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Research Article

A Hybrid Proximal Point Three-Step Algorithm for Nonlinear Set-Valued Quasi-Variational Inclusions System Involving (A, η)-Accretive Mappings

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The main purpose of this paper is to introduce and study a new class of generalized nonlinear set-valued quasi-variational inclusions system involving (A, η)-accretive mappings in Banach spaces. By using the resolvent operator due to Lan-Cho-Verma associated with (A, η)-accretive mappings and the matrix analysis method, we prove the convergence of a new hybrid proximal point three-step iterative algorithm for this system of set-valued variational inclusions and an existence theorem of solutions for this kind of the variational inclusions system. The results presented in this paper generalize, improve, and unify some recent results in this field.

1. Introduction

The variational inclusion, which was introduced and studied by Hassouni and Moudafi [1], is a useful and important extension of the variational inequality. It provides us with a unified, natural, novel, innovative, and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences. Various variational inclusions have been intensively studied in recent years. Ding and Luo [2], Verma [3, 4], Huang [5], Fang et al. [6], Fang and Huang [7], Fang et al. [8], Lan et al. [9], Zhang et al. [10] introduced the concepts of η-subdifferential operators, maximal η-monotone operators, H-monotone operators, A-monotone operators, (H, η)-monotone operators, (A, η)-accretive mappings, (G, η)-monotone operators, and defined resolvent operators associated with them, respectively. Moreover, by using the resolvent operator technique, many authors constructed some approximation algorithms for some nonlinear variational inclusions in Hilbert spaces or Banach spaces. Verma has developed a hybrid version of the Eckstein-Bertsekas [11] proximal point algorithm, introduced the algorithm based on the (A,η)-maximal monotonicity framework [12], and studied convergence of the algorithm. For the past few years, many existence results and iterative algorithms for various variational inequalities and variational inclusion problems have been studied. For details, please see [1–37] and the references therein.

On the other hand, some new and interesting problems for systems of variational inequalities were introduced and studied. Peng and Zhu [14], Cohen and Chaplais [15], Bianchi [16], and Ansari and Yao [17] considered a system of scalar variational inequalities. Ansari et al. [18] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities using a fixed point theorem. Allevi et al. [19] considered a system of generalized vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [20] introduced a system of variational inequalities and proved an existence theorem through the Ky Fan lemma. Kassay et al. [21] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani-Fan-Glicksberg fixed point theorem. J. K. Kim and D. S. Kim [22] introduced a new system of generalized nonlinear quasi-variational inequalities and obtained some existence and uniqueness results on solutions for this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Cho et al. [23] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems for solutions for the system of nonlinear variational inequalities. As generalizations of a system of variational inequalities, Agarwal et al. [24] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for this system of generalized nonlinear mixed quasi-variational inclusions in Hilbert spaces. Kazmi and Bhat [25] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [26], Fang et al. [8] introduced and studied a new system of variational inclusions involving H-monotone operators and (H,η)-monotone operators, respectively. Yan et al. [27] introduced and studied a system of set-valued variational inclusions which is more general than the model in [3].

Inspired and motivated by recent research work in this field, in this paper, a general set-valued quasi-variational inclusions system with (A,η)-accretive mappings is studied in Banach spaces, which includes many variational inclusions (inequalities) as special cases. By using the resolvent operator associated with (A,η)-accretive operator due to Lan, an existence theorem of solution for this class of variational inclusions is proved, and a new hybrid proximal point algorithm is established and suggested, and the convergence of iterative sequences generated by the algorithm is discussed in q-uniformly smooth Banach spaces. The results presented in this paper generalize, and unify some recent results in this field.

2. Preliminaries

Let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, 2^X denote the family of all the nonempty subsets of X, and $CB(X)$ denote the family of all nonempty closed bounded subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \ \|f^*\| = \|x\|^{q-1} \right\}, \quad \forall x \in X, \tag{2.1}$$

where $q > 1$ is a constant.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \ \|y\| \leq t \right\}. \tag{2.2}$$

A Banach space X is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0. \tag{2.3}$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad (q > 1). \tag{2.4}$$

Remark 2.1. In particular, J_2 is the usual normalized duality mapping. It is known that, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$, J_q is single-valued if X^* is strictly convex [10], or X is uniformly smooth (Hilbert space and L_p $(2 \leq p < \infty)$ space are 2 uniformly Banach space), and if $X = \mathscr{H}$, the Hilbert space, then J_2 becomes the identity mapping on H. In what follows we always denote the single-valued generalized duality mapping by J_q in real uniformly smooth Banach space X unless otherwise states.

Let us recall the following results and concepts.

Definition 2.2. A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in X. \tag{2.5}$$

Definition 2.3. A single-valued mapping $A : X \to X$ is said to be

(i) accretive if

$$\langle A(x_1) - A(x_2), J_q(x_1 - x_2) \rangle \geq 0, \quad \forall x_1, x_2 \in X; \tag{2.6}$$

(ii) strictly accretive if A is accretive and $\langle A(x_1) - A(x_2), J_q(x_1 - x_2) \rangle = 0$ if and only if $x_1 = x_2$ for all $x_1, x_2 \in X$;

(iii) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle A(x_1) - A(x_2), J_q(\eta(x_1, x_2)) \rangle \geq \ r\|x_1 \ - x_2\|^q, \quad \forall x_1, x_2 \in X; \tag{2.7}$$

(iv) α-Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$\|A(x_1) - A(x_2)\| \leq \alpha \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X. \tag{2.8}$$

Definition 2.4. A set-valued mapping $S : X \to CB(X)$ is said to be

(i) D-Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$D(S(x), S(y)) \leq \alpha \|x - y\| \quad \forall x, y \in X, \tag{2.9}$$

where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

(ii) β-strongly η-accretive if there exists a constant $\beta > 0$ such that

$$\langle u_1 - u_2, J_q(\eta(x_1, y_1)) \rangle \geq \beta \|x_1 - y_1\|^q, \quad \forall x_1, y_1 \in X, \ u_1 \in S(x_1), \quad u_2 \in S(y_1). \tag{2.10}$$

(iii) (σ, ζ)-relaxed cocoercive if there exist two constants $\sigma, \zeta > 0$ such that

$$\begin{aligned} \langle u_1 - u_2, J_q(x_1 - y_1) \rangle &\geq -\sigma \|u_1 - u_2\|^q + \zeta \|x_1 - y_1\|^q, \\ &\forall x_1, y_1 \in X, \quad u_1 \in S(x_1), \quad u_2 \in S(y_1). \end{aligned} \tag{2.11}$$

(iv) γ_i-strongly η_i-accretive with respect to the first argument of the mapping $F_i : X_1 \times X_2 \times X_3 \to X_i$, if there exists a constant $\gamma_i > 0$ such that

$$\begin{aligned} \langle F_i(u_1, \cdot\,, \cdot) - F_i(u_2, \cdot, \cdot), J_q(\eta_i(x_1, y_1)) \rangle &\geq \gamma_i \|x_1 - y_1\|^2 \\ \forall x_1, y_1 \in X, \quad u_1 \in S(x_1), \quad &u_2 \in S(y_1), \end{aligned} \tag{2.12}$$

where $i = 1, 2, 3$.

Definition 2.5. Let $A_i : X_i \to X_i$ be a single-valued mapping and $S_i : X_i \to BC(X_i)$ be a set-valued mapping $(i = 1, 2, 3)$. For $i = 1, 2, 3$, a single-valued mapping $F_i : X_1 \times X_2 \times X_3 \to X_i$ is said to be

(i) $(\mu_{i1}, \mu_{i2}, \mu_{i3})$-Lipschitz continuous if there exist three constants $\mu_{i1}, \mu_{i2}, \mu_{i3} > 0$ such that

$$\begin{aligned} \|F_i(x_1, x_2, x_3) - F_i(y_1, y_2, y_3)\| \leq \mu_{i1} \|x_1 - y_1\| + \mu_{i2} \|x_2 - y_2\| + \mu_{i3} \|x_3 - y_3\| \\ \forall x_i, y_i \in X_i; \end{aligned} \tag{2.13}$$

(ii) (ψ_i, κ_i)-relaxed cocoercive with respect to $A_i S_i$ in the first argument, if there exist constants $\psi_i, \kappa_i > 0$ such that

$$\begin{aligned} &\langle F_i(x_1, \cdot, \cdot) - F_i(x_2, \cdot, \cdot), J_q(A_i(a_1) - A_i(b_1)) \rangle \\ &\quad \geq -\psi_i \|F_i(x_1, \cdot, \cdot) - F(x_2, \cdot, \cdot)\|^{q_i} + \kappa_i \|x_1 - x_2\|^{q_i}, \\ &\qquad \forall x_1, x_2 \in X_i, \ a_1 \in S_i(x_1), \ b_1 \in S_i(x_2). \end{aligned} \tag{2.14}$$

In a similar way, we can define Lipschitz continuity and (ψ_i, κ_i)-relaxed cocoercive with respect to $A_i S_i$ of $F_i(\cdot,\cdot,\cdot)$in the second, or the three argument.

Definition 2.6. Let $A : X \to X$ and $\eta : X \times X \to X$ be single-valued mappings. A set-valued mapping $M : X \times X \to 2^X$ is said to be

(i) accretive if

$$\langle u_1 - u_2, J_q(x_1 - y_1)\rangle \geq 0, \quad \forall x_1, y_1 \in X,\ u_1 \in M(x_1, \cdot\ ),\ u_2 \in M(y_1, \cdot\ ); \tag{2.15}$$

(ii) η-accretive if

$$\langle u_1 - u_2, J_q(\eta(x_1, y_1))\rangle \geq 0, \quad \forall x_1, y_1 \in X,\ u_1 \in M(x_1, \cdot),\ u_2 \in M(y_1, \cdot); \tag{2.16}$$

(iii) m-relaxed η-accretive, if there exists a constant $m > 0$ such that

$$\langle u_1 - u_2, J_q(\eta(x_1, y_1))\rangle \geq -m\|x_1 - y_1\|^q, \quad \forall x_1, y_1 \in X,\ u_1 \in M(x_1, \cdot),\ u_2 \in M(y_1, \cdot); \tag{2.17}$$

(iv) A-accretive if M is accretive and $(A + \rho M)(X) = X$ for all $\rho > 0$;

(v) (A, η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(X) = X$ for every $\rho > 0$.

Based on [9], we can define the resolvent operator $R^{A,\eta}_{\rho,M}$ as follows.

