lemma 2.1 (see [14]). *Let E be a smooth Banach space, C a nonempty subset of E, and $T : C \to E$ a mapping of type (P). Then the following hold.*

(1) *If C is closed and convex, then so is $F(T)$.*

(2) $\widehat{F}(T) = F(T)$.

Theorem 2.2 (see [31]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, and $T : C \to E$ a mapping of type (P). Then the following hold.*

(1) *If $\{x_n\}$ is a sequence of C such that $x_n \to x \in C$, then $Tx_n \rightharpoonup Tx, J(x_n - Tx_n) \rightharpoonup J(x - Tx)$, and $\|x_n - Tx_n\| \to \|x - Tx\|$.*

(2) *If E has the Kadec-Klee property, then T is norm-to-norm continuous.*

(3) *If E is uniformly convex, then T is uniformly norm-to-norm continuous on each nonempty bounded subset of C.*

Theorem 2.3 (see [31]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty bounded closed convex subset of E, and $T : C \to E$ a mapping of type (P). Then $F(P_C T)$ is nonempty. Furthermore, if T is a self-mapping, then $F(T)$ is nonempty.*

Lemma 2.4 (see [14]). *Let E be a smooth and uniformly convex Banach space, $\{M_n\}$ and $\{N_n\}$ sequences of nonempty closed convex subsets of E, $x \in E$, and $\{x_n\}$ a sequence of E such that $x_n = P_{N_n}(x)$ and $x_{n+1} \in N_n$ for all $n \in \mathbb{N}$. Then the following hold.*

(1) *If $\{x_n\}$ is bounded, then $x_n - x_{n+1} \to 0$.*

(2) *If $x_{n+1} = P_{M_n}(x)$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} M_n$ is nonempty, then $\{x_n\}$ is bounded.*

Theorem 2.5 (see [14]). *Let E be a smooth and uniformly convex Banach space, C a nonempty closed convex subset of E, and $\{T_n\}$ a sequence of mappings of type (P) of C into itself such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that each weak subsequential limit of $\{z_n\}$ belongs to F whenever*

$\{z_n\}$ is a bounded sequence of C such that $z_n - T_n z_n \to 0$ and $z_n - z_{n+1} \to 0$. Let $\{x_n\}$ be a sequence defined by $x \in E$, $x_1 = P_C(x)$, and

$$C_n = \{z \in C : \langle z - T_n x_n, J(x_n - T_n x_n)\rangle \le 0\},$$
$$D_n = \{z \in C : \langle z - x_n, J(x - x_n)\rangle \le 0\}, \tag{2.4}$$
$$x_{n+1} = P_{C_n \cap D_n}(x)$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F(x)$.

Let E be a reflexive Banach space and $\{D_n\}$ a sequence of nonempty closed convex subsets of E. Then subsets s-$\mathrm{Li}_n D_n$ and w-$\mathrm{Ls}_n D_n$ of E are defined as follows.

(i) $z \in s$-$\mathrm{Li}_n D_n$ if there exists a sequence $\{x_n\}$ of E such that $x_n \in D_n$ for all $n \in \mathbb{N}$ and $x_n \to z$.

(ii) $z \in w$-$\mathrm{Ls}_n D_n$ if there exists a subsequence $\{D_{n_i}\}$ of $\{D_n\}$ and a sequence $\{y_i\}$ of E such that $y_i \in D_{n_i}$ for all $i \in \mathbb{N}$ and $y_i \rightharpoonup z$.

The sequence $\{D_n\}$ is said to be *Mosco convergent* to a subset D of E if s-$\mathrm{Li}_n D_n = \mathrm{w}$-$\mathrm{Ls}_n D_n = D$ holds. We represent this by M-$\lim_n D_n = D$. We know that if $\{D_n\}$ is a sequence of nonempty closed convex subsets of E such that $D_n \supset D_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} D_n$ is nonempty, then M-$\lim_n D_n = \bigcap_{n=1}^{\infty} D_n$. We also know the following theorem.

