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Research Article

Some Krasnonsel'skiĭ-Mann Algorithms and the Multiple-Set Split Feasibility Problem

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Some variable Krasnonsel'skiĭ-Mann iteration algorithms generate some sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, respectively, via the formula $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_N \cdots T_2 T_1 x_n$, $y_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \lambda_i T_i y_n$, $z_{n+1} = (1-\gamma_{n+1})z_n + \gamma_{n+1} T_{[n+1]} z_n$, where $T_{[n]} = T_{n \bmod N}$ and the mod function takes values in $\{1, 2, \ldots, N\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{T_1, T_2, \ldots, T_N\}$ are sequences of nonexpansive mappings. We will show, in a fairly general Banach space, that the sequence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by the above formulas converge weakly to the common fixed point of $\{T_1, T_2, \ldots, T_N\}$, respectively. These results are used to solve the multiple-set split feasibility problem recently introduced by Censor et al. (2005). The purpose of this paper is to introduce convergence theorems of some variable Krasnonsel'skiĭ-Mann iteration algorithms in Banach space and their applications which solve the multiple-set split feasibility problem.

1. Introduction

The Krasnonsel'skiĭ-Mann (K-M) iteration algorithm [1, 2] is used to solve a fixed point equation

$$Tx = x, \tag{1.1}$$

where T is a self-mapping of closed convex subset C of a Banach space X. The K-M algorithm generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ and the initial guess $x_0 \in C$ is chosen arbitrarily. It is known [3] that if X is a uniformly convex Banach space with a Frechet differentiable norm (in particular, a Hilbert space), if $T : C \to C$ is nonexpansive, that is, T satisfies the property

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C \tag{1.3}$$

and if T has a fixed point, then the sequence $\{x_n\}$ generated by the K-M algorithm (1.2) converges weakly to a fixed point of T provided that $\{\alpha_n\}$ fulfils the condition

$$\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty. \tag{1.4}$$

(See [4, 5] for details on the fixed point theory for nonexpansive mappings.)

Many problems can be formulated as a fixed point equation (1.1) with a nonexpansive T and thus K-M algorithm (1.2) applies. For instance, the split feasibility problem (SFP) introduced in [6–8], which is to find a point

$$x \in C \quad \text{such that } Ax \in Q, \tag{1.5}$$

where C and Q are closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and A is a linear bounded operator from H_1 to H_2. This problem plays an important role in the study of signal processing and image reconstruction. Assuming that the SFP (1.5) is consistent (i.e., (1.5) has a solution), it is not hard to see that $x \in C$ solves (1.5) if and only if it solves the fixed point equation

$$x = P_C (I - \gamma A^* (I - P_Q) A) x, \quad x \in C, \tag{1.6}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. Moreover, for sufficiently small $\gamma > 0$, the operator $P_C(I - \gamma A^*(I - P_Q)A)$ which defines the fixed point equation (1.6) is nonexpansive.

To solve the SFP (1.5), Byrne [7, 8] proposed his CQ algorithm (see also [9]) which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C (I - \gamma A^* (I - P_Q) A) x_n, \quad n \ge 0, \tag{1.7}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A. In 2005, Zhao and Yang [10] considered the following perturbed algorithm:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_{C_n} (I - \gamma A^* (I - P_{Q_n}) A) x_n, \tag{1.8}$$

where C_n and Q_n are sequences of closed and convex subsets of H_1 and H_2, respectively, which are convergent to C and Q, respectively, in the sense of Mosco (c.f. [11]). Motivated by (1.8), Zhao and Yang [10, 12] also studied the following more general algorithm which generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \tag{1.9}$$

where $\{T_n\}$ is a sequence of nonexpansive mappings in a Hilbert space H, under certain conditions, they proved convergence of (1.9) essentially in a finite-dimensional Hilbert space. Furthermore, with regard to (1.9), Xu [13] extended the results of Zhao and Yang [10] in the framework of fairly general Banach space.

