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Research Article

On Some Geometric Constants and the Fixed Point Property for Multivalued Nonexpansive Mappings

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We show some geometric conditions on a Banach space X concerning the Jordan-von Neumann constant, Zbăganu constant, characteristic of (separation) noncompact convexity, and the coefficient $R(1, X)$, the weakly convergent sequence coefficient, which imply the existence of fixed points for multivalued nonexpansive mappings.

1. Introduction

Fixed point theory for multivalued mappings has many useful applications in Applied Sciences, in particular, in game theory and mathematical economics. Thus it is natural to try of extending the known fixed point results for single-valued mappings to the setting of multivalued mappings.

In 1969, Nadler [1] established the multivalued version of Banach's contraction principle. One of the most celebrated results about multivalued mappings was given by Lim [2] in 1974. Using Edelstein's method of asymptotic centers, he proved the existence of a fixed point for a multivalued nonexpansive self-mapping $T : C \to K(C)$ where C is a nonempty bounded closed convex subset of a uniformly convex Banach space. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Some other classical fixed point theorems for single-valued mappings have been extended to multivalued mappings. However, many questions remain open, for instance, the possibility of extending the well-known Kirk's theorem, that is, do Banach spaces with weak normal structure have the fixed point property (FPP, in short) for multivalued nonexpansive mappings?

Since weak normal structure is implied by different geometrical properties of Banach spaces, it is natural to study if those properties imply the FPP for multivalued mappings.

Dhompongsa et al. [3, 4] introduced the Domnguez-Lorenzo condition ((DL) condition, in short) and property (D) which imply the FPP for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL) condition or property (D). In this setting the following results have been obtained.

(1) Dhompongsa et al. [3] proved that uniformly nonsquare Banach spaces with property WORTH satisfy the (DL) condition.

(2) Dhompongsa et al. [4] showed that the condition

$$C_{NJ}(X) < 1 + \frac{\mathrm{WCS}(X)^2}{4} \tag{1.1}$$

implies property (D).

(3) Satit Saejung [5] proved that the condition $\varepsilon_0(X) < \mathrm{WCS}(X)$ implies property (D).

(4) Gavira [6] showed that the condition

$$J(X) < 1 + \frac{1}{R(1,X)} \tag{1.2}$$

implies (DL) condition.

In 2007, Domínguez Benavides and Gavira [7] have established FFP for multivalued nonexpansive mappings in terms of the modulus of squareness, universal infinite-dimensional modulus, and Opia modulus. Attapol Kaewkhao [8] has established FFP for multivalued nonexpansive mappings in terms of the James constant, the Jordan-von Neumann Constants, weak orthogonality.

Besides, In 2010, Domínguez Benavides and Gavira [9] have given a survey of this subject and presented the main known results and current research directions.

In this paper, in terms of the Jordan-von Neumann constant, Zbăganu constant, $\varepsilon_\beta(X)$ and the coefficient $R(1,X)$, the weakly convergent sequence coefficient, we show some geometrical properties which imply the property (D) or (DL) condition and so the FPP for multivalued nonexpansive mappings.

2. Preliminaries

Let X be a Banach space and C be a nonempty subset of X; we denote all nonempty bounded closed subsets of X by $CB(X)$ and all nonempty compact convex subsets of X by $KC(X)$.

A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if the inequality

$$H(Tx,Ty) \le \|x-y\| \tag{2.1}$$

holds for every $x,y \in C$, where $H(\cdot,\cdot)$ is the Hausdorff distance on $CB(X)$, that is,

$$H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \quad A,B \in \mathrm{CB}(X). \tag{2.2}$$

Let $C \subset X$ be a nonempty bounded closed convex subset and $\{x_n\} \in X$ a bounded sequence; we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, that is,

$$r(C, \{x_n\}) = \inf\left\{ \limsup_n \|x_n - x\| : x \in C \right\},$$
$$A(C, \{x_n\}) = \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}. \tag{2.3}$$

It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex as C is.