Definition 2.7 (see [9]). Let $\eta : X \times X \longrightarrow X$ be a single-valued mapping $A : X \to X$ be a strictly η-accretive single-valued mapping and $M : X \times X \to 2^X$ be a (A, η)-accretive mapping. The resolvent operator $R^{A,\eta}_{\rho,M} : X \to X$ is defined by

$$R^{A,\eta}_{\rho,M}(x) = (A + \rho M)^{-1}(x) \quad \forall x \in X, \tag{2.18}$$

where $\rho > 0$ is a constant.

Remark 2.8. The (A, η)-accretive mappings are more general than (H, η)-monotone mappings, H-accretive mappings, A-monotone operators, η-subdifferential operators, and m-accretive mappings in Banach space or Hilbert space, and the resolvent operators associated with (A, η)-accretive mappings include as special cases the corresponding resolvent operators associated with (H, η)-monotone operators, m-accretive mappings, H-accretive mappings, A-monotone operators, η-subdifferential operators [5, 6, 11, 14, 15, 26, 27, 35–37].

Lemma 2.9 (see [9]). *Let $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschtiz continuous mapping, $A_i : X_i \to X_i$ be an r_i-strongly η_i-accretive mapping, and $M_i : X_i \times X_j \to 2^{X_1}$ $(j \equiv (1 + i) \operatorname{mode} 3)$ be set-valued (A_i, η_i)-accretive mapping, respectively. Then the generalized resolvent operator $R^{A_i,\eta_i}_{\rho_i,M_i} : X_i \to X_i$ is*

$\tau_i^{q_i-1}/(r_i - m_i\rho_i)$*-Lipschitz continuous, that is,*

$$\left\| R_{\rho_i,M_i}^{A_i,\eta_i}(x_i) - R_{\rho_i,M_i}^{A_i,\eta_i}(y_i) \right\| \leq \frac{\tau_i^{q_i-1}}{r_i - m_i\rho_i} \|x_i - y_i\| \quad \forall x_i, y_i \in X_i, \tag{2.19}$$

where $\rho_i \in (0, r_i/m_i)$, $q_i > 1$, *and* $i = 1, 2, 3$.

In the study of characteristic inequalities in q_i-uniformly smooth Banach spaces X_i, Xu [29] proved the following result.

Lemma 2.10 (see [29]). *Let* X_i *be a real uniformly smooth Banach space. Then* X_i *is* q_i*-uniformly smooth if andonly if there exists a constant* $c_{q_i} > 0$ *such that for all* $x_i, y_i \in X_i$

$$\|x_i + y_i\|^{q_i} \leq \|x_i\|^{q_i} + q_i\langle y_i, J_{q_i}(x_i)\rangle + c_{q_i}\|y_i\|^{q_i} \quad (i = 1, 2, 3). \tag{2.20}$$

Theorem 2.11. *Let the function* $g(x, y) = (x + y)^q - x^q - y^q$, *where* $x, y > 0$ *and* $q > 0$, *then*

$$\begin{aligned} &\text{(i) } g(x,y) > 0, \quad \textit{as } q > 1, \\ &\text{(ii) } g(x,y) = 0, \quad \textit{as } q = 1, \\ &\text{(iii) } g(x,y) < 0, \quad \textit{as } q < 1. \end{aligned} \tag{2.21}$$

Proof. Let $h(a, b) = a^q + b^q - 1$, where $0 < a, b < 1$, $a + b = 1$, $0 < q$. Then $h(a, b) = a^q + (1 - a)^q - 1$ by the $b = 1 - a$. We can obtain

$$\begin{gathered} h(a,b) \geq 1 - 2\left(\frac{1}{2}\right)^q > 0, \quad (\forall 1 < q), \\ h(a,b) \leq 1 - 2\left(\frac{1}{2}\right)^q < 0, \quad (\forall 0 < q < 1), \\ h(a,b) = 0, \quad (q = 1). \end{gathered} \tag{2.22}$$

Let $a = x/(x + y)$, and $b = y/(x + y)$, where $x, y > 0$. It follows that

$$\begin{aligned} &\text{(i) } g(x,y) = (x+y)^q h(a,b) > 0, \quad \text{as } \ q > 1, \\ &\text{(ii) } g(x,y) = (x+y)^q h(a,b) = 0, \quad \text{as } \ q = 1, \\ &\text{(iii) } g(x,y) = (x+y)^q h(a,b) < 0, \quad \text{as } \ q < 1. \end{aligned} \tag{2.23}$$

This completes the proof. □

Corollary 2.12. *Let* $a, b, c > 0$ *be real, for any real* $q \geq 1$, *if* $a^q \leq b^q + c^q$, *then*

$$a \leq b + c. \tag{2.24}$$

Proof. The proof directly follows from the (i) in the Theorem 2.11. □

Definition 2.13 (see [38]). Let $\tilde{\Xi} = \{(c_{ij})_{m\times n} : c_{ij} \in R \text{ is a real}\}$ be a real matrix set, then the mappings

$$\left\|(c_{ij})_{m\times n}\right\|_q = \left\{\sum_{i,j=1,1}^{n,m} \left|c_{ij}\right|^q\right\}^{1/q} \quad (q > 0),$$
$$\left\|(c_{ij})_{m\times n}\right\|_\infty = \max_{1\le i\le n, 1\le j\le m} \left|c_{ij}\right| \tag{2.25}$$

is called the q-norm, and ∞-norm, respectively.

Obviously, $(\tilde{\Xi}, \|\cdot\|)$ may be a Banach space on real field R, which is called the real matrix-Banach space.

Definition 2.14 (see [38]). Let $\tilde{\Xi} = \{(c_{ij})_{m\times n} : c_{ij} \in R \text{ is a real}\}$ be a real matrix-Banach Space with the matrix-norm $\|\cdot\|$ ($\|\cdot\| = \|\cdot\|_q$ $(q > 0)$, or $\|\cdot\|_\infty$). If

$$\left(\lim_{k\to\infty} c_{ij}(k)\right)_{m\times n} = (d_{ij})_{m\times n} \in \tilde{\Xi}, \tag{2.26}$$

then the matrix $(d_{ij})_{m\times n}$ is called the limit matrix of matrix sequence $\{(c_{ij}(k))_{m\times n}\}$, noted by $\lim_{k\to\infty}(c_{ij}(k))_{m\times n} = (d_{ij})_{m\times n}$, where $\{c_{ij}(k)\}$ is a real sequence, $\lim_{k\to\infty} c_{ij}(k) = d_{ij}$, $m, n, i, j = 1, 2, \ldots$, and $i \le m$, $j \le n$.

Lemma 2.15 (see [38]). $\lim_{k\to\infty}(c_{ij}(k))_{m\times n} = (d_{ij})_{m\times n}$, *if and only if*

$$\lim_{k\to\infty}\left\|(c_{ij}(k))_{m\times n} - (d_{ij})_{m\times n}\right\| = 0. \tag{2.27}$$

Hence, if $\lim_{k\to\infty}(c_{ij}(k))_{m\times n} = (d_{ij})_{m\times n}$, *then* $\lim_{k\to\infty}\|(c_{ij}(k))_{m\times n}\| = \|(d_{ij})_{m\times n}\|$.

In this paper, the matrix norm symbol $\|\cdot\|_1$ is noted by $\|\cdot\|$.

Definition 2.16. Let a_i, b_i $(i = 1, \cdot, \cdot, \cdot, n)$ be real numbers, and $\vec{a} = (a_1, \cdot, \cdot, \cdot, a_n)^T$ and $\vec{b} = (b_1, \cdot, \cdot, \cdot, b_n)^T$ be two real vectors, then $\vec{a} = (a_1, \cdot, \cdot, \cdot, a_n)^T \le \vec{b} = (b_1, \cdot, \cdot, \cdot, b_n)^T$ if and only if $a_i \le b_i$ $(i = 1, \cdot, \cdot, \cdot, n)$.

3. Quasi-Variational Inclusions System Problem and Hybrid Proximal Point Algorithm

Let X_i be a real q_i-uniformly smooth Banach space with dual space X_i^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X_i and X_i^*, 2^{X_i} denote the family of all the nonempty subsets of X_i, and $CB(X_i)$

denote the family of all nonempty closed bounded subsets of X_i. The generalized duality mapping $J_{q_i} : X_i \to 2^{X_i^*}$ is defined by

$$J_{q_i}(x) = \left\{ f_i^* \in X_i^* : \langle x, f_i^* \rangle = \|x\|^{q_i}, \|f^*\| = \|x\|^{q_i - 1} \right\}, \quad \forall x \in X_i, \tag{3.1}$$

where $q_i > 1$ is a constant. Now, we consider the following generational nonlinear set-valued quasi-variational inclusions system problem with (A, η)-accretive mappings (GNSVQVIS) problem.

Let $A_i : X_i \to X_i$, $\eta_i : X_i \times X_i \to X_i$, and $F_i, G_i : X_1 \times X_2 \times X_3 \to X_i$ be single-valued mappings for $i = 1, 2, 3$. Let $M_i : X_i \times X_j \to 2^{X_1}$ ($j \equiv (1 + i)$ mode 3) be a set-valued (A_i, η_i)-accretive mapping and $S_i, T_i, U_i, V_i : X_{\rm i} \to CB(X_i)$ be set-valued mappings for $i = 1, 2, 3$.

For any $\varepsilon_i \in X_i$, finding $(x_i, a_i, b_i, c_i, d_i)$ such that $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$, $a_i \in S_i(x_i)$, $b_i \in T_i(x_i)$, $c_i \in U_i(x_i)$, $d_i \in V_i(x_i)$, and

$$\begin{aligned} \varepsilon_1 &\in F_1(x_1, x_2, x_3) - G_1(a_1, a_2, a_3) + M_1(d_1, d_2), \\ \varepsilon_2 &\in F_2(x_1, x_2, x_3) - G_2(b_1, b_2, b_3) + M_2(d_2, d_3), \\ \varepsilon_3 &\in F_3(x_1, x_2, x_3) - G_3(c_1, c_2, c_3) + M_3(d_3, d_1), \end{aligned} \tag{3.2}$$

where $i = 1, 2, 3$.

Remark 3.1. Some special cases of problem (3.2) are as follows.

(i) If $G_i \equiv 0$, $M_i(\cdot, \cdot) \equiv M_i(\cdot)$, $V_i \equiv g_i$, $A_i \equiv H_i$, $\varepsilon_i = 0$ and X_i is a Hilbert space, then the problem (3.2) reduces to the problem associated with the system of variational inclusions with (H_i, η_i)-monotone operators, which is finding $(x, y, z, a, b, c, d, e, p, q, u, v)$ such that $(x, y, z) \in {\rm H}_1 \times {\rm H}_2 \times {\rm H}_3$, $a \in A(x) = S_1$, $b \in B(y) = S_2$, $c \in C(z) = S_3$, $d \in D(x) = T_1$, $e \in E(y) = T_2$, $p \in P(z) = T_3$, $q \in Q(x) = U_1$, $u \in U(y) = U_2$, $v \in V(z) = U_3$ and

$$\begin{aligned} 0 &\in F_1(x, y, z) - G_1(a, b, c) + M_1(g_1(x)), \\ 0 &\in F_2(x, y, z) - G_2(d, e, p) + M_2(g_2(y)), \\ 0 &\in F_3(x, y, z) - G_3(q, u, v) + M_3(g_3(z)), \end{aligned} \tag{3.3}$$

where $i = 1, 2, 3$.