The multiple-set split feasibility problem (MSSFP) which finds application in intensity-modulated radiation therapy [14] has recently been proposed in [15] and is formulated as finding a point

$$x \in C = \bigcap_{i=1}^{N} C_i \quad \text{such that } Ax \in Q = \bigcap_{j=1}^{M} Q_j, \tag{1.10}$$

where N and M are positive integers, $\{C_1, C_2, \dots, C_N\}$ and $\{Q_1, Q_2, \dots, Q_M\}$ are closed and convex subsets of H_1 and H_2, respectively, and A is a linear bounded operator from H_1 to H_2.

Assuming consistency of the MSSFP (1.10), Censor et al. [15] introduced the following projection algorithm:

$$x_{n+1} = P_\Omega\left(x_n - \gamma\left(\sum_{i=1}^{N}\alpha_i(x_n - P_{C_i}x_n) + \sum_{j=1}^{M}\beta_j A^*\left(Ax_n - P_{Q_j}Ax_n\right)\right)\right), \tag{1.11}$$

where Ω is another closed and convex subset of H_1, $0 < \gamma < 2/L$ with $L = \sum_{i=1}^{N}\alpha_i + \rho(A^*A)\sum_{j=1}^{M}\beta_j$ and $\rho(A^*A)$ being the spectral radius of A^*A, and $\alpha_i > 0$ for all i and $\beta_j > 0$ for all j. They studied convergence of the algorithm (1.11) in the case where both H_1 and H_2 are finite dimensional. In 2006, Xu [13] demonstrated some projection algorithms for solving the MSSFP (1.10) in Hilbert space as follows:

$$\begin{aligned} x_{n+1} &= \left[P_{C_N}(I-\gamma\nabla q)\right]\cdots\left[P_{C_1}(I-\gamma\nabla q\right] x_n, \quad n \geq 0, \\ y_{n+1} &= \sum_{i=1}^{N}\lambda_i P_{C_i}\left(y_n - \gamma\sum_{j=1}^{M}\beta_j A^*\left(I-P_{Q_j}\right)Ay_n\right), \quad n \geq 0, \\ z_{n+1} &= P_{C_{[n+1]}}\left(z_n - \gamma\sum_{j=1}^{M}\beta_j A^*\left(I-P_{Q_j}\right)Az_n\right), \quad n \geq 0, \end{aligned} \tag{1.12}$$

where $q(x) = (1/2)\sum_{j=1}^{M}\beta_j\|P_{Q_j}Ax - Ax\|^2$, $\nabla q(x) = \sum_{j=1}^{M}\beta_j A^*(I-P_{Q_j})Ax$, $x \in C$, and $C_{[n]} = C_{n \bmod N}$ and the mod function takes values in $\{1, 2, \dots, N\}$. This is a motivation for us to study the following more general algorithm which generate the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, respectively, via the formulas

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_N \cdots T_2 T_1 x_n, \tag{1.13}$$

$$y_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \lambda_i T_i y_n, \tag{1.14}$$

$$z_{n+1} = (1-\gamma_{n+1})z_n + \gamma_{n+1} T_{[n+1]} z_n, \tag{1.15}$$

where $T_{[n]} = T_{n \bmod N}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{T_1, T_2, \ldots, T_N\}$ are sequences of nonexpansive mappings. We will show, in a fairly general Banach space X, that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (1.13), (1.14), and (1.15) converge weakly to the common fixed point of $\{T_1, T_2, \ldots, T_N\}$, respectively. The applications of these results are used to solve the multiple-set split feasibility problem recently introduced by [15].

Note that, letting C be a nonempty subset of Banach space X and A, B are self-mappings of C, we use $D_\rho(A,B)$ to denote $\sup\{\|Ax - Bx\| : \|x\| \le \rho\}$, that is,

$$D_\rho(A,B) := \sup\{\|Ax - Bx\| : \|x\| \le \rho\}. \tag{1.16}$$

This paper is organized as follows. In the next section, we will prove a weak convergence theorems for the three variable K-M algorithms (1.13), (1.14), and (1.15) in a uniformly convex Banach space with a Frechet differentiable norm (the class of such Banach spaces include Hilbert space and L^p and l^p space for $1 < p < \infty$). In the last section, we will present the applications of the weak convergence theorems for the three variable K-M algorithms (1.13), (1.14), and (1.15).

2. Convergence of Variable Krasnonsel'skiĭ-Mann Iteration Algorithm

To solve the multiple-set split feasibility problem (MSSFP) in Section 3, we firstly present some theorems of the general variable Krasnonsel'skiĭ-Mann iteration algorithms.

