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Research Article

Convergence of Three-Step Iterations Scheme for Nonself Asymptotically Nonexpansive Mappings

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Weak and strong convergence theorems of three-step iterations are established for nonself asymptotically nonexpansive mappings in uniformly convex Banach space. The results obtained in this paper extend and improve the recent ones announced by Suantai (2005), Khan and Hussain (2008), Nilsrakoo and Saejung (2006), and many others.

1. Introduction

Suppose that X is a real uniformly convex Banach space, K is a nonempty closed convex subset of X. Let T be a self-mapping of K.

A mapping T is called *nonexpansive* provided

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$.

T is called *asymptotically nonexpansive* mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.2}$$

for all $x, y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive maps which is an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [1]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point.

T is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for all $x, y \in K$, the following inequality holds:

$$\left\|T^n x - T^n y\right\| \leq L\|x - y\| \tag{1.3}$$

for all $n \geq 1$.

Asymptotically nonexpansive self-mappings using Ishikawa iterative and the Mann iterative processes have been studied extensively by various authors to approximate fixed points of asymptotically nonexpansive mappings (see [1, 2]). Noor [3] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [4] applied a three-step iterative process for finding the approximate solutions of liquid crystal theory, and eigenvalue computation. It has been shown in [1] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Xu and Noor [5] introduced and studied a three-step scheme to approximate fixed point of asymptotically nonexpansive mappings in a Banach space. Very recently, Nilsrakoo and Saejung [6] and Suantai [7] defined new three-step iterations which are extensions of Noor iterations and gave some weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in Banach space. It is clear that the modified Noor iterations include Mann iterations [8], Ishikawa iterations [9], and original Noor iterations [3] as special cases. Consequently, results obtained in this paper can be considered as a refinement and improvement of the previously known results

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + \gamma_n T^n x_n + \left(1 - \alpha_n - \beta_n - \gamma_n\right) x_n, \quad \forall n \geq 1, \end{aligned} \tag{1.4}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0, 1]$ satisfy certain conditions.

If $\{\gamma_n\} = 0$, then (1.4) reduces to the modified Noor iterations defined by Suantai [7] as follows:

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + \left(1 - \alpha_n - \beta_n\right) x_n, \quad \forall n \geq 1, \end{aligned} \tag{1.5}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\alpha_n + \beta_n\}$ in $[0, 1]$ satisfy certain conditions.

If $\{c_n\} = \{\beta_n\} = \{\gamma_n\} = 0$, then (1.4) reduces to Noor iterations defined by Xu and Noor [5] as follows:

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad \forall n \geq 1. \end{aligned} \tag{1.6}$$

If $\{a_n\} = \{c_n\} = \{\beta_n\} = \{\gamma_n\} = 0$, then (1.4) reduces to modified Ishikawa iterations as follows:

$$\begin{aligned} y_n &= b_n T^n z_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad \forall n \geq 1. \end{aligned} \tag{1.7}$$

If $\{a_n\} = \{b_n\} = \{c_n\} = \{\beta_n\} = \{\gamma_n\} = 0$, then (1.4) reduces to Mann iterative process as follows:

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad \forall n \geq 1. \tag{1.8}$$

Let X be a real normed space and K be a nonempty subset of X. A subset K of X is called a *retract* of X if there exists a continuous map $P : X \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a rectract. A map $P : X \to K$ is called a retraction if $P^2 = P$. In particular, a subset K is called a *nonexpansive retract* of X if there exists a *nonexpansive retraction* $P : X \to K$ such that $Px = x$ for all $x \in K$.

Iterative techniques for converging fixed points of nonexpansive nonself-mappings have been studied by many authors (see, e.g., Khan and Hussain [10], Wang [11]). Evidently, we can obtain the corresponding nonself-versions of (1.5)–(1.7). We will obtain the weak and strong convergence theorems using (1.12) for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space. Very recently, Suantai [7] introduced iterative process and used it for the weak and strong convergence of fixed points of self-mappings in a uniformly convex Banach space. As remarked earlier, Suantai [7] has established weak and strong convergence criteria for asymptotically nonexpansive self-mappings, while Chidume et al. [12] studied the Mann iterative process for the case of nonself-mappings. Our results will thus improve and generalize corresponding results of Suantai [7] and others for nonself-mappings and those of Chidume et al. [12] in the sense that our iterative process contains the one used by them. The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume et al. [12] as the generalization of asymptotically nonexpansive self-mappings and obtained some strong and weak convergence theorems for such mappings given (1.9) as follows: for $x_1 \in K$,

$$\begin{aligned} y_n &= P\Big(\beta_n T (PT)^{n-1} x_n + (1 - \beta_n) x_n\Big), \\ x_{n+1} &= P\Big(\alpha_n T (PT)^{n-1} y_n + (1 - \alpha_n) x_n\Big), \quad \forall n \geq 1, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$.