Problem (3.3) contains the system of variational inclusions with (H, η)-monotone operators in Peng and Zhu [14], and the system of variational inclusions with (H, η)-monotone operators in [8] as special cases.

(ii) If $G_i \equiv 0$, $\varepsilon_i = 0$, $X_i \equiv {\rm H}_i$ (Hilbert space) and, $M_i(\cdot, \cdot) \equiv M_i(\cdot) \equiv \Delta_{\eta_i}\varphi_i(\cdot)$, where $\varphi_i : {\rm H}_i \to R \cup \infty$ is a proper, η_i-subdifferentiable functional and $\Delta_{\eta_i}\varphi_i$ denotes the η_i-subdifferential operator of φ_i, then problem(3.3) changes to the problem associated with the following system of variational-like inequalities, which is finding $(x, y, z) \in \mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3$ such that

$$\begin{aligned} &\langle F_1(x,y,z), \eta_1(a,x)\rangle + \varphi_1(a) - \varphi_1(x) \geq 0, \quad \forall a \in \mathrm{H}_1, \\ &\langle F_2(x,y,z), \eta_2(b,x)\rangle + \varphi_2(b) - \varphi_2(x) \geq 0, \quad \forall b \in \mathrm{H}_2, \\ &\langle F_3(x,y,z), \eta_3(c,x)\rangle + \varphi_3(c) - \varphi_3(x) \geq 0, \quad \forall c \in \mathrm{H}_3, \end{aligned} \tag{3.4}$$

where $i = 1, 2, 3$.

(iii) If $G_i \equiv 0$, $\eta_i(x_i, y_i) = x_i - y_i$, $\varepsilon_i = 0$, $X_i \equiv \mathrm{H}_i$ (Hilbert space) and $M_i(\cdot,\cdot) \equiv M_i(\cdot) \equiv \partial\varphi_i(\cdot)$, where $\varphi_i : \mathrm{H}_i \to R \cup \infty$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_i(\cdot)$ denotes the subdifferential operator of $\varphi_i(\cdot)$, then problem (3.3) changes to the problem associated with the following system of variational inequalities, which is finding $(x, y, z) \in \mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3$ such that

$$\begin{aligned} &\langle F_1(x,y,z), (a-x)\rangle + \varphi_1(a) - \varphi_1(x) \geq 0, \quad \forall a \in \mathrm{H}_1, \\ &\langle F_2(x,y,z), (b-x)\rangle + \varphi_2(b) - \varphi_2(x) \geq 0, \quad \forall b \in \mathrm{H}_2, \\ &\langle F_3(x,y,z), (c-x)\rangle + \varphi_3(c) - \varphi_3(x) \geq 0, \quad \forall c \in \mathrm{H}_3, \end{aligned} \tag{3.5}$$

where $i = 1, 2, 3$.

(iv) If $\varepsilon_i = 0$, $X_i \equiv \mathrm{H}_i$ (Hilbert space), and $M_i(\cdot,\cdot) \equiv M_i(\cdot) \equiv \partial\delta_{K_i}(\cdot)$, where $K_i \subset \mathrm{H}_i$ is a nonempty, closed, and convex subsets and δ_{K_i} denotes the indicator of K_i, then problem (3.5) reduces to the problem associated with the following system of variational inequalities, which is finding $(x, y, z) \in \mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3$ such that

$$\begin{aligned} &\langle F_1(x,y,z), (a-x)\rangle \geq 0, \quad \forall a \in K_1, \\ &\langle F_2(x,y,z), (b-x)\rangle \geq 0, \quad \forall b \in K_2, \\ &\langle F_3(x,y,z), (c-x)\rangle \geq 0, \quad \forall c \in K_3, \end{aligned} \tag{3.6}$$

where $i = 1, 2, 3$.

(v) If $\varepsilon_i = 0$, and $\mathrm{H}_i = \mathrm{H}$ is a Hilbert space, $K_i = K$ is a nonempty, closed and convex subset, for all $x_i \in K$, $F_i(x_1, x_2, x_3) = \sigma_i T(y, x) + x_i - x_{i+1}$ $(i + 1 \equiv n \bmod (3),\ n = 1, 2, 3, \cdot, \cdot, \cdot)$, where $T : K \times K \to H$ is a mapping on $K \times K$, $\sigma_i > 0$ is a constant, then problem (3.6) changes to the following problem: find $x_i \in K$ such that

$$\begin{aligned} &\langle \sigma_1 T(x_2, x) + x_1 - x_2, (a - x_1)\rangle \geq 0, \quad \forall a \in K, \\ &\langle \sigma_2 T(x_3, x) + x_2 - x_3, (b - x_2)\rangle \geq 0, \quad \forall b \in K, \\ &\langle \sigma_3 T(x_1, x) + x_3 - x_1, (c - x_3)\rangle \geq 0, \quad \forall c \in K, \end{aligned} \tag{3.7}$$

where $i = 1, 2, 3$.

Moreover, if $\sigma_3 = 0$, problem (3.7) becomes the problem introduced and studied by Verma [31].

We can see that problem (3.2) includes a number of known classes of system of variational inequalities and variational inclusions as special cases (see, e.g., [2–9, 11–27, 29, 32–37]). It is worth noting that problems (3.2)–(3.7) are all new mathematical models.

Theorem 3.2. *Let X_i be a Banach space, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschtiz continuous mapping, $A_i : X_i \to X_i$ be an r_i-strongly η_i-accretive mapping, and $M_i : X_i \times X_j \to 2^{X_1}$ $(j \equiv (1+i) \bmod 3)$ be a set-valued (A_i, η_i)-accretive mapping for $i = 1, 2, 3$. Then the following statements are mutually equivalent.*

(i) *An element $(x_i, a_i, b_i, c_i, d_i)$ is a solution of problem (3.2), $i = 1, 2, 3$.*

(ii) *For $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ and $a_i \in S_i(x_i)$, $b_i \in T_i(x_i)$, $c_i \in U_i(x_i)$, $d_i \in V_i(x_i)$, the following relations hold:*

$$\begin{aligned}
d_1 &= R^{A_1,\eta_1}_{\rho_1, M_1(d_1,\cdot)}\left(A_1(d_1) + \rho_1\varepsilon_1 + \rho_1 G_1(a_1, a_2, a_3) - \rho_1 F_1(x_1, x_2, x_3)\right),\\
d_2 &= R^{A_2,\eta_2}_{\rho_2, M_1(d_2,\cdot)}\left(A_1(d_2) + \rho_2\varepsilon_2 + \rho_2 G_2(b_1, b_2, b_3) - \rho_2 F_2(x_1, x_2, x_3)\right),\\
d_3 &= R^{A_3,\eta_3}_{\rho_3, M_3(d_3,\cdot)}\left(A_3(d_3) + \rho_3\varepsilon_3 + \rho_3 G_3(c_1, c_2, c_3) - \rho_3 F_3(x_1, x_2, x_3)\right),
\end{aligned} \tag{3.8}$$

where $\rho_i > 0$ is a constant and $i = 1, 2, 3$, respectively.

Proof. This directly follows from definition of $R^{A_i,\eta_i}_{\rho_i, M_i(\cdot,\cdot)}$ and the problem (3.2) for $i = 1, 2, 3$. □

Algorithm 3.3. Let $\{\alpha_i^n\}_{n=0}^{\infty}$, $\{\xi_i^n\}_{n=0}^{\infty}$ and $\{\rho_i^n\}_{n=0}^{\infty}$ be three nonnegative sequences such that

$$\lim_{n\to\infty} \xi_i^n = 0, \quad \alpha = \limsup_{n\to\infty} \alpha_i^n < 1, \quad \rho_i^n \uparrow \rho_i \le \infty \quad (n = 0, 1, 2, \ldots;\ i = 1, 2, 3). \tag{3.9}$$

Step 1. For arbitrarily chosen initial points $x_i^0 \in X_i$, $a_i^0 \in S_i(x_i^0)$, $b_i^0 \in T_i(x_i^0)$, $c_i^0 \in U_i(x_i^0)$, $d_i^0 \in V_i(x_i^0)$, $y_i^0 \in X_i$ $(1 \le i \le 3)$, Set

$$\begin{aligned}
x_i^1 &= \left(1 - \alpha_i^0\right) x_i^0 + \alpha_i^0\left(x_i^0 - d_i^0 + y_i^0\right),\\
\Omega_i^1 &= A_i(d_i^0) + \rho_i^0 \varepsilon_i + \rho_i^0 G_i(a_1^0, a_2^0, a_3^0) - \rho_i^0 F_i(x_1^0, x_2^0, x_3^0),
\end{aligned} \tag{3.10}$$

where the y_i^0 satisfies

$$\left\| y_i^0 - R^{A_1,\eta_1}_{\rho_1, M_1(d_1,\cdot)}\left(\Omega_i^1\right) \right\| \le \xi_i^0 \left\| y_i^0 - d_i^0 \right\|. \tag{3.11}$$

By using [39], we can choose suitable $a_i^1 \in S_i(x_i^1)$, $b_i^1 \in T_i(x_i^1)$, $c_i^1 \in U_i(x_i^1)$ and $d_i^1 \in V_i(x_i^1)$ such that

$$\begin{aligned}
\left\|a_i^0 - a_i^1\right\| &\leq \left(1 + \frac{1}{1}\right) D\left(S_i\left(x_i^0\right), S_i\left(x_i^1\right)\right), \\
\left\|b_i^0 - b_i^1\right\| &\leq \left(1 + \frac{1}{1}\right) D\left(T_i\left(x_i^0\right), T_i\left(x_i^1\right)\right), \\
\left\|c_i^0 - c_i^1\right\| &\leq \left(1 + \frac{1}{1}\right) D\left(U_i\left(x_i^0\right), U_i\left(x_i^1\right)\right), \\
\left\|d_i^0 - d_i^1\right\| &\leq \left(1 + \frac{1}{1}\right) D\left(V_i\left(x_i^0\right), V_i\left(x_i^1\right)\right)
\end{aligned} \tag{3.12}$$

for $i = 1, 2, 3$.