Theorem 2.1. *Let X be a uniformly convex Banach space with a Frechet differentiable norm, let C be a nonempty closed and convex subset of X, and let $T_i : C \to C$ be nonexpansive mapping, $i = 1, 2, \ldots, N$. Assume that the set of common fixed point of $\{T_1, T_2, \ldots, T_N\}$, $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$, is nonempty. Let $\{x_n\}$ be any sequence generated by* (1.13), *where $0 < \alpha_n < 1$ satisfy the conditions*

(i) $\sum_{n=0}^{\infty} \alpha_n (1-\alpha_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n D_\rho(T_N \cdots T_1, T_i) < \infty$ *for every $\rho > 0$ and $i = 1, 2, \ldots, N$, where* $D_\rho(T_N \cdots T_1, T_i) = \sup\{\|T_N \cdots T_1 x - T_i x\| : \|x\| \le \rho\}$.

Then $\{x_n\}$ converges weakly to a common fixed point p of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Since $T_i : C \to C$ is nonexpansive mapping, for $i = 1, 2, \ldots, N$, then, the composition $T_N \cdots T_2 T_1$ is nonexpansive mapping from C to C. Let $U := T_N \cdots T_2 T_1$.

Take $x \in \bigcap_{j=1}^{N} \mathrm{Fix}(T_j)$ $(x \in \mathrm{Fix}(U))$ to deduce that

$$\begin{aligned} \|x_{n+1} - x\| &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n \|Ux_n - x\| \\ &\leq \|x_n - x\|. \end{aligned} \tag{2.1}$$

Thus, $\{\|x_n - x\|\}$ is a decreasing sequence, and we have that $\lim_{n\to\infty}\|x_n - x\|$ exists. Hence, $\{x_n\}$ is bounded, so are $\{T_i x_n\}$, $i = 1, 2, \ldots, N$, and $\{Ux_n\}$. Let $\rho = \sup\{\|x_n\|, \|Ux_n - T_i x_n\| : n \geq 0, i = 1, 2, \ldots, N\} < \infty$, and let $r = 2\rho + \|x\| < \infty$.

Now since X is uniformly convex, by [16, Theorem 2], there exists a continuous strictly convex function φ, with $\varphi(0) = 0$, so that

$$\left\|\lambda x + (1-\lambda)y\right\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\left\|y\right\|^2 - \lambda(1-\lambda)\varphi\left(\left\|x - y\right\|\right), \tag{2.2}$$

for all $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$ and for all $\lambda \in [0, 1]$. Let $Ux_n - T_i x_n$, $i = 1, 2, \ldots, N$, be replaced by $e_{n,i}$ (note that $\|e_{n,i}\| \leq D_\rho(U, T_i)$), and taking a constant M so that $M \geq \sup\{2\|x_n - x\| + \alpha_n\|e_{n,i}\| : n \geq 0\}$, by the above (2.2), we obtain that

$$\begin{aligned} \|x_{n+1} - x\|^2 &= \|(1-\alpha_n)(x_n - x + \alpha_n e_{n,i}) + \alpha_n(T_i x_n - x + \alpha_n e_{n,i})\|^2 \\ &\leq (1-\alpha_n)\|x_n - x + \alpha_n e_{n,i}\|^2 + \alpha_n\|T_i x_n - x + \alpha_n e_{n,i}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\varphi(\|x_n - T_i x_n\|) \\ &\leq (1-\alpha_n)\Big(\|x_n - x\|^2 + 2\alpha_n\|x_n - x\|\|e_{n,i}\| + \alpha_n^2\|e_{n,i}\|^2\Big) \\ &\quad + \alpha_n\Big(\|T_i x_n - x\|^2 + 2\alpha_n\|e_{n,i}\|\|T_i x_n - x\| + \alpha_n^2\|e_{n,i}\|^2\Big) \\ &\quad - \alpha_n(1-\alpha_n)\varphi(\|x_n - T_i x_n\|) \\ &\leq \|x_n - x\|^2 + M\alpha_n D_\rho(U, T_i) - \alpha_n(1-\alpha_n)\varphi(\|x_n - T_i x_n\|). \end{aligned} \tag{2.3}$$