A nonself-mapping T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq k_n\|x-y\| \tag{1.10}$$

for all $x, y \in K$, and $n \geq 1$. T is called *uniformly L-Lipschitzian* if there exists constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\|x-y\| \tag{1.11}$$

for all $x, y \in K$, and $n \geq 1$. From the above definition, it is obvious that nonself asymptotically nonexpansive mappings are uniformly L-Lipschitzian.

Now, we give the following nonself-version of (1.4):

for $x_1 \in K$,

$$\begin{aligned} z_n &= P\Big(a_nT(PT)^{n-1}x_n + (1-a_n)x_n\Big),\\ y_n &= P\Big(b_nT(PT)^{n-1}z_n + c_nT(PT)^{n-1}x_n + (1-b_n-c_n)x_n\Big),\\ x_{n+1} &= P\Big(\alpha_nT(PT)^{n-1}y_n + \beta_nT(PT)^{n-1}z_n + \gamma_nT(PT)^{n-1}x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n\Big), \end{aligned} \tag{1.12}$$

for all $n \geq 1$, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ satisfy certain conditions.

The aim of this paper is to prove the weak and strong convergence of the three-step iterative sequence for nonself asymptotically nonexpansive mappings in a real uniformly convex Banach space. The results presented in this paper improve and generalize some recent papers by Suantai [7], Khan and Hussain [10], Nilsrakoo and Saejung [6], and many others.

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space, K is a nonempty closed convex subset of X, and $F(T)$ is the set of fixed points of mapping T. A Banach space X is said to be *uniformly convex* if the modulus of convexity of X is as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\} > 0, \tag{2.1}$$

for all $0 < \varepsilon \leq 2$ (i.e., $\delta(\varepsilon)$ is a function $(0,2] \to (0,1)$).

Recall that a Banach space X is said to satisfy *Opial's condition* [13] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|. \tag{2.2}$$

Lemma 2.1 (see [12]). *Let X be a uniformly convex Banach space, K a nonempty closed convex subset of X and $T : K \to X$ a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, then $I - T$ is demiclosed at zero.*

Lemma 2.2 (see [12]). *Let X be a real uniformly convex Banach space, K a nonempty closed subset of X with P as a sunny nonexpansive retraction and $T : K \to X$ a mapping satisfying weakly inward condition, then $F(PT) = F(T)$.*

Lemma 2.3 (see [14]). *Let $\{s_n\}$, $\{t_n\}$, and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: for all $n \geq 1$, $s_{n+1} \leq (1 + \sigma_n)s_n + t_n$, where $\sum_{n=0}^{\infty} \sigma_n < \infty$ and $\sum_{n=0}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

Lemma 2.4 (see [6]). *Let X be a uniformly convex Banach space and $B_R := \{x \in X : \|x\| \leq R\}$, $R > 0$, then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\begin{aligned} \|\lambda x + \mu y + \xi z + \nu w\|^2 \leq{}& \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \nu\|w\|^2 \\ & - \frac{1}{3}\nu\left(\lambda g(\|x - w\|) + \mu g(\|y - w\|) + \xi g(\|z - w\|)\right), \end{aligned} \tag{2.3}$$

for all $x, y, z, w \in B_r$, and $\lambda, \mu, \xi, \nu \in [0, 1]$ with $\lambda + \mu + \xi + \nu = 1$.

Lemma 2.5 (See [7], Lemma 2.7). *Let X be a Banach space which satisfies Opial's condition and let x_n be a sequence in X. Let $q_1, q_2 \in X$ be such that $\lim_{n\to\infty}\|x_n - q_1\|$ and $\lim_{n\to\infty}\|x_n - q_2\|$. If $\{x_{n_k}\}$, $\{x_{n_j}\}$ are the subsequences of $\{x_n\}$ which converge weakly to $q_1, q_2 \in X$, respectively, then $q_1 = q_2$.*

3. Main Results

In this section, we prove theorems of weak and strong of the three-step iterative scheme given in (1.12) to a fixed point for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space. In order to prove our main results the followings lemmas are needed.