Step 2. The sequences $\{x_i^n\}_{n=0}^{\infty}, \{a_i^n\}_{n=0}^{\infty}, \{b_i^n\}_{n=0}^{\infty}, \{c_i^n\}_{n=0}^{\infty}$, and $\{d_i^n\}_{n=0}^{\infty}$ are generated by an iterative procedure:

$$x_i^{n+1} = (1 - \alpha_i^n) x_i^n + \alpha_i^n (x_i^n - d_i^n + y_i^n), \tag{3.13}$$

$$\Omega_i^{n+1} = A_i(d_i^n) + \rho_i^n \varepsilon_i + \rho_i^n G_i(a_1^n, a_2^n, a_3^n) - \rho_i^n F_i(x_1^n, x_2^n, x_3^n), \tag{3.14}$$

where

$$\left\| y_i^n - R_{\rho_i, M_i(d_i, \cdot)}^{A_i, \eta_i}(\Omega_i^n) \right\| \leq \xi_i^n \|y_i^n - d_i^n\|. \tag{3.15}$$

Thus, we can choose suitable $a_i^{n+1} \in S_i(x_i^1)$, $b_i^{n+1} \in T_i(x_i^n)$, $c_i^{n+1} \in U_i(x_i^n)$ and $d_i^{n+1} \in V_i(x_i^n)$ such that

$$\begin{aligned}
\left\|a_i^n - a_i^{n+1}\right\| &\leq \left(1 + \frac{1}{n+1}\right) D\left(S_i(x_i^n), S_i\left(x_i^{n+1}\right)\right), \\
\left\|b_i^n - b_i^{n+1}\right\| &\leq \left(1 + \frac{1}{n+1}\right) D\left(T_i(x_i^n), T_i\left(x_i^{n+1}\right)\right), \\
\left\|c_i^n - c_i^{n+1}\right\| &\leq \left(1 + \frac{1}{n+1}\right) D\left(U_i(x_i^n), U_i\left(x_i^{n+1}\right)\right), \\
\left\|d_i^n - d_i^{n+1}\right\| &\leq \left(1 + \frac{1}{n+1}\right) D\left(V_i(x_i^n), V_i\left(x_i^{n+1}\right)\right)
\end{aligned} \tag{3.16}$$

for $i = 1, 2, 3$ and $n = 0, 1, 2, \ldots$.

Remark 3.4. If we choose suitable some operators $A_i, \eta_i, F_i, G_i, S_i, T_i, \mathrm{U}_i, V_i, M_i, \varepsilon_i$, and space X_i, then Algorithm 3.3 can be degenerated to a number of known algorithms for solving the system of variational inequalities and variational inclusions (see, e.g., [2–9, 11–27, 29, 31–35, 38, 39]).

4. Existence and Convergence

In this section, we prove the existence of solutions for problem (3.2) and the convergence of iterative sequences generated by Algorithm 3.3.

Theorem 4.1. *Let X_i be a q_i-uniformly smooth Banach space, $\eta_i : X_i \times X_i \to X_i$ be a τ_i-Lipschtiz continuous mapping, and $A_i : X_i \to X_i$ be a r_i-strongly η_i-accretive mapping and γ_i-Lipschitz continuous. Let $S_i, T_i, U_i, V_i : X_i \to CB(X_i)$ be a set-valued mappings of D-Lipschitz continuous with constants $s_i, t_i \cdot u_i, v_i$, and S_i, T_i, U_i, V_i be (σ_i, ζ_i)-relaxed cocoercive, respectively. Let $F_i : X_1 \times X_2 \times X_3 \to X_i$ be Lipschitz continuous with constants $(\mu_{i1}, \mu_{i2}, \mu_{i3})$ and $\|F_i(x_1, x_2, x_3)\| \le h_i$ $(h_i \ge 0)$ for all $(x_1, x_2, x_3) \in (X_1 \times X_2 \times X_3)$, and (ψ_i, κ_i)-relaxed cocoercive with respect to $A_i S_i$ in the first, second and third arguments, respectively. Let $G_i : X_1 \times X_2 \times X_3 \to X_i$ be Lipschitz continuous with constants $(\nu_{i1}, \nu_{i2}, \nu_{i3})$ and $\|G_i(x_1, x_2, x_3)\| \le g_i$ $(g_i \ge 0)$ for all $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$. Let $M_1 : X_1 \times X_2 \to 2^{X_1}$, $M_2 : X_2 \times X_3 \to 2^{X_2}$, $M_3 : X_3 \times X_1 \to 2^{X_3}$ be some set-valued mappings such that for each given $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$, $\operatorname{range}(V_1) \cap \operatorname{dom} M_1(\cdot, x_2) \neq \emptyset$, $\operatorname{range}(V_2) \cap \operatorname{dom} M_2(\cdot, x_3) \neq \emptyset$, $\operatorname{range}(V_3) \cap \operatorname{dom} M_3(\cdot, x_1) \neq \emptyset$ and $M_i : X_i \times X_j \to 2^{X_1}$ $(j \equiv (1+i) \operatorname{mode} 3)$ be an (A_i, η_i)-accretive mapping, respectively. Suppose that $\{\alpha_i^n\}_{n=0}^{\infty}$, $\{\xi_i^n\}_{n=0}^{\infty}$ and $\{\rho_i^n\}_{n=0}^{\infty}$ are three nonnegative sequences with*

$$\lim_{n\to\infty} \xi_i^n = 0, \qquad \alpha = \limsup_{n\to\infty} \alpha_i^n < 1, \qquad \rho_i = \lim_{n\to\infty} \rho_i^n \le \infty, \tag{4.1}$$

$$\begin{gathered}
\left(1 + q_1\sigma_1 v_1^{q_1} - q_1\zeta_1 + c_{q_1} v_1^{q_1}\right)^{1/q_1} - \lambda_1 + \rho_1 \nu_{11} s_1 < 1, \\
\alpha_1\rho_1\left(\mu_{12} + \nu_{12}s_2 + \mu_{13} + \nu_{13}s_3\right) < 1, \\
\left(1 + q_2\sigma_2 v_2^{q_2} - q_2\zeta_2 + c_{q_2} v_2^{q_2}\right)^{1/q_2} - \lambda_2 + \rho_2 \nu_{22} t_2 < 1, \\
\alpha_2\rho_2\left(\mu_{21} + \nu_{21}t_1 + \mu_{23} + \nu_{23}t_3\right) < 1, \\
\left(1 + q_3\sigma_3 v_3^{q_3} - q_3\zeta_3 + c_{q_3} v_3^{q_3}\right)^{1/q_3} - \lambda_3 + \rho_3 \nu_{33} u_3 < 1. \\
\alpha_3\rho_3\left(\mu_{31} + \nu_{31}u_1 + \mu_{32} + \nu_{32}u_2\right) < 1, \\
\lambda_i = \frac{\tau_i^{q_i-1}}{r_i - m_i\rho_i}\left(c_{q_i}\rho_i^{q_i}\mu_{ii}^{q_i} + \gamma_i^{q_i} v_i^{q_i} + q_i\rho_i\psi_i\mu_{ii}^{q_i} - \rho_i\kappa_i\right)^{1/q_i},
\end{gathered} \tag{4.2}$$

where $c_{q_i} > 0$ is the same as in Lemma 2.10, $\rho_i \in (0, r_i/m_i)$, and $i = 1, 2, 3$. Then the problem (3.2) has a solution $(x_i^*, a_i^*, b_i^*, c_i^*, d_i^*)$ $(i = 1, 2, 3)$.

Proof. Let

$$\Omega_i^{n+1} = A_i(d_i^n) + \rho_i^n \varepsilon_i + \rho_i^n G_i(a_1^n, a_2^n, a_3^n) - \rho_i^n F_i(x_1^n, x_2^n, x_3^n) \tag{4.3}$$

for $i = 1, 2, 3$. Then it follows from (3.13) that

$$\left\|x_1^{n+1} - x_1^n\right\| \le (1 - \alpha_1^n)\left\|x_1^n - x_1^{n-1}\right\| + \alpha_1^n\left\|x_1^n - x_1^{n-1} - \left(d_1^n - d_1^{n-1}\right)\right\| + \alpha_1^n\left\|y_1^n - y_1^{n-1}\right\|. \tag{4.4}$$

Since $V_1 : X_1 \to CB(X_1)$ is D-Lipschitz continuous with constants v_1 and (σ_1, ζ_1)-relaxed cocoercive,

$$\begin{aligned}
&\left\| x_1^n - x_1^{n-1} - \left(d_1^n - d_1^{n-1} \right) \right\|^{q_1} \\
&\quad = \left\| x_1^n - x_1^{n-1} \right\|^{q_1} - q_1 \left\langle x_1^n - x_1^{n-1}, J_{q_1}\left(d_1^n - d_1^{n-1} \right) \right\rangle + c_{q_1} \left\| d_1^n - d_1^{n-1} \right\|^{q_1} \\
&\quad \leq \left(1 + q_1 \sigma_1 \left(1 + \frac{1}{n+1} \right)^{q_1} v_1^{q_1} - q_1 \zeta_1 + c_{q_1} \left(1 + \frac{1}{n+1} \right)^{q_1} v_1^{q_1} \right) \left\| x_1^n - x_1^{n-1} \right\|^{q_1}.
\end{aligned} \tag{4.5}$$

By (3.15), we have

$$\begin{aligned}
\left\| y_1^n - y_1^{n-1} \right\| &\leq \left\| y_1^n - R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}(\Omega_1^n) \right\| + \left\| R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}\left(\Omega_1^{n-1} \right) - y_1^{n-1} \right\| \\
&\quad + \left\| R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}(\Omega_1^n) - R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}\left(\Omega_1^{n-1} \right) \right\| \\
&\leq \xi_1^n \left\| y_1^n - d_1^n \right\| + \xi_1^{n-1} \left\| y_1^{n-1} - d_1^{n-1} \right\| \\
&\quad + \left\| R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}(\Omega_1^n) - R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}\left(\Omega_1^{n-1} \right) \right\|.
\end{aligned} \tag{4.6}$$