It follows that

$$\alpha_n(1-\alpha_n)\varphi(\|x_n - T_i x_n\|) \leq \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + M\alpha_n D_\rho(U, T_i). \tag{2.4}$$

Since $\lim_{n\to\infty}\|x_n - x\|$ exists, by condition (ii) and (2.4), it implies that

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\varphi\left(\left\|x_n - T_i y_n\right\|\right) < \infty \tag{2.5}$$

which further implies that by (i) $\liminf_{n\to\infty}\varphi(\|x_n - T_ix_n\|) = 0$, hence,

$$\liminf_{n\to\infty}\|x_n - T_ix_n\| = 0. \tag{2.6}$$

On the other hand, it is not hard to deduce from (1.13) that

$$\begin{aligned}
\|x_{n+1} - T_ix_{n+1}\| &= \|(1-\alpha_n)x_n + \alpha_nUx_n - T_ix_{n+1}\| \\
&= \|(1-\alpha_n)x_n + \alpha_nUx_n - T_ix_n + T_ix_n - T_ix_{n+1}\| \\
&\le (1-\alpha_n)\|x_n - T_ix_n\| + \alpha_n\|Ux_n - T_ix_n\| + \|x_{n+1} - x_n\| \\
&= (1-\alpha_n)\|x_n - T_ix_n\| + \alpha_n\|Ux_n - T_ix_n\| + \alpha_n\|x_n - Ux_n\| \\
&\le (1-\alpha_n)\|x_n - T_ix_n\| + \alpha_n\|Ux_n - T_ix_n\| \\
&\quad + \alpha_n\|x_n - T_ix_n\| + \alpha_n\|T_ix_n - Ux_n\| \\
&= \|x_n - T_ix_n\| + 2\alpha_n\|Ux_n - T_ix_n\| \\
&\le \|x_n - T_ix_n\| + 2\alpha_nD_\rho(T_i, U).
\end{aligned} \tag{2.7}$$

Since $\sum_{n=1}^{\infty}\alpha_nD_\rho(U,T_i) < \infty$, we see that $\lim_{n\to\infty}\|x_n - T_ix_n\|$ exists. This together with (2.6) implies that

$$\lim_{n\to\infty}\|x_n - T_ix_n\| = 0. \tag{2.8}$$

The demiclosedness principle for nonexpansive mappings (see [5, 17]) implies that

$$\omega_w(x_n) \subset \bigcap_{i=1}^{N}F(T_i), \tag{2.9}$$

where $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

To prove that $\{x_n\}$ is weakly convergent to a common fixed point p of $\{T_1, T_2, \ldots, T_N\}$, it now suffices to prove that $\omega_w(x_n)$ consists of exactly one point.

Indeed, if there are $\overline{x}$, $\widetilde{x} \in \omega_w(x_n)(x_{n_i} \rightharpoonup \overline{x},\ x_{m_j} \rightharpoonup \widetilde{x})$, since $\lim_{n\to\infty}\|x_n - \overline{x}\|$ and $\lim_{n\to\infty}\|x_n - \widetilde{x}\|$ exist, if $\widetilde{x} \neq \overline{x}$, then

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - \widetilde{x}\|^2 &= \lim_{j\to\infty}\left\|\left(x_{m_j} - \overline{x}\right) + (\overline{x} - \widetilde{x})\right\|^2 \\
&= \lim_{j\to\infty}\left\|x_{m_j} - \overline{x}\right\|^2 + \|\overline{x} - \widetilde{x}\|^2 + 2\lim_{j\to\infty}\left\langle x_{m_j} - \overline{x}, \overline{x} - \widetilde{x}\right\rangle \\
&= \lim_{j\to\infty}\left\|x_{m_j} - \overline{x}\right\|^2 + \|\overline{x} - \widetilde{x}\|^2 \\
&> \lim_{i\to\infty}\|x_{m_i} - \overline{x}\|^2 = \lim_{i\to\infty}\|x_{n_i} - \overline{x}\|^2 \\
&= \lim_{i\to\infty}\|(x_{n_i} - \widetilde{x}) + (\widetilde{x} - \overline{x})\|^2 \\
&= \lim_{i\to\infty}\|x_{n_i} - \widetilde{x}\|^2 + \|\widetilde{x} - \overline{x}\|^2 + 2\lim_{j\to\infty}\langle x_{n_i} - \widetilde{x}, \widetilde{x} - \overline{x}\rangle \\
&= \lim_{i\to\infty}\|x_{n_i} - \widetilde{x}\|^2 + \|\widetilde{x} - \overline{x}\|^2 \\
&> \lim_{i\to\infty}\|x_{n_i} - \widetilde{x}\|^2 = \lim_{n}\|x_n - \widetilde{x}\|^2.
\end{aligned} \tag{2.10}$$