Theorem 2.6 (see [37]). *Let E be a strictly convex and reflexive Banach space and $\{D_n\}$ a sequence of nonempty closed convex subsets of E such that* M-$\lim_n D_n = D$ *exists and nonempty. Then $\{P_{D_n}(x)\}$ converges weakly to $P_D(x)$ for all $x \in E$. Furthermore, if E has the Kadec-Klee property, then $\{P_{D_n}(x)\}$ converges strongly to $P_D(x)$ for all $x \in E$.*

Kimura et al. [18] obtained the following strong convergence theorem by using Theorem 2.6; see also Kimura and Takahashi [19] for related results which were obtained by using Mosco convergence.

Theorem 2.7 (see [18]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $\{T_n\}$ a sequence of mappings of C into itself such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that there exists a sequence $\{a_n\}$ of $(-\infty, 0)$ such that*

$$\langle z - x, J(x - T_n x)\rangle \le a_n \|x - T_n x\|^2 \tag{2.5}$$

for all $n \in \mathbb{N}$, $x \in C$, and $z \in F(T_n)$. Let $\{x_n\}$ be a sequence defined by

$$x_1 = x \in C = C_0,$$
$$C_n = \left\{z \in C_{n-1} : \langle z - x_n, J(x_n - T_n x_n)\rangle \ \le a_n \|x_n - T_n x_n\|^2\right\}, \tag{2.6}$$
$$x_{n+1} = P_{C_n}(x)$$

for all $n \in \mathbb{N}$. Then the following hold.

(1) *$F \subset C_n \subset C_{n-1}$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is well defined.*

(2) *If E has the Kadec-Klee property, $\sup_n a_n < 0$, and $\{T_n\}$ satisfies the condition that $z \in F$ whenever $\{z_n\}$ is a sequence of C such that $z_n \to z$ and $T_n z_n \to z$, then $\{x_n\}$ converges strongly to $P_F(x)$.*

Using Theorems 2.2 and 2.7, we obtain the following strong convergence theorem for mappings of type (P).

Corollary 2.8. *Let E be a smooth, strictly convex, and reflexive Banach space which has the Kadec-Klee property, C a nonempty closed convex subset of E, and $T : C \to C$ a mappings of type (P) such that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by*

$$\begin{gathered} x_1 = x \in C = C_0, \\ C_n = \{z \in C_{n-1} : \langle z - Tx_n, J(x_n - Tx_n)\rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(x) \end{gathered} \tag{2.7}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

Proof. We first show that $\{T_n\}$ defined by $T_n = T$ for all $n \in \mathbb{N}$ satisfies (2.5). Let $x \in C$ and $z \in F(T)$ be given. Since T is of type (P), we have

$$\langle z - Tx, J(x - Tx)\rangle = -\langle Tz - Tx, J(z - Tz) - J(x - Tx)\rangle \leq 0. \tag{2.8}$$

This implies that

$$\langle z - x, J(x - Tx)\rangle = \langle z - Tx, J(x - Tx)\rangle - \|x - Tx\|^2 \leq -\|x - Tx\|^2. \tag{2.9}$$

Hence $\{T_n\}$ satisfies (2.5) with $\{a_n\}$ given by $a_n = -1$ for all $n \in \mathbb{N}$.

We next show that $\{T_n\}$ satisfies the assumption in (2) of Theorem 2.7. Let $\{z_n\}$ be a sequence of C such that $z_n \to z$ and $Tz_n \to z$. Since T is demicontinuous by (1) of Theorem 2.2, $Tz_n \rightharpoonup Tz$. Hence we have $z \in F(T)$. Therefore, Theorem 2.7 implies the conclusion. □

3. Existence Theorems

Using the techniques in [12, 20, 21], we show the following two lemmas.

Lemma 3.1. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, $T : C \to C$ a mapping of type (P), and D a nonempty bounded closed convex subset of E such that $D \subset C$. Then there exists $z \in C$ such that*

$$\langle z - Ty, J(I - T)y\rangle \leq 0 \tag{3.1}$$

for all $y \in T^{-1}(D)$.