Since $\|F_1(x_1, x_2, x_3)\| \leq h_1$ and $\|G_1(x_1, x_2, x_3)\| \leq g_1$, by Lemma 2.9, we have

$$\begin{aligned}
&\left\| R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}(\Omega_1^n) - R^{A_1,\eta_1}_{\rho_1^n, M_1(d_1,\cdot)}\left(\Omega_1^{n-1} \right) \right\| \\
&\quad \leq \frac{\tau_1^{q_1-1}}{r_1 - m_1 \rho_1^n} \left\| \Omega_1^n - \Omega_1^{n-1} \right\| \\
&\quad \leq \frac{\tau_1^{q_1-1}}{r_1 - m_1 \rho_1^n} \Big[\Big\| A_1(d_1^n) - A_1\left(d_1^{n-1} \right) - \rho_1^n \Big(F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^{n-1} \right) \Big) \Big\| \\
&\qquad\qquad + \rho_1^n \left\| G_1(a_1^n, a_2^n, a_3^n) - G_1\left(a_1^{n-1}, a_2^{n-1}, a_3^{n-1} \right) \right\| + \left| \rho_1^n - \rho_1^{n-1} \right| (h_1 + g_1 + \|\varepsilon_1\|) \Big]. \\
&\quad \leq \frac{\tau_1^{q_1-1}}{r_1 - m_1 \rho_1^n} \Big[\Big\| A_1(d_1^n) - A_1\left(d_1^{n-1} \right) - \rho_1^n \Big(F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^n, x_3^n \right) \Big) \Big\| \\
&\qquad\qquad + \rho_1^n \left\| F_1\left(x_1^{n-1}, x_2^n, x_3^n \right) - F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^n \right) \right\| \\
&\qquad\qquad + \rho_1^n \left\| F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^n \right) - F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^{n-1} \right) \right\| \\
&\qquad\qquad + \rho_1^n \left\| G_1(a_1^n, a_2^n, a_3^n) - G_1\left(a_1^{n-1}, a_2^{n-1}, a_3^{n-1} \right) \right\| \Big].
\end{aligned} \tag{4.7}$$

Since $F_1 : X_1 \times X_2 \times X_3 \to X_1$ is Lipschitz continuous with constants $\mu_{11}, \mu_{12}, \mu_{13}$, and (ψ_1, κ_1)-relaxed cocoercive with respect to A_1S_1 in the first arguments, $G_i : X_1 \times X_2 \times X_3 \to X_i$ be Lipschitz continuous with constants $(\nu_{i1}, \nu_{i2}, \nu_{i3})$, respectively, Lemma 2.10, we have

$$\begin{aligned}
&\left\|A_1(d_1^n) - A_1\left(d_1^{n-1}\right) - \rho_1^n\left(F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^n, x_3^n\right)\right)\right\|^{q_1} \\
&\quad\le c_{q_1}(\rho^n)_1^{q_1}\left\|F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^n, x_3^n\right)\right\|^{q_1} + \left\|A_1(d_1^n) - A_1\left(d_1^{n-1}\right)\right\|^{q_1} \\
&\qquad - q_1\rho_1^n\left\langle F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^n, x_3^n\right), J_q\left(A_1(d_1^n) - A_1\left(d_1^{n-1}\right)\right)\right\rangle \\
&\quad\le c_{q_1}(\rho^n)_1^{q_1}\left\|F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^n, x_3^n\right)\right\|^{q_1} + \left\|A_1(d_1^n) - A_1\left(d_1^{n-1}\right)\right\|^{q_1} \\
&\qquad + q_1\rho_1^n\psi_1\left\|F_1(x_1^n, x_2^n, x_3^n) - F_1\left(x_1^{n-1}, x_2^n, x_3^n\right)\right\|^{q_1} - \rho_1^n\kappa_1\left\|x_1^n - x_1^{n-1}\right\|^{q_1} \\
&\quad\le \left[c_{q_1}(\rho^n)_1^{q_1}\mu_{11}^{q_1} + \gamma_1^{q_1}\left(1 + (n+1)^{-1}\right)^{q_1} v_1^{q_1} + q_1\rho_1^n\psi_1\mu_{11}^{q_1} - \rho_1^n\kappa_1\right]\left\|x_1^n - x_1^{n-1}\right\|^{q_1}, \\
&\qquad\qquad \left\|F_1\left(x_1^{n-1}, x_2^n, x_3^n\right) - F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^n\right)\right\| \le \mu_{12}\left\|x_2^{n-1} - x_2^n\right\|, \\
&\qquad\quad \left\|F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^n\right) - F_1\left(x_1^{n-1}, x_2^{n-1}, x_3^{n-1}\right)\right\| \le \mu_{13}\left\|x_3^{n-1} - x_3^n\right\|, \\
&\left\|G_1(a_1^n, a_2^n, a_3^n) - G_1\left(a_1^{n-1}, a_2^{n-1}, a_3^{n-1}\right)\right\| \\
&\quad\le \left\|G_1(a_1^n, a_2^n, a_3^n) - G_1\left(a_1^{n-1}, a_2^n, a_3^n\right)\right\| \\
&\qquad + \left\|G_1\left(a_1^{n-1}, a_2^n, a_3^n\right) - G_1\left(a_1^{n-1}, a_2^{n-1}, a_3^n\right)\right\| \\
&\qquad + \left\|G_1\left(a_1^{n-1}, a_2^{n-1}, a_3^{n-1}\right) - G_1\left(a_1^{n-1}, a_2^{n-1}, a_3^n\right)\right\| \\
&\quad\le \nu_{11}\left\|a_1^{n-1} - a_1^n\right\| + \nu_{12}\left\|a_2^{n-1} - a_2^n\right\| + \nu_{13}\left\|a_3^{n-1} - a_3^n\right\|.
\end{aligned} \tag{4.8}$$

By (3.13), we know that $\|x_1^{n+1} - x_1^n\| = \alpha_1^n\|y_1^n - d_1^n\|$. Since $G_i : X_1 \times X_2 \times X_3 \to X_i$ is Lipschitz continuous with constants $(\nu_{i1}, \nu_{i2}, \nu_{i3})$, and $S_i : X_i \to CB(X_i)$ is D-Lipschitz continuous with constants s_i, respectively, combing (4.4)–(4.8) and using Corollary 2.12, we have

$$\begin{aligned}
&(1 - \xi_1^n)\left\|x_1^{n+1} - x_1^n\right\| \\
&\quad\le (1 - \alpha_1^n)\left\|x_1^n - x_1^{n-1}\right\| \\
&\qquad + \alpha_1^n\left[\left(1 + q_1\sigma_1\left(1 + \frac{1}{n+1}\right)^{q_1} v_1^{q_1} - q_1\zeta_1 + c_{q_1}\left(1 + \frac{1}{n+1}\right)^{q_1} v_1^{q_1}\right)^{1/q_1} \xi_1^{n-1}\frac{1}{\alpha_1^{n-1}}\right. \\
&\qquad\qquad + \frac{\tau_1^{q_1-1}}{r_1 - m_1\rho_1^n}\left(\left(c_{q_1}(\rho^n)_1^{q_1}\mu_{11}^{q_1} + \gamma_1^{q_1}\left(1 + \frac{1}{n+1}\right)^{q_1} v_1^{q_1} + q_1\rho_1^n\psi_1\mu_{11}^{q_1} - \rho_1^n\kappa_1\right)^{1/q_1}\right. \\
&\qquad\qquad\qquad \left.\left. + \left|\rho_1^n - \rho_1^{n-1}\right|\left(h_1 + g_1 + \|\varepsilon_1\|\right)\right)\right]\left\|x_1^n - x_1^{n-1}\right\|
\end{aligned}$$

$$+\alpha_1^n\rho_1^n\Bigg[\mu_{12}\left\|x_2^{n-1}-x_2^n\right\|+\mu_{13}\left\|x_3^{n-1}-x_3^n\right\|+\nu_{11}\left(1+\frac{1}{n+1}\right)s_1\left\|x_1^{n-1}-x_1^n\right\|$$
$$+\nu_{12}\left(1+\frac{1}{n+1}\right)s_2\left\|x_2^{n-1}-x_2^n\right\|+\nu_{13}\left(1+\frac{1}{n+1}\right)s_3\left\|x_3^{n-1}-x_3^n\right\|\Bigg], \tag{4.9}$$

and so

$$\begin{aligned}
&\left\|x_1^{n+1}-x_1^n\right\| \\
&\le \frac{1}{1-\xi_1^n}\Bigg\{1-\alpha_1^n\Bigg[1-\left(1+q_1\sigma_1\left(1+\frac{1}{n+1}\right)^{q_1}v_1^{q_1}-q_1\zeta_1+c_{q_1}\left(1+\frac{1}{n+1}\right)^{q_1}v_1^{q_1}\right)^{1/q_1}-\frac{\xi_1^{n-1}}{\alpha_1^{n-1}} \\
&\qquad -\frac{\tau_1^{q_1-1}}{r_1-m_1\rho_1^n}\Bigg(\left(c_{q_1}(\rho^n)_1^{q_1}\mu_{11}^{q_1}+\gamma_1^{q_1}\left(1+\frac{1}{n+1}\right)^{q_1}v_1^{q_1}+q_1\rho_1^n\psi_1\mu_{11}^{q_1}-\rho_1^n\kappa_1\right)^{1/q_1} \\
&\qquad +\rho_1^n\nu_{11}\left(1+\frac{1}{n+1}\right)s_1+\left|\rho_1^n-\rho_1^{n-1}\right|(h_1+g_1+\|\varepsilon_1\|)\Bigg)\Bigg]\Bigg\} \\
&\quad\times\left\|x_1^n-x_1^{n-1}\right\|+\frac{1}{1-\xi_1^n}\alpha_1^n\rho_1^n\left[\left(\mu_{12}+\nu_{12}\left(1+\frac{1}{n+1}\right)s_2\right)\right]\left\|x_2^{n-1}-x_2^n\right\| \\
&\quad+\frac{1}{1-\xi_1^n}\alpha_1^n\rho_1^n\left[\left(\mu_{13}+\nu_{13}\left(1+\frac{1}{n+1}\right)s_3\right)\left\|x_3^{n-1}-x_3^n\right\|\right].
\end{aligned} \tag{4.10}$$

For the sequences $\{x_2^n\}_{n=0}^{\infty}$, we have

$$\left\|x_2^{n+1}-x_2^n\right\|\le(1-\alpha_2^n)\left\|x_2^n-x_2^{n-1}\right\|+\alpha_2^n\left\|x_2^n-x_2^{n-1}-\left(d_2^n-d_2^{n-1}\right)\right\|+\alpha_2^n\left\|y_2^n-y_2^{n-1}\right\|. \tag{4.11}$$

Since $V_2 : X_2 \to CB(X_2)$ is D-Lipschitz continuous with constant v_2 and (σ_2,ζ_2)-relaxed cocoercive, we have

$$\begin{aligned}
&\left\|x_2^n-x_2^{n-1}-\left(d_2^n-d_2^{n-1}\right)\right\|^{q_2} \\
&\quad=\left\|x_2^n-x_2^{n-1}\right\|^{q_2}-q_2\left\langle x_2^n-x_2^{n-1},J_{q_2}\left(d_2^n-d_2^{n-1}\right)\right\rangle+c_{q_2}\left\|d_2^n-d_2^{n-1}\right\|^{q_2} \\
&\quad\le\left(1+q_2\sigma_2\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}-q_2\zeta_2+c_{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}\right)\left\|x_2^n-x_2^{n-1}\right\|^{q_2}.
\end{aligned} \tag{4.12}$$