This is a contradiction.

The proof is completed. □

Theorem 2.2. *Let X be a uniformly convex Banach space with a Frechet differentiable norm, let C be a nonempty closed and convex subset of X, and let $T_i : C \to C$ be nonexpansive mapping, $i = 1, 2, \ldots, N$, assume that the set of common fixed point of $\{T_1, T_2, \ldots, T_N\}$, $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$, is nonempty. Let $\{y_n\}$ be defined by* (1.14)*, where $0 < \beta_n < 1$ satisfy the following conditions*

(i) $\sum_{n=0}^{\infty}\beta_n(1-\beta_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}\beta_n D_\rho(\sum_{i=1}^N \lambda_i T_i, T_i) < \infty$ *for every* $\rho > 0$ *and* $i = 1, 2, \ldots, N$, *where* $D_\rho(\sum_{i=1}^N \lambda_i T_i, T_i) = \sup\{\|\sum_{i=1}^N \lambda_i T_i x - T_i x\| : \|x\| \le \rho\}$.

Then $\{y_n\}$ converges weakly to a common fixed point q of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Since $T_i : C \to C$ is a nonexpansive mapping, $i = 1, 2, \ldots, N$, then, it is not hard to see that $\sum_{i=1}^N \lambda_i T_i$ is a nonexpansive mapping from C to C.

The remainder of the proof is the same as Theorem 2.1.

The proof is completed. □

Theorem 2.3. *Let X be a uniformly convex Banach space with a Frechet differentiable norm, let C be a nonempty closed convex subset of X, and let $T_i : C \to C$ be nonexpansive mapping, $i = 1, 2, \ldots, N$, assume that the set of common fixed point of $\{T_1, T_2, \ldots, T_N\}$, $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$, is nonempty. Let $\{z_n\}$ be defined by* (1.15)*, where $0 < \gamma_n < 1$ satisfy the conditions*

(i) $\sum_{n=0}^{\infty}\gamma_n(1-\gamma_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}\gamma_n D_\rho(T_{[n+1]}, T_i) < \infty$ *for every* $\rho > 0$ *and* $i = 1, 2, \ldots, N$, *where* $D_\rho(T_{[n+1]}, T_i) = \sup\{\|T_{[n+1]}x - T_i x\| : \|x\| \le \rho\}$.

Then $\{z_n\}$ converges weakly to a common fixed point w of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Since $T_{[n]} = T_{n \bmod N}$ and $\{T_1, T_2, \ldots, T_N\}$ is a sequence of nonexpansive mappings from C to C, so, the proof of this theorem is similar to Theorems 2.1 and 2.2.

The proof is completed. □

3. Applications for Solving the Multiple-Set Split Feasibility Problem (MSSFP)

Recall that a mapping T in a Hilbert space H is said to be averaged if T can be written as $(1-\lambda)I + \lambda S$, where $\lambda \in (0,1)$ and S is nonexpansive. Recall also that an operator A in H is said to be γ-inverse strongly monotone (γ-ism) for a given constant $\gamma > 0$ if

$$\langle x - y,\, Ax - Ay\rangle \geq \gamma \left\| Ax - Ay \right\|^2, \quad \forall x, y \in H. \tag{3.1}$$

A projection P_K of H onto a closed convex subset K is both nonexpansive and 1-ism. It is also known that a mapping T is averaged if and only if the complement $I - T$ is γ-ism for some $\gamma > 1/2$; see [8] for more property of averaged mappings and γ-ism.