Let $\{x_n\}$ and C be as above; then $\{x_n\}$ is called regular relative to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$; further, $\{x_n\}$ is called asymptotically uniform relative to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In Banach spaces, we have the following results:

(1) (Goebel [10] and Lim [2]) there always exists a subsequence of $\{x_n\}$ which is regular relative to C;

(2) (Kirk [11]) if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to C.

If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x \in C} \sup_{y \in D} \|x - y\|. \tag{2.4}$$

In 2006, Dhompongsa et al. [3] introduced the Domnguez-Lorenzo condition ((DL) condition, in short) in the following way.

Definition 2.1 (see [3]). We say that a Banach space X satisfies the (DL) condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C, \{x_n\})) \leq \lambda r(C, \{x_n\}). \tag{2.5}$$

The (DL) condition implies weak normal structure [3]. We recll that a Banach space X is said to have a weak normal structure (w-NS) if for every weakly compact convex subset C of X with $\operatorname{diam}(C) := \sup\{\|x - y\| : x, y \in C\} > 0$ there exist $x \in C$ such that $\sup\{\|x - y\| : y \in C\} < \operatorname{diam}(C)$.

The (DL) condition also implies the existence of fixed points for multivalued nonexpansive mappings.

Theorem 2.2 (see [3]). *Let C be a weakly compact convex subset of Banach space X; if C satisfies (DL) condition, then multivalued nonexpansive mapping $T : C \to KC(C)$ has a fixed point.*

Definition 2.3 (see [4]). A Banach space X is said to have property (D) if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every sequence $\{x_n\} \subset C$ and for every $\{y_n\} \subset A(C,\{x_n\})$ which are regular asymptotically uniform relative to C,

$$r(C,\{y_n\}) \leq \lambda r(C,\{x_n\}). \tag{2.6}$$

It was observed that property (D) is weaker than the (DL) condition and stronger than weak normal structure, and Dhompongsa et al. [4] proved that property (D) implies the w-MFPP.

Theorem 2.4 (see [4]). *Let C be a weakly compact convex subset of Banach space X; if C satisfies property (D), then multivalued nonexpansive mapping $T : C \to KC(C)$ has a fixed point.*

Before going to the results, let us recall some more definitions. Let X be a Banach space.

The Benavides coefficient $R(1,X)$ is defined by Domínguez Benavides [12] as

$$R(1,X) = \sup\left\{\liminf_{n\to\infty}\{\|x_n + x\|\}\right\}, \tag{2.7}$$

where the supremum is taken over all $x \in X$ with $\|X\| \leq 1$ and all weakly null sequence $\{x_n\}$ in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \leq 1. \tag{2.8}$$

Obviously, $1 \leq R(1,X) \leq 2$.

The weakly convergent sequence coefficient WCS(X) is equivalently defined by (see [13])

$$\mathrm{WCS}(X) = \inf\left\{\frac{\lim_{n\neq m}\|x_n - x_m\|}{\limsup_n \|x_n\|}\right\}, \tag{2.9}$$

where the infimum is taken over all weakly (not strongly) null sequences $\{x_n\}$ with $\lim_{n\neq m}\|x_n - x_m\|$ existing.

The ultrapower of a Banach space has proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting.

First we recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on an index set $\mathbb{N}$ and let X be a Banach space. A sequence x_n in X *convergers to x with respect to $\mathcal{F}$*, denoted by $\lim_{\mathcal{F}} x_n = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset \mathbb{N}, i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$; otherwise, it is called nontrivial. Let $l_\infty(X)$ denote the subspace of the product space $\Pi_{i\in\mathbb{N}} X_i$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty. \tag{2.10}$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \left\{ (x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0 \right\}. \tag{2.11}$$

The ultrapower of X, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of ultrapower. It follows from the definition of the quotient norm that

$$\|(x_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|. \tag{2.12}$$

Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\tilde{X}$ isometrically. For more details see [14].