Proof. By Theorem 2.3, there exists $u \in D$ such that $P_D Tu = u$. This implies that $\langle u - p, J(I - T)u\rangle \leq 0$ for all $p \in D$. Fix $y \in T^{-1}(D)$. Then we have

$$\langle u - Ty, J(I - T)u\rangle \leq 0. \tag{3.2}$$

Since T is of type (P), we also know that

$$\langle Tu - Ty, J(I - T)u - J(I - T)y\rangle \geq 0. \tag{3.3}$$

Using (3.2) and (3.3), we obtain

$$\begin{aligned}\langle Tu - Ty, J(I - T)y\rangle &\leq \langle Tu - Ty, J(I - T)u\rangle \\ &= -\|Tu - u\|^2 + \langle u - Ty, J(I - T)u\rangle \leq 0.\end{aligned} \tag{3.4}$$

Since $Tu \in C$, by putting $z = Tu$, we obtain the desired result. □

Lemma 3.2. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping of type (P). Let $\{x_n\}$ be a sequence defined by*

$$\begin{gathered} x_1 = x \in C, \\ C_n = \{z \in C : \langle z - Tx_n, J(x_n - Tx_n)\rangle \leq 0\}, \\ D_n = \{z \in C : \langle z - x_n, J(x - x_n)\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}(x) \end{gathered} \tag{3.5}$$

for all $n \in \mathbb{N}$. Then the following hold.

(1) *$C_n \cap D_n \supset \bigcap_{k=1}^{n} C_k$ and $\bigcap_{k=1}^{n} C_k$ is nonempty for all $n \in \mathbb{N}$.*

(2) *$\{x_n\}$ is well defined.*

(3) *$\bigcap_{n=1}^{\infty}(C_n \cap D_n) = \bigcap_{n=1}^{\infty} C_n \supset F(T)$.*

Proof. We first show (1) by induction on $n \in \mathbb{N}$. It is obvious that $D_1 = C$ and $Tx_1 \in C_1$. Thus $C_1 \cap D_1 = C_1 \neq \emptyset$. Fix $n \geq 2$ and suppose that $C_k \cap D_k \supset \bigcap_{j=1}^{k} C_j \neq \emptyset$ for all $k \in \{1, 2, \ldots, n-1\}$. Then $x_1, x_2, \ldots, x_n$ are defined. Note that $D_n \supset C_{n-1} \cap D_{n-1}$ by the definitions of D_n and x_n. This implies that

$$C_n \cap D_n \supset C_n \cap (C_{n-1} \cap D_{n-1}) \supset C_n \cap \bigcap_{k=1}^{n-1} C_k = \bigcap_{k=1}^{n} C_k. \tag{3.6}$$

We next show that $\bigcap_{k=1}^{n} C_k$ is nonempty. Let r be a positive real number such that $\|Tx_k\| \leq r$ for all $k \in \{1, 2, \ldots, n\}$ and put $D = C \cap \{w \in E : \|w\| \leq r\}$. It is clear that $x_k \in T^{-1}(D)$ for all $k \in \{1, 2, \ldots, n\}$. By Lemma 3.1, we have $z \in C$ such that $\langle z - Ty, J(I - T)y \rangle \leq 0$ for all $y \in T^{-1}(D)$. This implies that

$$\langle z - Tx_k, J(I - T)x_k \rangle \leq 0 \tag{3.7}$$

for all $k \in \{1, 2, \ldots, n\}$ and hence $z \in \bigcap_{k=1}^{n} C_k$. The part (2) is a direct consequence of (1).

We finally show (3). By (1), we have

$$\bigcap_{n=1}^{\infty} C_n \supset \bigcap_{n=1}^{\infty} (C_n \cap D_n) \supset \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{n} C_k = \bigcap_{n=1}^{\infty} C_n. \tag{3.8}$$

On the other hand, if $u \in F(T)$, then, by the assumption that T is of type (P), we have

$$\langle u - Tx_n, J(x_n - Tx_n) \rangle \leq 0 \tag{3.9}$$

for all $n \in \mathbb{N}$. Thus $u \in \bigcap_{n=1}^{\infty} C_n$. Therefore we obtain the desired result. □

Similarly, we can also show the following lemma.