To solve the MSSFP (1.10), Censor et al. [15] proposed the following projection algorithm (1.11), the algorithm (1.11) involves an additional projection P_Ω. Though the MSSFP, (1.10) includes the SFP (1.5) as a special case, which does not reduced to (1.7), let alone (1.8). In this section, we will propose some new projection algorithms which solve the MSSFP (1.10) and which are the application of algorithms (1.13), (1.14), and (1.15) for solving the MSSFP. These projection algorithms can also reduce to the algorithm (1.8) when the MSSFP (1.10) is reduced to the SFP (1.5).

The first one is a K-M type successive iteration method which produces a sequence $\{x_n\}$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \left[P_{C_N}(I - \gamma\nabla q)\right] \cdots \left[P_{C_1}(I - \gamma\nabla q\right] x_n, \quad n \geq 0. \tag{3.2}$$

Theorem 3.1. *Assume that the MSSFP* (1.10) *is consistent. Let $\{x_n\}$ be the sequence generated by the algorithm* (3.2), *where $0 < \gamma < 2/L$ with $L = \|A\|^2 \sum_{j=1}^{M} \beta_j$ and $0 < \alpha_n < 1$ satisfy the condition: $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a solution of the MSSFP* (1.10).

Proof. Let $T_i := P_{C_i}(I - \gamma\nabla q)$, $i = 1, 2, \ldots, N$.

Hence,

$$U = T_N \cdots T_1 = \left[P_{C_N}(I - \gamma\nabla q)\right] \cdots \left[P_{C_1}(I - \gamma\nabla q)\right]. \tag{3.3}$$

Since

$$\nabla q(x) = \sum_{j=1}^{M} \beta_j A^* \left(I - P_{Q_j}\right) Ax, \quad x \in C, \tag{3.4}$$

and $I - P_{Q_j}$ is nonexpansive, it is easy to see that ∇q is L-Lipschitzian, with $L = \|A\|^2 \sum_{j=1}^{M} \beta_j$.

Therefore, ∇q is $(1/L)$-ism [18]. This implies that for any $0 < \gamma < 2/L$, $I - \gamma\nabla q$ is averaged. Hence, for any closed and convex subset K of H_1, the composite $P_K(I - \gamma\nabla q)$ is averaged.

So $U = T_N \cdots T_1 = [P_{C_N}(I - \gamma\nabla q)] \cdots [P_{C_1}(I - \gamma\nabla q)]$ is averaged, thus U is nonexpansive.

By the position 2.2 [8], we see that the fixed point set of U, $\mathrm{Fix}(U)$, is the common fixed point set of the averaged mappings $\{T_N \cdots T_1\}$.

By Reich [3], we have $\{x_n\}$ converges weakly to a fixed point of U which is also a common fixed point of $\{T_N \cdots T_1\}$ or a solution of the MSSFP (1.10).

The proof is completed. $\square$

The second algorithm is also a K-M type method which generates a sequence $\{y_n\}$ by

$$y_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \lambda_i P_{C_i}\left(y_n - \gamma \sum_{j=1}^{M} \beta_j A^* \left(I - P_{Q_j}\right) A y_n \right), \quad n \geq 0. \tag{3.5}$$

Theorem 3.2. *Assume that the MSSFP* (1.10) *is consistent. Let $\{x_n\}$ be any sequence generated by the algorithm* (3.5), *where $0 < \gamma < 2/L$ with $L = \|A\|^2 \sum_{j=1}^{M} \beta_j$ and $0 < \beta_n < 1$ satisfy the condition: $\sum_{n=0}^{\infty} \beta_n(1-\beta_n) = \infty$. Then $\{y_n\}$ converges weakly to a solution of the MSSFP* (1.10).

Proof. From the proof of Theorem 3.1, it is easy to know that $T_i := P_{C_i}(I - \gamma \nabla q)$ is averaged, so, the convex combination $S := \sum_{i=1}^{N} \lambda_i T_i$ is also averaged.

Thus S is nonexpansive.

By Reich [3], we have $\{y_n\}$ converges weakly to a fixed point of S.

Next, we only need to prove the fixed point of S is also the common fixed point of $\{T_N \cdots T_1\}$ which is the solution of the MSSFP (1.10), that is, $\mathrm{Fix}(S) = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

Indeed, it suffices to show that $\bigcap_{n=1}^{N} \mathrm{Fix}(T_i) \supset \mathrm{Fix}(\sum_{i=1}^{N} \lambda_i T_i)$.