3. Main Results

We first give some sufficient conditions which imply (DL) condition. The Jordan-von Neumann constant $C_{NJ}(X)$ was defined in 1937 by Clarkson [15] as

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2 + \|y\|^2\right)} : x, y \in X,\ \|x\| + \|y\| \neq 0 \right\}. \tag{3.1}$$

Theorem 3.1. *Let X be a Banach space and C a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in C which is regulary relative to C. Then*

$$r_C(A(C,\{x_n\})) \leq \frac{R(1,X)\sqrt{2C_{NJ}(X)}}{R(1,X)+1} r(C,\{x_n\}). \tag{3.2}$$

Proof. Denote $r = r(C,\{x_n\})$ and $A = A(C,\{x_n\}$. We can assume that $r > 0$. Since $\{x_n\} \subset C$ is bounded and C is a weakly compact set, by passing through a subsequence if necessary, we can also assume that x_n converges weakly to some element in $x \in C$ and $d = \lim_{n\neq m}\|x_n - x_m\|$ exists. We note that since $\{x_n\}$ is regular, $r(C,\{x_n\}) = r(C,\{y_n\})$ for any subsequence $\{y_n\}$ of $\{x_n\}$. Observe that, since the norm is weak lower semicontinuity, we have

$$\liminf_n \|x_n - x\| \leq \liminf_n \liminf_m \|x_n - x_m\| = \liminf_{n\neq m} \|x_n - x_m\| = d. \tag{3.3}$$

Let $\eta > 0$; taking a subsequence if necessary, we can assume that $\|x_n - x\| < \mathrm{d} + \eta$ for all n.

Let $z \in A$. Then we have $\limsup_n \|x_n - z\| = r$ and $\|x - z\| \leq \liminf_n \|x_n - z\| \leq r$. Denote $R = R(1,X)$; by definition, we have

$$R \geq \liminf_n \left\| \frac{x_n - x}{d+\eta} + \frac{z-x}{r} \right\| = \liminf_n \left\| \frac{x_n - x}{d+\eta} - \frac{x-z}{r} \right\|. \tag{3.4}$$

On the other hand, observe that the convexity of C implies $((R-1)/(R+1))x+(2/(R+1))z\in C$; since the norm is weak lower semicontinuity, we have

$$\begin{aligned}
&\liminf_n\left\|\frac{1}{r}(x_n-z)+\frac{1}{R}\left(\frac{x_n-x}{d+\eta}-\frac{x-z}{r}\right)\right\| \\
&\quad=\liminf_n\left\|\left(\frac{1}{r}+\frac{1}{R(d+\eta)}\right)x_n-\left(\frac{1}{R(d+\eta)}+\frac{1}{Rr}\right)x-\left(\frac{1}{r}-\frac{1}{Rr}\right)z\right\| \\
&\quad\geq\left\|\left(\frac{1}{r}-\frac{1}{Rr}\right)x+\frac{2}{Rr}z-\left(\frac{1}{r}+\frac{1}{Rr}\right)z\right\|=\left(\frac{1}{r}+\frac{1}{Rr}\right)\left\|\frac{R-1}{R+1}x+\frac{2}{R+1}z-z\right\| \\
&\quad\geq\left(\frac{1}{r}+\frac{1}{Rr}\right)r_C(A), \qquad (3.5)\\
&\liminf_n\left\|\frac{1}{r}(x_n-z)-\frac{1}{R}\left(\frac{x_n-x}{d+\eta}-\frac{x-z}{r}\right)\right\| \\
&\quad=\liminf_n\left\|\left(\frac{1}{r}-\frac{1}{R(d+\eta)}\right)(x_n-x)-\left(\frac{1}{r}+\frac{1}{Rr}\right)(z-x)\right\| \\
&\quad\geq\left(\frac{1}{r}+\frac{1}{Rr}\right)\|z-x\|\geq\left(\frac{1}{r}+\frac{1}{Rr}\right)r_C(A).
\end{aligned}$$