Lemma 3.3. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping of type (P). Let $\{x_n\}$ be a sequence defined by*

$$\begin{gathered} x_1 = x \in C = C_0, \\ C_n = \{ z \in C_{n-1} : \langle z - Tx_n, J(x_n - Tx_n) \rangle \leq 0 \}, \\ x_{n+1} = P_{C_n}(x) \end{gathered} \tag{3.10}$$

for all $n \in \mathbb{N}$. Then the following hold.

(1) *C_n is nonempty for all $n \in \mathbb{N}$.*

(2) *$\{x_n\}$ is well defined.*

(3) $\bigcap_{n=1}^{\infty} C_n \supset F(T)$.

Proof. We first show (1). It is obvious that $Tx_1 \in C_1$ and hence C_1 is nonempty. Fix $n \geq 2$ and suppose that C_k is nonempty for all $k \in \{1, 2, \ldots, n-1\}$. Then $x_1, x_2, \ldots, x_n$ are defined. We next show that C_n is nonempty. Let r be a positive real number such that $\|Tx_k\| \leq r$ for all $k \in \{1, 2, \ldots, n\}$ and put $D = C \cap \{w \in E : \|w\| \leq r\}$. It is clear that $x_k \in T^{-1}(D)$ for all $k \in \{1, 2, \ldots, n\}$. By Lemma 3.1, we have $z \in C$ such that $\langle z - Ty, J(I - T)y \rangle \leq 0$ for all $y \in T^{-1}(D)$. This implies that

$$\langle z - Tx_k, J(I - T)x_k \rangle \leq 0 \tag{3.11}$$

for all $k \in \{1, 2, \ldots, n\}$ and hence $z \in C_n$. Part (2) is a direct consequence of (1). Part (3) follows from the assumption that T is of type (P). □

Using Lemmas 2.4, 3.2, and Theorem 2.5, we can prove the following existence theorem.

Theorem 3.4. *Let E be a smooth and uniformly convex Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping of type (P). Let $\{x_n\}, \{C_n\}$, and $\{D_n\}$ be defined by* (3.5). *Then $\{x_n\}$ is well defined and the following are equivalent.*

(1) *$F(T)$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ is bounded.*

(4) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

Proof. By (3) of Lemma 3.2, we know that (1) implies (2). We first show that (2) implies (3). Suppose that $\bigcap_{n=1}^{\infty} C_n$ is nonempty and let $M_n = C_n \cap D_n$ and $N_n = D_n$ for all $n \in \mathbb{N}$. It is clear that $x_n = P_{N_n}(x)$ and $x_{n+1} = P_{M_n}(x) \in N_n$ for all $n \in \mathbb{N}$. By (3) of Lemma 3.2 and assumption, the equality $\bigcap_{n=1}^{\infty} M_n = \bigcap_{n=1}^{\infty} C_n$ holds and this set is nonempty. Thus, (2) of Lemma 2.4 implies that $\{x_n\}$ is bounded.

We next show that (3) implies (1). Suppose that $\{x_n\}$ is bounded. Then (1) of Lemma 2.4 implies that $\|x_n - x_{n+1}\| \to 0$. Since E is reflexive and C is weakly closed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u \in C$. By $Tx_n = P_{C_n}(x_n)$ and $x_{n+1} \in C_n$, we have

$$\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| \longrightarrow 0. \tag{3.12}$$

This gives us that $u \in \widehat{F}(T)$. By (2) of Lemma 2.1, we get $u \in F(T)$.

It follows from Theorem 2.5 that (1) implies that $\{x_n\}$ converges strongly to $P_{F(T)}(x)$. Thus (1) implies (4). It is obvious that (4) implies (3). This completes the proof. □

Using Lemmas 2.4, 3.3, and Corollary 2.8, we can also show the following existence theorem. We employ the methods, based on Mosco convergence, which were developed by Kimura et al. [18] and Kimura and Takahashi [19].

Theorem 3.5. *Let E be a smooth, strictly convex, and reflexive Banach space which has the Kadec-Klee property, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping of type (P). Let $\{x_n\}$ and $\{C_n\}$ be defined by* (3.10). *Then $\{x_n\}$ is well defined and the following are equivalent.*

(1) *$F(T)$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{F(T)}(x)$. Moreover, if E is uniformly convex, then these conditions are also equivalent to the following.