Pick an arbitrary $x \in \mathrm{Fix}(\sum_{i=1}^{N} \lambda_i T_i)$, thus $\sum_{i=1}^{N} \lambda_i T_i x = x$. Also pick a $y \in \mathrm{Fix}(\bigcap_{n=1}^{N} T_i)$, thus $T_i y = y$, $i = 1, 2, \ldots, N$.

Write $T_i = (1-\beta_i)I + \beta_i \widetilde{T}_i$, $i = 1, 2, \ldots, N$ with $\beta_i \in (0,1)$ and $\widetilde{T}_i$ is nonexpansive.

We claim that if z is such that $T_i z \neq z$, then $\|T_i x - y\| < \|x - y\|$, $i = 1, 2, \ldots, N$.

Indeed, we have

$$\begin{aligned} \left\|T_i z - y\right\|^2 &= \left\|(1-\beta_i)(z-y) + \beta_i\left(\widetilde{T}_i z - y\right)\right\|^2 \\ &= (1-\beta_i)\left\|z-y\right\|^2 + \beta_i\left\|\widetilde{T}_i z - y\right\|^2 - \beta_i(1-\beta_i)\left\|z - \widetilde{T}_i z\right\|^2 \\ &\leq \left\|z-y\right\|^2 - (1-\beta_i)\|z - T_i z\|^2 \\ &< \left\|z-y\right\|^2, \quad \text{as } \|z - T_i z\| > 0. \end{aligned} \tag{3.6}$$

If we can show that $T_i x = x$, then we are done. So assume that $Tx \neq x$. Now since $\sum_{i=1}^{N} \lambda_i T_i x = x \neq Tx$, we have

$$\begin{aligned} \left\|x-y\right\| &= \left\|\sum_{i=1}^{N} \lambda_i T_i x - y\right\| \\ &\leq \sum_{i=1}^{N} \lambda_i \left\|T_i x - y\right\| \\ &< \left\|x-y\right\|. \end{aligned} \tag{3.7}$$

This is a contradiction. Therefore, we must have $T_i x = x$, $i = 1, 2, \ldots, N$, that is, $\bigcap_{n=1}^{N} \mathrm{Fix}(T_i)x = x$.

This proof is completed. □

We now apply Theorem 2.3 to solve the MSSFP (1.10). Recall that the ρ-distance between two closed and convex subsets E_1 and E_2 of a Hilbert space H is defined by

$$d_\rho(E_1, E_2) = \sup_{\|x\| \le \rho} \{\|P_{E_1} x - P_{E_2} x\|\}. \tag{3.8}$$

The third method is a K-M type cyclic algorithm which produces a sequence $\{z_n\}$ in the following manner: apply T_1 to the initial guess z_0 to get $z_1 = (1-\gamma_1)z_0 + \gamma_1 P_{C_1}(z_0 - \gamma \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Az_0)$, next apply T_2 to z_1 to get $z_2 = (1-\gamma_2)z_1 + \gamma_2 P_{C_2}(z_1 - \gamma \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Az_1)$, and continue this way to get $z_N = (1-\gamma_N)z_0 + \gamma_N P_{C_N}(z_{N-1} - \gamma \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Az_{N-1})$; then repeat this process to get $z_{N+1} = (1-\gamma_{N+1})z_0 + \gamma_N P_{C_1}(z_N - \gamma \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Az_N)$, and so on. Thus, the sequence $\{z_n\}$ is defined and we write it in the form

$$z_{n+1} = (1-\gamma_{n+1})z_0 + \gamma_{n+1} P_{C_{[n+1]}} \left(z_n - \gamma \sum_{j=1}^{M} \beta_j A^* \left(I - P_{Q_j}\right) A z_n \right), \quad n \ge 0, \tag{3.9}$$

where $C_{[n]} = C_{n \bmod N}$.