It follows from (3.15) that

$$\begin{aligned}\left\|y_2^n-y_2^{n-1}\right\| &\leq \left\|y_2^n-R_{\rho_2^n,M_2(d_2,\cdot)}^{A_21,\eta_2}\left(\Omega_2^n\right)\right\|+\left\|R_{\rho_2^n,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^{n-1}\right)-y_2^{n-1}\right\| \\ &\quad+\left\|R_{\rho_2^n,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^n\right)-R_{\rho_2^n,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^{n-1}\right)\right\| \\ &\leq \xi_2^n\left\|y_2^n-d_2^n\right\|+\xi_2^{n-1}\left\|y_2^{n-1}-d_2^{n-1}\right\| \\ &\quad+\left\|R_{\rho_2,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^n\right)-R_{\rho_2,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^{n-1}\right)\right\|.\end{aligned} \tag{4.13}$$

Since $\|F_2(x_1,x_2,x_3)\|\leq h_2$ and $\|G_2(x_1,x_2,x_3)\|\leq g_2$, by using Lemma 2.9, we obtain

$$\begin{aligned}&\left\|R_{\rho_2^n,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^n\right)-R_{\rho_2^n,M_2(d_2,\cdot)}^{A_2,\eta_2}\left(\Omega_2^{n-1}\right)\right\| \\ &\quad\leq \frac{\tau_2^{q_2-1}}{r_2-m_2\rho_2^n}\left\|\Omega_2^n-\Omega_2^{n-1}\right\| \\ &\quad\leq \frac{\tau_2^{q_2-1}}{r_2-m_2\rho_2^n}\left[\left\|A_2\left(d_2^n\right)-A_2\left(d_2^{n-1}\right)-\rho_2\left(F_2\left(x_2^n,x_2^n,x_2^n\right)-F_2\left(x_2^{n-1},x_2^{n-1},x_2^{n-1}\right)\right)\right\|\right. \\ &\qquad\qquad+\rho_2^n\left\|G_2\left(a_2^n,a_2^n,a_2^n\right)-G_2\left(a_2^{n-1},a_2^{n-1},a_2^{n-1}\right)\right\| \\ &\qquad\qquad\left.+\left|\rho_2^n-\rho_2^{n-1}\right|\left(h_2+g_2+\|\varepsilon_2\|\right)\right] \\ &\quad\leq \frac{\tau_2^{q_2-1}}{r_2-m_2\rho_2^n}\left[\left\|A_2\left(d_2^n\right)-A_2\left(d_2^{n-1}\right)-\rho_2^n\left(F_2\left(x_1^n,x_2^n,x_3^n\right)-F_1\left(x_1^n,x_2^{n-1},x_3^n\right)\right)\right\|\right. \\ &\qquad\qquad+\rho_2^n\left\|F_2\left(x_1^n,x_2^{n-1},x_3^n\right)-F_2\left(x_1^{n-1},x_2^{n-1},x_3^n\right)\right\| \\ &\qquad\qquad+\rho_2^n\left\|F_2\left(x_1^{n-1},x_2^{n-1},x_3^n\right)-F_2\left(x_1^{n-1},x_2^{n-1},x_3^{n-1}\right)\right\| \\ &\qquad\qquad\left.+\rho_2^2\left\|G_2\left(b_1^n,b_2^n,b_3^n\right)-G_2\left(b_1^{n-1},b_2^{n-1},b_3^{n-1}\right)\right\|+\left|\rho_2^n-\rho_2^{n-1}\right|\left(h_2+g_2+\|\varepsilon_2\|\right)\right].\end{aligned} \tag{4.14}$$

Since $F_2: X_1\times X_2\times X_3\to X_2$ is Lipschitz continuous with constants $\mu_{21},\mu_{22},\mu_{23}$, and (ψ_2,κ_2)-relaxed cocoercive with respect to A_2S_2 in the first arguments, $G_i: X_1\times X_2\times X_3\to X_i$ is

Lipschitz continuous with constants $(\nu_{i1},\nu_{i2},\nu_{i3})$, respectively, it follows from Lemma 2.10 that

$$\begin{aligned}
&\left\|A_2(d_2^n)-A_2\left(d_2^{n-1}\right)-\rho_2\left(F_2(x_1^n,x_2^n,x_3^n)-F_2\left(x_1^n,x_2^{n-1},x_3^n\right)\right)\right\|^{q_2}\\
&\quad\le c_{q_2}(\rho_2^n)^{q_2}\left\|F_2(x_1^n,x_2^n,x_3^n)-F_2\left(x_1^n,x_2^{n-1},x_3^n\right)\right\|^{q_2}+\left\|A_2(d_2^n)-A_2\left(d_2^{n-1}\right)\right\|^{q_2}\\
&\qquad-q_2\rho_2^n\left\langle F_2(x_1^n,x_2^n,x_3^n)-F_2\left(x_1^n,x_2^{n-1},x_3^n\right),J_q\left(A_2(d_2^n)-A_2\left(d_2^{n-1}\right)\right)\right\rangle\\
&\quad\le c_{q_2}(\rho_2^n)^{q_2}\left\|F_2(x_1^n,x_2^n,x_3^n)-F_2\left(x_1^n,x_2^{n-1},x_3^n\right)\right\|^{q_2}+\left\|A_2(d_2^n)-A_2\left(d_2^{n-1}\right)\right\|^{q_2}\\
&\qquad+q_2\rho_2^n\psi_2\left\|F_2(x_1^n,x_2^n,x_3^n)-F_2\left(x_1^n,x_2^{n-1},x_3^n\right)\right\|^{q_2}-\rho_2^n\kappa_2\left\|x_2^n-x_2^{n-1}\right\|^{q_2}\\
&\quad\le\left[c_{q_2}(\rho_2^n)^{q_2}\mu_{21}^{q_2}+\gamma_2^{q_2}\left(1+\frac{1}{(n+1)}\right)^{q_2}v_2^{q_2}+q_2\rho_2^n\psi_2\mu_{21}^{q_2}-\rho_2^n\kappa_2\right]\left\|x_2^n-x_2^{n-1}\right\|^{q_2},\\
&\qquad\qquad\left\|F_2\left(x_1^n,x_2^{n-1},x_3^n\right)-F_2\left(x_1^{n-1},x_2^{n-1},x_3^n\right)\right\|\le\mu_{21}\left\|x_1^{n-1}-x_1^n\right\|,\\
&\qquad\qquad\left\|F_2\left(x_1^{n-1},x_2^{n-1},x_3^n\right)-F_2\left(x_1^{n-1},x_2^{n-1},x_3^{n-1}\right)\right\|\le\mu_{23}\left\|x_3^{n-1}-x_3^n\right\|,\\
&\left\|G_2(b_1^n,b_2^n,b_3^n)-G_2\left(b_1^{n-1},b_2^{n-1},b_3^{n-1}\right)\right\|\\
&\quad\le\left\|G_2(b_1^n,b_2^n,b_3^n)-G_2\left(b_1^{n-1},b_2^n,b_3^n\right)\right\|+\left\|G_2\left(b_1^{n-1},b_2^n,b_3^n\right)-G_2\left(b_1^{n-1},b_2^{n-1},b_3^n\right)\right\|\\
&\qquad+\left\|G_2\left(b_1^{n-1},b_2^{n-1},b_3^{n-1}\right)-G_2\left(b_1^{n-1},b_2^{n-1},b_3^n\right)\right\|\\
&\quad\le\nu_{21}\left\|b_1^{n-1}-b_1^n\right\|+\nu_{22}\left\|b_2^{n-1}-b_2^n\right\|+\nu_{23}\left\|b_3^{n-1}-b_3^n\right\|.
\end{aligned}\tag{4.15}$$

By (3.13), we know that $\|x_2^{n+1}-x_2^n\|=\alpha_2^n\|y_2^n-d_2^n\|$. Since $G_i:X_1\times X_2\times X_3\to X_i$ is Lipschitz continuous with constants $(\nu_{i1},\nu_{i2},\nu_{i3})$, and $T_i:X_i\to CB(X_i)$ is D-Lipschitz continuous with constants t_i, respectively, combing (4.11)–(4.20) and using Corollary 2.12, we have

$$\begin{aligned}
&(1-\xi_2^n)\left\|x_2^{n+1}-x_2^n\right\|\\
&\quad\le(1-\alpha_2^n)\left\|x_2^n-x_2^{n-1}\right\|\\
&\qquad+\alpha_2^n\Bigg[\left(1+q_2\sigma_2\left(1+\frac{1}{n+1}\right)^{q_2}\mathrm{v}_1^{q_2}-q_2\zeta_2+c_{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}\right)^{1/q_2}\xi_2^{n-1}\frac{1}{\alpha_2^{n-1}}\\
&\qquad\quad+\frac{\tau_2^{q_2-1}}{r_2-m_2\rho_2^n}\left(\left(c_{q_2}(\rho_2^n)^{q_2}\mu_{22}^{q_2}+\gamma_2^{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}+q_2\rho_2^n\psi_2\mu_{22}^{q_2}-\rho_2^n\kappa_2\right)^{1/q_2}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left|\rho_2^n-\rho_2^{n-1}\right|\left(h_2+g_2+\|\varepsilon_2\|\right)\Big)\Big]\left\|x_2^n-x_2^{n-1}\right\| \\
&+\alpha_2^n\rho_2^n\left[\mu_{21}\left\|x_1^{n-1}-x_1^n\right\|+\mu_{23}\left\|x_3^{n-1}-x_3^n\right\|+\nu_{21}\left(1+\frac{1}{n+1}\right)t_1\left\|x_1^{n-1}-x_1^n\right\|\right. \\
&\left.+\nu_{22}\left(1+\frac{1}{n+1}\right)t_2\left\|x_2^{n-1}-x_2^n\right\|+\nu_{23}\left(1+\frac{1}{n+1}\right)t_3\left\|x_3^{n-1}-x_3^n\right\|\right],
\end{aligned}
\tag{4.16}
$$

and so

$$
\begin{aligned}
&\left\|x_2^{n+1}-x_2^n\right\| \\
&\leq \frac{1}{1-\xi_2^n}\left\{1-\alpha_2^n\left[1-\left(1+q_2\sigma_2\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}-q_2\zeta_2+c_{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}\right)^{1/q_2}-\frac{\xi_2^{n-1}}{\alpha_2^{n-1}}\right.\right. \\
&\qquad -\frac{\tau_2^{q_2-1}}{r_2-m_2\rho_2^n}\left(\left(c_{q_2}(\rho_2^n)^{q_2}\mu_{22}^{q_2}+\gamma_2^{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}+q_2\rho_2^n\psi_2\mu_{22}^{q_2}-\rho_2^n\kappa_2\right)^{1/q_2}\right. \\
&\qquad\qquad \left.\left.\left.+\rho_2^n\nu_{22}\left(1+\frac{1}{n+1}\right)t_2+\left|\rho_2^n-\rho_2^{n-1}\right|\left(h_2+g_2+\|\varepsilon_2\|\right)\right)\right]\right\} \\
&\quad\times\left\|x_2^n-x_2^{n-1}\right\|+\frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\left[\left(\mu_{21}+\nu_{21}\left(1+\frac{1}{n+1}\right)t_1\right)\right]\left\|x_1^{n-1}-x_1^n\right\| \\
&\quad+\frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\left[\left(\mu_{23}+\nu_{23}\left(1+\frac{1}{n+1}\right)t_3\right)\right]\left\|x_3^{n-1}-x_3^n\right\|.
\end{aligned}
\tag{4.17}
$$