In the ultrapower $\widetilde{X}$ of X, we consider

$$\widetilde{u}=\frac{1}{r}\{x_n-z\}_{\mathcal{U}}\in S_{\widetilde{X}}, \qquad \widetilde{v}=\frac{1}{R}\left\{\frac{x_n-x}{d+\eta}-\frac{x-z}{r}\right\}_{\mathcal{U}}\in B_{\widetilde{X}}. \qquad (3.6)$$

Using the above estimates, we obtain

$$\begin{aligned}
\|\widetilde{u}+\widetilde{v}\|&=\lim_{\mathcal{U}}\left\|\frac{1}{r}(x_n-z)+\frac{1}{R}\left(\frac{x_n-x}{d+\eta}-\frac{x-z}{r}\right)\right\|\geq\left(\frac{1}{r}+\frac{1}{Rr}\right)r_C(A),\\
\|\widetilde{u}-\widetilde{v}\|&=\lim_{\mathcal{U}}\left\|\frac{1}{r}(x_n-z)-\frac{1}{R}\left(\frac{x_n-x}{d+\eta}-\frac{x-z}{r}\right)\right\|\geq\left(\frac{1}{r}+\frac{1}{Rr}\right)r_C(A).
\end{aligned} \qquad (3.7)$$

Therefore, we have

$$\begin{aligned}
C_{NJ}\left(\widetilde{X}\right)&\geq\frac{\|\widetilde{u}+\widetilde{v}\|^2+\|\widetilde{u}-\widetilde{v}\|^2}{2\left(\|\widetilde{u}\|^2+\|\widetilde{v}\|^2\right)}\\
&\geq\frac{2(1/r+1/(Rr))^2r_C(A)^2}{2(1+1)}\\
&=\frac{1}{2}\left(\frac{1}{r}+\frac{1}{Rr}\right)^2r_C(A)^2.
\end{aligned} \qquad (3.8)$$

Since Jordan-von Neumann constant $C_{NJ}(\widetilde{X})$ of $\widetilde{X}$ equals to $C_{NJ}(X)$ of X, we obtain

$$C_{NJ}(X) \geq \frac{1}{2}\left(\frac{1}{r}+\frac{1}{Rr}\right)^2 r_C(A)^2. \tag{3.9}$$

Hence we deduce the desired inequality. □

By Theorems 2.2 and 3.1, we have the following result.

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{NJ}(X) < (1/R(1,X)+1)^2/2$ and $T : C \to KC(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. since $R(1,X) \geq 1$, if $C_{NJ}(X) < (1/R(1,X)+1)^2/2$, then we have $C_{NJ}(X) < 2$ which implies that X is uniformly nonsquare; hence X is reflexive. Thus by Theorems 2.2 and 3.1, the result follows. □

Remark 3.3. Note that $J(X)^2/2 \leq C_{NJ}(X)$; it is easy to see that Theorem 3.1 includes [6, Theorem 3] and Corollary 3.2 includes [6, Corollary 2].

To characterize Hilbert space, Zbăganu defined the following Zbăganu constant: (see [16])

$$C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2+\|y\|^2} : x,y \in X,\ \|x\|+\|y\|>0\right\}. \tag{3.10}$$

We first give the following tool.

Proposition 3.4. $C_Z(X) = C_Z(\widetilde{X})$.

Proof. Clearly, $C_Z(X) \leq C_Z(\widetilde{X})$. To show $C_Z(\widetilde{X}) \leq C_Z(X)$, suppose $\widetilde{x},\ \widetilde{y} \in \widetilde{X}$ are not all zero. Without loss of generality, we assume $\|\widetilde{x}\| = a > 0$.