Theorem 3.3. *Assume that the MSSFP* (1.10) *is consistent. Let $\{x_n\}$ be the sequence generated by the algorithm* (3.9)*, where $0 < \gamma < 2/L$ with $L = \|A\|^2 \sum_{j=1}^{M} \beta_j$ and $0 < \gamma_n < 1$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \gamma_n (1-\gamma_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} \gamma_n d_\rho(C_{[n+1]}, C_i) < \infty$ *and* $\sum_{n=0}^{\infty} \gamma_n d_\rho(Q_{[n+1]}, Q_i) < \infty$ *for each* $\rho > 0$, $i = 1, 2, \ldots, N$.

Then $\{z_n\}$ converges weakly to a solution of the MSSFP (1.10).

Proof. From the proof of application (3.2), it is easy to verify that $T_i := P_{C_i}(I - \gamma \nabla q)$ is averaged, so, $T_{[n+1]} := T_{n+1 \bmod N}$ is also averaged.

Thus $T_{[n+1]}$ is nonexpansive.

The projection iteration algorithm (3.9) can also be written as

$$z_{n+1} = (1-\gamma_{n+1})z_n + \gamma_{n+1} T_{[n+1]} z_n. \tag{3.10}$$

Given $\rho > 0$, let

$$\tilde{\rho} = \sup\left\{\max\left\{\|Ax\|, \left\|x - \gamma A^*(I - P_Q)Ax\right\|\right\} : \|x\| \le \rho\right\} < \infty. \tag{3.11}$$

We compute, for $x \in H_1$, such that $\|x\| \le \rho$,

$$\begin{aligned}
&\left\|T_{[n+1]}x - T_i x\right\| \\
&\quad \le \left\|P_{C_{[n+1]}}\left(x - \gamma A^*\left(I - P_{Q_{[n+1]}}\right)Ax\right) - P_{C_{[n+1]}}\left(x - \gamma A^*\left(I - P_{Q_i}\right)Ax\right)\right\| \\
&\qquad + \left\|P_{C_{[n+1]}}\left(x - \gamma A^*\left(I - P_{Q_i}\right)Ax\right) - P_{C_i}\left(x - \gamma A^*\left(I - P_{Q_i}\right)Ax\right)\right\| \\
&\quad \le \left\|P_{C_{[n+1]}}\left(x - \gamma A^*\left(I - P_{Q_i}\right)Ax\right) - P_{C_i}\left(x - \gamma A^*\left(I - P_{Q_i}\right)Ax\right)\right\| \\
&\qquad + \gamma\left\|A^*\left(P_{Q_{[n+1]}}Ax - P_{Q_i}Ax\right)\right\| \\
&\quad \le d_{\tilde{\rho}}\left(C_{[n+1]}, C_i\right) + \gamma\|A\| d_{\tilde{\rho}}\left(Q_{[n+1]}, Q_i\right).
\end{aligned} \tag{3.12}$$

This shows that

$$D_\rho\left(T_{[n+1]}, T_i\right) \le d_{\tilde{\rho}}\left(C_{[n+1]}, C_i\right) + \gamma\|A\| d_{\tilde{\rho}}\left(Q_{[n+1]}, Q_i\right). \tag{3.13}$$

It then follows from condition (ii) that

$$\sum_{n=0}^{\infty}\gamma_n D_\rho\left(T_{[n+1]}, T_i\right) \le \sum_{n=0}^{\infty}\gamma_n d_{\tilde{\rho}}\left(C_{[n+1]}, C_i\right) + \sum_{n=0}^{\infty}\gamma_n d_{\tilde{\rho}}\left(Q_{[n+1]}, Q_i\right) < \infty. \tag{3.14}$$

Now we cam apply Theorem 2.3 to conclude that the sequence $\{z_n\}$ given by the projection Algorithm (3.9) converges weakly to a solution of the MSSFP (1.10).

The proof is completed. □

Remark 3.4. The algorithms (3.12), (3.13), and (3.15) of Xu [13] are some projection algorithms for solving the MSSEP (1.10), which are concrete projection algorithms. In this paper, firstly, we present some general variable K-M algorithms (1.13), (1.14), and (1.15), and prove the weak convergence for them in Section 2. Secondly, through the applications of the weak convergence for three general variable K-M algorithms (1.13), (1.14), and (1.15), we solve the MSSEP (1.10) by the algorithms (3.2), (3.5), and (3.9).

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