Using the same as the method, we can obtain

$$
\begin{aligned}
&\|x_3^{n+1}-x_3^n\| \\
&\leq \frac{1}{1-\xi_3^n}\left\{1-\alpha_3^n\left[1-\left(1+q_3\sigma_3\left(1+\frac{1}{n+1}\right)^{q_3}v_3^{q_3}-q_3\zeta_3+c_{q_3}\left(1+\frac{1}{n+1}\right)^{q_3}v_3^{q_3}\right)^{1/q_3}-\frac{\xi_3^{n-1}}{\alpha_3^{n-1}}\right.\right. \\
&\qquad -\frac{\tau_3^{q_3-1}}{r_3-m_3\rho_3^n}\left(\left(c_{q_3}(\rho_3^n)^{q_3}\mu_{33}^{q_3}+\gamma_3^{q_3}\left(1+\frac{1}{n+1}\right)^{q_3}v_3^{q_3}+q_3\rho_3^n\psi_3\mu_{33}^{q_3}-\rho_3^n\kappa_3\right)^{1/q_3}\right. \\
&\qquad\qquad \left.\left.\left.+\rho_3^n\nu_{33}\left(1+\frac{1}{n+1}\right)u_3+\left|\rho_3^n-\rho_3^{n-1}\right|\left(h_3+g_3+\|\varepsilon_3\|\right)\right)\right]\right\} \\
&\quad\times\left\|x_3^n-x_3^{n-1}\right\|+\frac{1}{1-\xi_3^n}\alpha_3^n\rho_3^n\left[\left(\mu_{31}+\nu_{31}\left(1+\frac{1}{n+1}\right)u_1\right)\right]\left\|x_1^{n-1}-x_1^n\right\| \\
&\quad+\frac{1}{1-\xi_3^n}\alpha_3^n\rho_3^n\left[\left(\mu_{32}+\nu_{32}\left(1+\frac{1}{n+1}\right)u_2\right)\right]\left\|x_2^{n-1}-x_2^n\right\|.
\end{aligned}
\tag{4.18}
$$

Let

$$\Gamma_{11}(n)=\frac{1}{1-\xi_1^n}\Bigg\{1-\alpha_1^n\Bigg[1-\left(1+q_1\sigma_1\left(1+\frac{1}{n+1}\right)^{q_1}v_1^{q_1}-q_1\zeta_1+c_{q_1}\left(1+\frac{1}{n+1}\right)^{q_1}v_1^{q_1}\right)^{1/q_1}-\frac{\xi_1^{n-1}}{\alpha_1^{n-1}}$$
$$-\frac{\tau_1^{q_1-1}}{r_1-m_1\rho_1^n}\left(\left(c_{q_1}(\rho_1^n)^{q_1}\mu_{11}^{q_1}+\gamma_1^{q_1}\left(1+\frac{1}{n+1}\right)^{q_1}v_1^{q_1}+q_1\rho_1^n\psi_1\mu_{11}^{q_1}-\rho_1\kappa_1\right)^{1/q_1}\right.$$
$$\left.+\rho_1^n\nu_{11}\left(1+\frac{1}{n+1}\right)s_1+()\left|\rho_1^n-\rho_1^{n-1}\right|\left(h_1+g_1+\|\varepsilon_1\|\right)\right)\Bigg]\Bigg\},$$

$$\Gamma_{12}(n)=\frac{1}{1-\xi_1^n}\alpha_1^n\rho_1^n\left[\left(\mu_{12}+\nu_{12}\left(1+\frac{1}{n+1}\right)s_2\right)\right],$$

$$\Gamma_{13}(n)=\frac{1}{1-\xi_1^n}\alpha_1^n\rho_1^n\left[\left(\mu_{13}+\nu_{13}\left(1+\frac{1}{n+1}\right)s_3\right)\right],$$

$$\Gamma_{21}(n)=\frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\left[\left(\mu_{21}+\nu_{21}\left(1+\frac{1}{n+1}\right)t_1\right)\right],$$

$$\Gamma_{22}(n)=\frac{1}{1-\xi_2^n}\Bigg\{1-\alpha_2^n\Bigg[1-\left(1+q_2\sigma_2\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}-q_2\zeta_2+c_{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}\right)^{1/q_2}-\frac{\xi_2^{n-1}}{\alpha_2^{n-1}}$$
$$-\frac{\tau_2^{q_2-1}}{r_2-m_2\rho_2^n}\left(\left(c_{q_2}(\rho_2^n)^{q_2}\mu_{22}^{q_2}+\gamma_2^{q_2}\left(1+\frac{1}{n+1}\right)^{q_2}v_2^{q_2}+q_2\rho_2^n\psi_2\mu_{22}^{q_2}-\rho_2^n\kappa_2\right)^{1/q_2}\right.$$
$$\left.+\rho_2^n\nu_{22}\left(1+\frac{1}{n+1}\right)t_2+\left|\rho_2^n-\rho_2^{n-1}\right|\left(h_2+g_2+\|\varepsilon_2\|\right)\right)\Bigg]\Bigg\},$$

$$\Gamma_{23}(n)=\frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\left[\left(\mu_{23}+\nu_{23}\left(1+\frac{1}{n+1}\right)t_3\right)\right],$$

$$\Gamma_{31}(n)=\frac{1}{1-\xi_3^n}\alpha_3^n\rho_3^n\left[\left(\mu_{31}+\nu_{31}\left(1+\frac{1}{n+1}\right)u_1\right)\right],$$

$$\Gamma_{32}(n)=\frac{1}{1-\xi_3^n}\alpha_3^n\rho_3^n\left[\left(\mu_{32}+\nu_{32}\left(1+\frac{1}{n+1}\right)u_2\right)\right],$$

$$\Gamma_{33}(n)=\frac{1}{1-\xi_3^n}\Bigg\{1-\alpha_3^n\Bigg[1-\left(1+q_3\sigma_3\left(1+\frac{1}{n+1}\right)^{q_3}v_3^{q_3}-q_3\zeta_3+c_{q_3}\left(1+\frac{1}{n+1}\right)^{q_3}v_3^{q_3}\right)^{1/q_3}-\frac{\xi_3^{n-1}}{\alpha_3^{n-1}}$$
$$-\frac{\tau_3^{q_3-1}}{r_3-m_3\rho_3^n}\left(\left(c_{q_3}(\rho_3^n)^{q_3}\mu_{33}^{q_3}+\gamma_3^{q_3}\left(1+\frac{1}{n+1}\right)^{q_3}v_3^{q_3}+q_3\rho_3^n\psi_3\mu_{33}^{q_3}-\rho_3^n\kappa_3\right)^{1/q_3}\right.$$
$$\left.+\rho_3^n\nu_{33}\left(1+\frac{1}{n+1}\right)u_3+\left|\rho_3^n-\rho_3^{n-1}\right|\left(h_3+g_3+\|\varepsilon_3\|\right)\right)\Bigg]\Bigg\}. \tag{4.19}$$

Letting $\overrightarrow{a}(n+1) = (\|x_1^{n+1} - x_1^n\|, \|x_2^{n+1} - x_2^n\|, \|x_3^{n+1} - x_3^n\|)^T$ $(n = 0, 1, 2, \ldots)$, then combining (4.10),(4.17)–(4.19), we have $\overrightarrow{a}(n+1) \leq \Psi(\omega, n)\overrightarrow{a}(n)$, where

$$\Psi(\omega, n) = \begin{pmatrix} \Gamma_{11}(n) & \Gamma_{12}(n) & \Gamma_{13}(n) \\ \Gamma_{21}(n) & \Gamma_{22}(n) & \Gamma_{23}(n) \\ \Gamma_{31}(n) & \Gamma_{32}(n) & \Gamma_{33}(n) \end{pmatrix}, \tag{4.20}$$

which is called the iterative matrix for Hybrid proximal point three-step algorithm of nonlinear set-valued quasi-variational inclusions system involving (A, η)-Accretive mappings. Using (4.20), $\lim_{n\to\infty}\xi_i^n = 0$, $\alpha = \limsup_{n\to\infty}\alpha_i^n < 1$, $\rho_i = \lim_{n\to\infty}\rho_i^n \leq \infty$, we have

$$\Psi(\omega) = \limsup_{n\to\infty} \begin{pmatrix} \Gamma_{11}(n) & \Gamma_{12}(n) & \Gamma_{13}(n) \\ \Gamma_{21}(n) & \Gamma_{22}(n) & \Gamma_{23}(n) \\ \Gamma_{31}(n) & \Gamma_{32}(n) & \Gamma_{33}(n) \end{pmatrix} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix}, \tag{4.21}$$