(4) *$\{x_n\}$ is bounded.*

Proof. By (3) of Lemma 3.3, we know that (1) implies (2). We first show that (2) implies (1). Suppose that $D = \bigcap_{n=1}^{\infty} C_n$ is nonempty. By this assumption and $C_{n-1} \supset C_n$ for all $n \in \mathbb{N}$, we know that M-$\lim_n C_n = D$. By Theorem 2.6, $\{P_{C_n}(x)\}$ converges strongly to $P_D(x)$. This implies that $\{x_n\}$ converges strongly to $P_D(x)$. Put $u = P_D(x)$. Since $u \in C_n$ for all $n \in \mathbb{N}$, we have

$$\langle u - Tx_n, J(x_n - Tx_n) \rangle \leq 0 \tag{3.13}$$

for all $n \in \mathbb{N}$. By Theorem 2.2, we have $Tx_n \to Tu$ and $J(x_n - Tx_n) \rightharpoonup J(u - Tu)$. Thus it follows from (3.13) that

$$\|u - Tu\|^2 = \langle u - Tu, J(u - Tu) \rangle = \lim_{n \to \infty} \langle u - Tx_n, J(x_n - Tx_n) \rangle \leq 0. \tag{3.14}$$

This gives us that $u \in F(T)$ and hence $F(T)$ is nonempty.

Using Corollary 2.8, we know that (1) implies that $\{x_n\}$ converges strongly to $P_{F(T)}(x)$. Hence (1) implies (3).

We next show that (3) implies (2). Suppose that $\{x_n\}$ converges strongly to $v \in C$. Let $m \in \mathbb{N}$. Then we have $x_{n+1} \in C_n \subset C_m$ for all $n \geq m$. Since C_m is closed and $x_n \to v$, we have $v \in C_m$. This gives us that $v \in \bigcap_{m=1}^{\infty} C_m$ and hence $\bigcap_{m=1}^{\infty} C_m$ is nonempty.

We next show that (4) implies (1). Suppose that E is uniformly convex and $\{x_n\}$ is bounded and let $M_n = C_n$ and $N_n = C_{n-1}$ for all $n \in \mathbb{N}$. Then it is clear that $x_n = P_{N_n}(x)$ and $x_{n+1} = P_{M_n}(x) \in N_n$. By (1) of Lemma 2.4, we know that $x_n - x_{n+1} \to 0$. Since E is reflexive and C is weakly closed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w \in C$. Let

$$D_n = \{z \in C : \langle z - Tx_n, J(x_n - Tx_n) \rangle \leq 0\} \tag{3.15}$$

for all $n \in \mathbb{N}$. Since $Tx_n = P_{D_n}(x_n)$ and $x_{n+1} \in C_n \subset D_n$ for all $n \in \mathbb{N}$, we have

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| \longrightarrow 0. \tag{3.16}$$

This gives us that $w \in \widehat{F}(T)$. By (2) of Lemma 2.1, we get $w \in F(T)$. Thus $F(T)$ is nonempty. It is obvious that (3) implies (4). This completes the proof. □

4. Deduced Results

In this section, we obtain some corollaries of Theorems 3.4 and 3.5. We first deduce the following corollary from Theorem 3.4.

Corollary 4.1. *Let E be a smooth and uniformly convex Banach space and $A : E \to 2^{E^*}$ a monotone operator such that there exists a nonempty closed convex subset C of E satisfying $D(A) \subset C \subset R(I + J^{-1}A)$. Let $T : C \to C$ be the mapping defined by $Tx = (I + J^{-1}A)^{-1}x$ for all $x \in C$ and $\{x_n\}$ a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ C_n &= \{z \in C : \langle z - Tx_n, J(x_n - Tx_n)\rangle \leq 0\}, \\ D_n &= \{z \in C : \langle z - x_n, J(x - x_n)\rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap D_n}(x) \end{aligned} \tag{4.1}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and the following are equivalent.