Let us choose $\eta \in (0,a)$. Since $\|\widetilde{x}\| = \lim_{\mathcal{U}}\|x_n\| = a$ and

$$c := \frac{\|\widetilde{x}+\widetilde{y}\|\|\widetilde{x}+\widetilde{y}\|}{\|\widetilde{x}\|^2+\|\widetilde{y}\|^2} = \lim_{\mathcal{U}}\frac{\|x_n+y_n\|\|x_n-y_n\|}{\|x_n\|^2+\|y_n\|^2} := \lim_{\mathcal{U}} c_n, \tag{3.11}$$

the set $A := \{n \in \mathbb{N} : |c_n - c| < \eta \text{ and} |\|x_n\| - a| < \eta\}$ belongs to $\mathcal{U}$. In particular, noticing that $x_n \neq 0$ for $n \in A$, there exists n such that

$$\begin{aligned}\frac{\|\widetilde{x}+\widetilde{y}\|\|\widetilde{x}+\widetilde{y}\|}{\|\widetilde{x}\|^2+\|\widetilde{y}\|^2} &< \frac{\|x_n+y_n\|\|x_n-y_n\|}{\|x_n\|^2+\|y_n\|^2}+\eta \\ &\leq C_Z(X)+\eta.\end{aligned} \tag{3.12}$$

Hence, the inequality $C_Z(\widetilde{X}) \leq C_Z(X)$ follows from the arbitrariness of η. □

Theorem 3.5. *Let X be a Banach space and C a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in C which is regulary relative to C. Then*

$$r_C(A(C,\{x_n\})) \le \frac{R(1,X)\sqrt{2C_Z(X)}}{R(1,X)+1} r(C,\{x_n\}). \tag{3.13}$$

Proof. Let $\tilde{u}, \tilde{v}$ be as in Theorem 3.1. Then

$$\|\tilde{u}+\tilde{v}\| \ge \left(\frac{1}{r}+\frac{1}{Rr}\right) r_C(A), \qquad \|\tilde{u}-\tilde{v}\| \ge \left(\frac{1}{r}+\frac{1}{Rr}\right) r_C(A). \tag{3.14}$$

Therefore, by the definition of Zbăganu constant, we have

$$\begin{aligned} C_Z\left(\tilde{X}\right) &\ge \frac{\|\tilde{u}+t\tilde{v}\|\|\tilde{u}-t\tilde{v}\|}{\|\tilde{u}\|^2+\|\tilde{v}\|^2} \\ &\ge \frac{1}{2}\left(\frac{1}{r}+\frac{1}{Rr}\right)^2 r_C(A)^2. \end{aligned} \tag{3.15}$$

Since Zbăganu constant $C_Z(\tilde{X}))$ of $\tilde{X}$ equals to $C_Z(X)$ of X, we obtain

$$C_Z(X) \ge \frac{1}{2}\left(\frac{1}{r}+\frac{1}{Rr}\right)^2 r_C(A)^2. \tag{3.16}$$

Hence we deduce the desired inequality. □

Using Theorem 2.2, we obtain the following corollary.

Corollary 3.6. *Let C be a nonempty weakly compact convex subset of a Banach space X such that $C_Z(X) < (1+1/R(1,X))^2/2$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

In the following, we present some properties concerning geometrical constants of Banach spaces which also imply the property (D).

Theorem 3.7. *Let X be a Banach space. If $C_Z(X) < WCS(X)$; then X has property (D).*

Proof. Let C be a weakly compact convex subset of X; suppose that $\{x_n\} \subset C$ and $\{y_n\} \subset A(C,\{x_n\})$ are regular and asymptotically uniform relative to C. Passing to a subsequence of $\{y_n\}$, still denoted by $\{y_n\}$, we may assume that $y_n \xrightarrow{w} y_0 \in C$ and $d = \lim_{n\neq m}\|y_n - y_m\|$ exists.