where $\limsup_{n\to\infty}\Gamma_{ij}(n) = \Gamma_{ij}$ $(i, j = 1, 2, 3)$, and

$$\begin{aligned}
\Gamma_{11} &= 1 - \alpha_1\Bigg[1 - \left(1 + q_1\sigma_1 v_1^{q_1} - q_1\zeta_1 + c_{q_1}v_1^{q_1}\right)^{1/q_1} - \frac{\tau_1^{q_1-1}}{r_1 - m_1\rho_1} \\
&\qquad \times\left(c_{q_1}\rho_1^{q_1}\mu_{11}^{q_1} + \gamma_1^{q_1}v_1^{q_1} + q_1\rho_1\psi_1\mu_{11}^{q_1} - \rho_1\kappa_1\right)^{1/q_1} + \rho_1\nu_{11}s_1\Bigg], \\
\Gamma_{12} &= \alpha_1\rho_1\left(\mu_{12} + \nu_{12}s_2\right), \\
\Gamma_{13} &= \alpha_1\rho_1\left(\mu_{13} + \nu_{13}s_3\right), \\
\Gamma_{21} &= \alpha_2\rho_2\left(\mu_{21} + \nu_{21}t_1\right), \\
\Gamma_{22} &= 1 - \alpha_2\Bigg[1 - \left(1 + q_2\sigma_2 v_2^{q_2} - q_2\zeta_2 + c_{q_2}v_2^{q_2}\right)^{1/q_2} - \frac{\tau_2^{q_2-1}}{r_2 - m_2\rho_2} \\
&\qquad \times\left(c_{q_2}\rho_2^{q_2}\mu_{22}^{q_2} + \gamma_2^{q_2}v_2^{q_2} + q_2\rho_2\psi_2\mu_{22}^{q_2} - \rho_2\kappa_2\right)^{1/q_2} + \rho_2\nu_{22}t_2\Bigg], \\
\Gamma_{23} &= \alpha_2\rho_2\left[\left(\mu_{23} + \nu_{23}t_3\right)\right], \\
\Gamma_{31} &= \alpha_3\rho_3\left[\left(\mu_{31} + \nu_{31}u_1\right)\right], \\
\Gamma_{32} &= \alpha_3^n\rho_3\left[\left(\mu_{32} + \nu_{32}u_2\right)\right], \\
\Gamma_{33} &= 1 - \alpha_3\Bigg[1 - \left(1 + q_3\sigma_3 v_3^{q_3} - q_3\zeta_3 + c_{q_3}v_3^{q_3}\right)^{1/q_3} - \frac{\tau_3^{q_3-1}}{r_3 - m_3\rho_3} \\
&\qquad \times\left(c_{q_3}\rho_3^{q_3}\mu_{33}^{q_3} + \gamma_3^{q_3}v_3^{q_3} + q_3\rho_3\psi_3\mu_{33}^{q_3} - \rho_3\kappa_3\right)^{1/q_3} + \rho_3\nu_{33}u_3\Bigg].
\end{aligned} \tag{4.22}$$

By using [38], we have

$$\left\|\overrightarrow{a}(n+1)\right\| \leq \|\Psi(\omega,n)\|\left\|\overrightarrow{a}(n)\right\|. \tag{4.23}$$

Letting

$$\|\Psi(\omega)\| = \max\{\Gamma_{11},\Gamma_{12},\Gamma_{13},\Gamma_{21},\Gamma_{22},\Gamma_{23},\Gamma_{31},\Gamma_{32},\Gamma_{33}\}. \tag{4.24}$$

It follows from (4.22) and assumption condition (4.2) that $0 < \|\Psi(\omega)\| < 1$ and hence there exists $N_0 > 0$ and $\|\Psi(\omega)\|_* \in (\|\Psi(\omega)\|,1)$ such that $\Psi(\omega,n) < \|\Psi(\omega)\|_*$ for all $n \geq N_0$. Therefore, by (4.23), we have

$$\left\|\overrightarrow{a}(n+1)\right\| \leq \|\Psi(\omega)\|_*^n\left\|\overrightarrow{a}(1)\right\|, \quad \forall n \geq N_0. \tag{4.25}$$

Without loss of generality we assume

$$\left\|\overrightarrow{a}(n+1)\right\| \leq \|\Psi(\omega)\|_*^n\left\|\overrightarrow{a}(1)\right\|, \quad \forall n \geq 1. \tag{4.26}$$

By the property of the matrix norm [38], for $n \geq 1$, we have

$$\left\|x_i^{n+1} - x_i^n\right\| \leq \left\|\overrightarrow{a}(n+1)\right\| \leq \|\Psi(\omega)\|_*^n\left\|\overrightarrow{a}(1)\right\|. \tag{4.27}$$

Hence, for any $m > n > 0$ and $i = 1,2,3$, we have

$$\|x_i^m - x_i^n\| \leq \sum_{k=n}^{m-1}\left\|x_i^{k+1} - x_i^k\right\| \leq \sum_{k=n}^{m-1}\|\Psi(\omega)\|_*^n\left\|\overrightarrow{a}(1)\right\|. \tag{4.28}$$

It follows that $\|x_i^m - x_i^n\| \to 0$, as $n \to \infty$, and so that the $\{x_i^n\}$ is a Cauchy sequence in $\mathbf{X}$. Let $x_i^n \to x_i^*$ as $n \to \infty$. By the Lipschitz continuity of $S_i(\cdot), T_i(\cdot), U_i(\cdot), V_i(\cdot)$, we can obtain

$$\begin{aligned}
\left\|a_i^{n+1} - a_i^n\right\| &\leq \left(1+n^{-1}\right)D\left(S_i\left(x_i^{n+1}\right), S_i(x_i^n)\right) \leq \left(1+n^{-1}\right)s_i\left\|x_i^{n+1} - x_i^n\right\|, \\
\left\|b_i^{n+1} - b_i^n\right\| &\leq \left(1+n^{-1}\right)D\left(T_i\left(x_i^{n+1}\right), T_i(x_i^n)\right) \leq \left(1+n^{-1}\right)t_i\left\|x_i^{n+1} - x_i^n\right\|, \\
\left\|c_i^{n+1} - c_i^n\right\| &\leq \left(1+n^{-1}\right)D\left(U_i\left(x_i^{n+1}\right), U_i(x_i^n)\right) \leq \left(1+n^{-1}\right)u_i\left\|x_i^{n+1} - x_i^n\right\|, \\
\left\|d_i^{n+1} - d_i^n\right\| &\leq \left(1+n^{-1}\right)D\left(V_i\left(x_i^{n+1}\right), V_i(x_i^n)\right) \leq \left(1+n^{-1}\right)v_i\left\|x_i^{n+1} - x_i^n\right\|.
\end{aligned} \tag{4.29}$$

It follows that $\{a_i^n\}$, $\{b_i^n\}$, $\{c_i^n\}$, and $\{d_i^n\}$ are also Cauchy sequences in X_i. We can assume that $a_i^n \to a_i^*$, $b_i^n \to b_i^*$, $c_i^n \to c_i^*$, and $d_n \to d_i^*$, respectively. Noting that $a_i^n \in S_i(x_i^n)$, we have

$$\begin{aligned} d(a_i^*, S_i(x_i^*)) &\leq \|a_i^* - a_i^n\| + d(a_i^n, S_i(x_i^*)) \\ &\leq \|a_i^* - a_i^n\| + D(S_i(x_i^n), S_i(x_i^*)) \\ &\leq \|a_i^* - a_i^n\| + s_i\|x_i^n - x_i^*\| \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{4.30}$$

Hence $d(a_i^*, S_i(x_i^*)) = 0$ and therefore $a_i^* \in S_i(x_i^*)$. Similarly, we can prove that $b_i^* \in T_i(x_i^*)$, $c_i^* \in U_i(x_i^*)$, and $d_i^* \in V_i(x_i^*)$. By the Lipschitz continuity of $S_i(x_i^*)$, $T_i(x_i^*)$, $U_i(x_i^*)$, and $V_i(x_i^*)$, we have

$$\begin{aligned} d_1^* &= R_{\rho_1, M_1(d_1^*, \cdot)}^{A_1, \eta_1}\left(A_1(d_1^*) + \rho_1\varepsilon_1 + \rho_1 G_1(a_1^*, a_2^*, a_3^*) - \rho_1 F_1(x_1^*, x_2^*, x_3^*)\right), \\ d_2^* &= R_{\rho_2, M_1(d_2^*, \cdot)}^{A_2, \eta_2}\left(A_1(d_2^*) + \rho_2\varepsilon_2 + \rho_2 G_2(b_1^*, b_2^*, b_3^*) - \rho_2 F_2(x_1^*, x_2^*, x_3^*)\right), \\ d_3^* &= R_{\rho_3, M_3(d_3^*, \cdot)}^{A_3, \eta_3}\left(A_3(d_3^*) + \rho_3\varepsilon_3 + \rho_3 G_3(c_1^*, c_2^*, c_3^*) - \rho_3 F_3(x_1^*, x_2^*, x_3^*)\right) \end{aligned} \tag{4.31}$$

for $i = 1,2,3$, where $\rho_i > 0$ is a constant. Thus, by Theorem (3.3), we know that $(x_i^*, a_i^*, b_i^*, c_i^*, d_i^*)_{i=1}^3$ is solution of problem (3.2). This completes the proof. □

Corollary 4.2. *Let X_i be a q_i-uniformly smooth Banach space, $\eta_i : X_i \times X_i \to X_i$ be a τ_i-Lipschtiz continuous mapping, and $A_i : X_i \to X_i$ be an r_i-strongly η_i-accretive mapping and γ_i-Lipschitz continuous. Let $S_i, T_i, U_i, V_i, F_i, G_i, M_1, M_2, M_3$ be the same as in Theorem 4.1. If*

$$\begin{aligned} &\sum_{i=1}^{3}\left(1 + q_i\sigma_i v_i^{q_i} - q_i\zeta_i + c_{q_i} v_i^{q_i}\right)^{1/q_i} - \sum_{i=1}^{3}\frac{\tau_i^{q_i-1}}{r_i - m_i\rho_i}\left(c_{q_i}\rho_i^{q_i}\mu_{ii}^{q_i} + \gamma_i^{q_i} v_i^{q_i} + q_i\rho_i\psi_i\mu_{11}^{q_i} - \rho_1\kappa_1\right)^{1/q_i} \\ &\quad + \alpha_1\rho_1(\mu_{12} + \nu_{12}s_2 + \mu_{13} + \nu_{13}s_3) + (\rho_3\nu_{33}u_3 + \rho_1\nu_{11}s_1 + \rho_2\nu_{22}t_2) \\ &\quad + \alpha_2\rho_2(\mu_{21} + \nu_{21}t_1 + \mu_{23} + \nu_{23}t_3) + \alpha_3\rho_3(\mu_{31} + \nu_{31}u_1 + \mu_{32} + \nu_{32}u_2) < 1, \end{aligned} \tag{4.32}$$

where $c_{q_i} > 0$ is the same as in Lemma 2.10, $\rho_i \in (0, r_i/m_i)$, and $i = 1,2,3$. Then problem (3.2) *has a solution* $(x_i^*, a_i^*, b_i^*, c_i^*, d_i^*)$ $(i = 1,2,3)$.

Remark 4.3. For a suitable choice of the mappings $A_i, \eta_i, F_i, M_i, G_i, \varepsilon_i, S_i, T_i, U_i, V_i (i = 1,2,3)$, we can obtain several known results in [2–5, 9, 11–27, 29, 32–37] as special cases of Theorem 4.1 and Corollary 4.2.

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