(1) *$A^{-1}0$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ is bounded.*

(4) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{A^{-1}0}(x)$.

Proof. By assumption, we know that T: $C \to C$ is a mapping of type (P) and $F(T) = A^{-1}0$. Therefore, Theorem 3.4 implies the conclusion. □

We can similarly deduce the following corollary from Theorem 3.5; see Kimura and Takahashi [19] for related results.

Corollary 4.2. *Let E be a smooth, strictly convex, and reflexive Banach space which has the Kadec-Klee property and $A : E \to 2^{E^*}$ a monotone operator such that there exists a nonempty closed convex subset C of E satisfying $D(A) \subset C \subset R(I + J^{-1}A)$. Let $T : C \to C$ be the mapping defined by $Tx = (I + J^{-1}A)^{-1}x$ for all $x \in C$ and $\{x_n\}$ a sequence generated by*

$$\begin{aligned} x_1 &= x \in C = C_0, \\ C_n &= \{z \in C_{n-1} : \langle z - Tx_n, J(x_n - Tx_n)\rangle \leq 0\}, \\ x_{n+1} &= P_{C_n}(x) \end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and the following are equivalent.

(1) *$A^{-1}0$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{A^{-1}0}(x)$. Moreover, if E is uniformly convex, then these conditions are also equivalent to the following.

(4) *$\{x_n\}$ is bounded.*

As direct consequences of Theorems 3.4 and 3.5, we also obtain the following corollaries.

Corollary 4.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a firmly nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} x_1 &= x \in C, \\ C_n &= \{z \in C : \langle z - Tx_n, x_n - Tx_n \rangle \leq 0\}, \\ D_n &= \{z \in C : \langle z - x_n, x - x_n \rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap D_n}(x) \end{aligned} \tag{4.3}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and the following are equivalent.

(1) *$F(T)$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ is bounded.*

(4) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

Corollary 4.4 (see [12]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a firmly nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} x_1 &= x \in C = C_0, \\ C_n &= \{z \in C_{n-1} : \langle z - Tx_n, x_n - Tx_n \rangle \leq 0\}, \\ x_{n+1} &= P_{C_n}(x) \end{aligned} \tag{4.4}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and the following are equivalent.

(1) *$F(T)$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ is bounded.*

(4) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

Using Corollary 4.3, we next show the following result; see also [21, 24].

Corollary 4.5. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} x_1 &= x \in C, \\ C_n &= \{z \in C : \|z - Tx_n\| \leq \|z - x_n\|\}, \\ D_n &= \{z \in C : \langle z - x_n, x - x_n \rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap D_n}(x) \end{aligned} \tag{4.5}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and the following are equivalent.

(1) *$F(T)$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ is bounded.*

(4) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

Proof. Let S be a mapping defined by $S = (I + T)/2$. Then S: $C \to C$ is a firmly nonexpansive mapping and $F(S) = F(T)$. We also know that

$$\begin{aligned} \|z - x_n\|^2 - \|z - Tx_n\|^2 &= \|x_n\|^2 - 2\langle z, x_n \rangle - \|Tx_n\|^2 + 2\langle z, Tx_n \rangle \\ &= \langle x_n + Tx_n - 2z, x_n - Tx_n \rangle \\ &= 4\langle Sx_n - z, x_n - Sx_n \rangle \end{aligned} \tag{4.6}$$

for all $n \in \mathbb{N}$. This implies that $C_n = \{z \in C : \langle z - Sx_n, x_n - Sx_n \rangle \leq 0\}$ for all $n \in \mathbb{N}$. Thus Corollary 4.3 implies the conclusion. □

Using Corollary 4.4, we can similarly show the following result.

Corollary 4.6. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} x_1 &= x \in C = C_0, \\ C_n &= \{z \in C_{n-1} : \|z - Tx_n\| \leq \|z - x_n\|\}, \\ x_{n+1} &= P_{C_n}(x) \end{aligned} \tag{4.7}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and the following are equivalent.

(1) *$F(T)$ is nonempty.*

(2) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

(3) *$\{x_n\}$ is bounded.*

(4) *$\{x_n\}$ converges strongly.*

In this case, $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

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