Let $r = r(C,\{x_n\})$. Again passing to a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, we assume in addition that

$$\lim_{n\to\infty}\|x_n - y_{2n}\| = \lim_{n\to\infty}\|x_n - y_{2n+1}\| = \lim_{n\to\infty}\left\|x_n - \frac{1}{2}(y_{2n}+y_{2n+1})\right\| = r. \tag{3.17}$$

Let us consider an ultrapower $\widetilde{X}$ of X. Put

$$\widetilde{u} = \frac{1}{r}\{x_n - y_{2n}\}_{\mathcal{U}}, \qquad \widetilde{v} = \frac{1}{r}\{x_n - y_{2n+1}\}_{\mathcal{U}}; \tag{3.18}$$

then we know that $\widetilde{u} \in S_{\widetilde{X}}, \widetilde{v} \in S_{\widetilde{X}}$. We see that

$$\|\widetilde{u} + \widetilde{v}\| = \lim_{\mathcal{U}}\left\|\frac{x_n - y_{2n}}{r} + \frac{x_n - y_{2n+1}}{r}\right\| = 2, \tag{3.19}$$

$$\|\widetilde{u} - \widetilde{v}\| = \lim_{\mathcal{U}}\left\|\frac{x_n - y_{2n}}{r} - \frac{x_n - y_{2n+1}}{r}\right\| = \lim_{\mathcal{U}}\left\|\frac{y_{2n} - y_{2n+1}}{r}\right\| = \frac{d}{r}. \tag{3.20}$$

Thus, By the definition of Zbăganu constant, we have

$$C_Z\left(\widetilde{X}\right) \geq \frac{\|\widetilde{u} + \widetilde{v}\|\|\widetilde{u} - \widetilde{v}\|}{\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2} \geq \frac{d}{r}. \tag{3.21}$$

Since the Zbăganu constants of X and of $\widetilde{X}$ are the same, we obtain $C_Z(X) \geq d/r$. Now we estimate d as follows:

$$\begin{aligned} d &= \lim_{n \neq m}\|y_n - y_m\| = \lim_{n \neq m}\|(y_n - y_0) - (y_m - y_0)\| \\ &\geq \mathrm{WCS}(X)\limsup_n\|y_n - y_0\| \\ &\geq \mathrm{WCS}(X) r(C, \{y_n\}). \end{aligned} \tag{3.22}$$

Hence $r(C, \{y_n\}) \leq (C_Z(X)/\mathrm{WCS}(X)) r(C, \{x_n\})$ and the assertion follows by the definition of property (D). □

Using Theorems 2.4 and 3.7, we obtain the follwing corollary.

Corollary 3.8. *Let C be a nonempty bounded closed convex subset of a reflexive Banach space X such that $C_Z(X) < WCS(X)$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

The separation measure of noncompactness is defined by

$$\beta(B) = \sup\{\varepsilon : \text{ there exists a sequence } \{x_n\} \text{in } B \text{ such that } \mathrm{sep}(\{x_n\}) \geq \varepsilon\} \tag{3.23}$$

for any bounded subset B of a Banach space X, where

$$\mathrm{sep}(\{x_\mathrm{n}\}) = \inf\{\|x_n - x_m\| : \ n \neq m\}. \tag{3.24}$$

The modulus of noncompact convexity associated to β is defined in the following way:

$$\Delta_{X,\beta}(\varepsilon) = \inf\{1 - d(0, A) : \ A \subset B_X \text{ is convex}, \ \beta(A) \geq \varepsilon\}. \tag{3.25}$$

The characteristic of noncompact convexity of X associated with the measure of noncompactness β is defined by

$$\varepsilon_\beta(X) = \sup\{\varepsilon \geq 0 : \ \Delta_{X,\beta}(\varepsilon) = 0\}. \tag{3.26}$$

When X is a reflexive Banach space, we have the following alternative expression for the modulus of noncompact convexity associated with β,

$$\varepsilon_\beta(X) = \inf\left\{1 - \|x\| : \ \{x_n\} \subset B_X, \quad x = w - \lim_n x_n, \ \operatorname{sep}(\{x_n\}) \geq \varepsilon\right\}. \tag{3.27}$$

It is known that X is NUC if and only if $\varepsilon_\beta(X) = 0$. The above-mentioned definitions and properties can be found in [17].

Theorem 3.9. *Let X be a reflexive Banach space. If $\varepsilon_\beta(X) < WCS(X)$, then X has property (D).*

Proof. Let C be a weakly compact convex subset of X; suppose that $\{x_n\} \subset C$ and $\{y_j\} \subset A(C, \{x_n\})$ are regular and asymptotically uniform relative to C. Passing to a subsequence of $\{y_j\}$, still denoted by $\{y_j\}$, we may assume that $y_j \xrightarrow{w} y_0 \in C$ and $d = \lim_{k \neq l}\|y_k - y_l\|$ exists. Let $r = r(C, \{x_n\})$.

Since $\{y_0, y_j\} \subset A(C, \{x_n\})$, we have

$$\limsup_n \|x_n - y_0\| = r, \qquad \limsup_n \|x_n - y_j\| = r, \quad \forall j \in \mathbb{N}. \tag{3.28}$$

So for any $\eta \geq 0$, there exists $N \in \mathbb{N}$ such that $\|x_N - y_0\| \geq r - \eta$ and $\|x_N - y_i\| \leq r + \eta$, for all $j \in \mathbb{N}$.

Without loss of generality, we suppose that $\|y_k - y_l\| \geq d - \eta$ for all $k \neq l$. Now we consider sequence $\{(x_N - y_j)/(r + \eta)\} \subset B_X$; notice that

$$\beta\left(\left\{\frac{x_N - y_j}{r + \eta}\right\}\right) \geq \frac{d - \eta}{r + \eta}, \qquad \frac{x_N - y_j}{r + \eta} \xrightarrow{w} \frac{x_N - y_0}{r + \eta}. \tag{3.29}$$

By the definition of $\Delta_{X,\beta}(\cdot)$, we have

$$\Delta_{X,\beta}\left(\frac{d - \eta}{r + \eta}\right) \leq 1 - \left\|\frac{x_N - y_0}{r + \eta}\right\| \leq 1 - \frac{r - \eta}{r + \eta}. \tag{3.30}$$

Since the last inequality is true for any $\eta > 0$, we obtain $\Delta_{X,\beta}(d/r) = 0$; thus $\varepsilon_\beta(X) \geq d/r$. Now we estimate d as follows:

$$\begin{aligned} d &= \lim_{k \neq l}\|y_k - y_l\| = \lim_{k \neq l}\|(y_k - y_0) - (y_l - y_0)\| \\ &\geq \mathrm{WCS}(X)\limsup_n \|y_n - y_0\| \\ &\geq \mathrm{WCS}(X) r(C, \{y_n\}). \end{aligned} \tag{3.31}$$

Hence,

$$r(C, \{y_n\}) \leq \frac{\varepsilon_\beta(X)}{\mathrm{WCS}(X)} r(C, \{x_n\}). \quad (3.32)$$

□

Remark 3.10. Since $\varepsilon_\beta(X) \leq \varepsilon_0(X)$, Theorem 3.9 implies the [5, Theorem 3]. Furthermore, it is easy to see $C_{NJ}(X) \geq 1 + (\varepsilon_0(X)^2/4 \geq 1 + (\varepsilon_\beta(X)^2/4$; then Theorem 3.9 also includes [4, Theorem 3.7].

By Theorem 3.9, we obtain the following Corollary.

Corollary 3.11. *Let C be a nonempty bounded closed convex subset of a reflexive Banach space X such that $\varepsilon_\beta(X) < WCS(X)$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Noticing $\mathrm{WCS}(X) \geq 1$, obviously, Corollary 3.11 extends the following well-known result.

Theorem 3.12 (see [18, Theorem 3.5]). *Let C be a nonempty bounded closed convex subset of a reflexive Banach space X such that $\varepsilon_\beta(X) < 1$ and let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

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