Lemma 3.1. *If $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n\to\infty}(b_n + c_n) < 1$ and $\{k_n\}$ is sequence of real numbers with $k_n \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} k_n = 1$, then there exists a positive integer N_1 and $\gamma \in (0, 1)$ such that $c_n k_n < \gamma$ for all $n \geq N_1$.*

Proof. By $\limsup_{n\to\infty}(b_n + c_n) < 1$, there exists a positive integer N_0 and $\delta \in (0, 1)$ such that

$$c_n \leq b_n + c_n < \delta, \quad \forall n \geq N_0. \tag{3.1}$$

Let $\delta' \in (0, 1)$ with $\delta' > \delta$. From $\lim_{n\to\infty} k_n = 1$, then there exists a positive integer $N_1 \geq N_0$ such that

$$k_n - 1 < \frac{1}{\delta'} - 1, \quad \forall n \geq N_1, \tag{3.2}$$

from which we have $k_n < 1/\delta'$, for all $n \geq N_1$. Put $\gamma = \delta/\delta'$, then we have $c_n k_n < \gamma$ for all $n \geq N_1$. □

Lemma 3.2. *Let X be a real Banach space and K a nonempty closed and convex subset of X. Let $T : K \to X$ be a nonself asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}$ be a sequence in K defined by (1.12), then we have, for any $q \in F(T)$, $\lim_{n\to\infty}\|x_n - q\|$ exists.*

Proof. Consider

$$
\begin{aligned}
\|z_n - q\| &= \left\|P\left(a_n T(PT)^{n-1}x_n + (1-a_n)x_n\right) - Pq\right\| \\
&\leq \left\| a_n T(PT)^{n-1}x_n + (1-a_n)x_n - q)\right\| \\
&\leq \left\|a_n\left(T(PT)^{n-1}x_n - q\right) + (1-a_n)(x_n - q)\right\| \\
&\leq a_n\left\|T(PT)^{n-1}x_n - q\right\| + (1-a_n)\|x_n - q\| \\
&\leq a_n k_n\|x_n - q\| + (1-a_n)\|x_n - q\| \\
&= (1 + a_n k_n - a_n)\|x_n - q\| = (1 + a_n(k_n - 1))\|x_n - q\|, \\
\|y_n - q\| &= \left\|P\left(b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1-b_n-c_n)x_n\right) - Pq\right\| \\
&\leq \left\|b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1-b_n-c_n)x_n - q\right\| \\
&\leq b_n\left\|T(PT)^{n-1}z_n - q\right\| + c_n\left\|T(PT)^{n-1}x_n - q\right\| + (1-b_n-c_n)\|x_n - q\| \\
&\leq b_n k_n\|z_n - q\| + c_n k_n\|x_n - q\| + (1-b_n-c_n)\|x_n - q\| \\
&\leq b_n k_n(1 + a_n(k_n - 1))\|x_n - q\| + (c_n k_n + (1-b_n-c_n))\|x_n - q\| \\
&= (1 + (k_n - 1)(b_n + c_n + a_n b_n k_n))\|x_n - q\|, \\
\|x_{n+1} - q\| &= \left\|P\left(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n\right.\right. \\
&\qquad \left.\left. + \gamma_n T(PT)^{n-1}x_n + \left(1-\alpha_n-\beta_n-\gamma_n\right)x_n\right) - Pq\right\| \\
&\leq \alpha_n\left\|T(PT)^{n-1}y_n - q\right\| + \beta_n\left\|T(PT)^{n-1}z_n - q\right\| + \gamma_n\left\|T(PT)^{n-1}x_n - q\right\| \\
&\quad + \left(1-\alpha_n-\beta_n-\gamma_n\right)\|x_n - q\| \\
&\leq \alpha_n k_n\|y_n - q\| + \beta_n k_n\|z_n - q\| + \gamma_n k_n\|x_n - q\| \\
&\quad + \left(1-\alpha_n-\beta_n-\gamma_n\right)\|x_n - q\| \\
&\leq \left[(\alpha_n k_n)(1 + (k_n - 1)(b_n + c_n + a_n b_n(k_n))) + \beta_n k_n(1 + a_n(k_n - 1))\right. \\
&\qquad \left. + \gamma_n k_n + \left(1-\alpha_n-\beta_n-\gamma_n\right)\right]\|x_n - q\| \\
&\leq \left[1 + (k_n - 1)\left[\alpha_n + \beta_n + \gamma_n\right] + (k_n - 1)\left[k_n\alpha_n b_n + k_n\alpha_n c_n\right]\right. \\
&\quad \left. + \left[(k_n - 1)\left(\alpha_n k_n^2 b_n a_n\right) + (k_n - 1)(\beta_n k_n a_n)\right]\right]\|x_n - q\|.
\end{aligned}
\tag{3.3}
$$

Thus, we have

$$\|x_{n+1}-q\| \le \Big(1+(k_n-1)\Big(\alpha_n+\beta_n+\gamma_n+\alpha_n k_n b_n+\alpha_n k_n c_n \\ +\alpha_n k_n^2 b_n a_n+\beta_n k_n a_n\Big)\Big)\|x_n-q\|. \tag{3.4}$$

Since $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and from Lemma 2.3, it follows that $\lim_{n\to\infty}\|x_n-q\|$ exits. □

Lemma 3.3. *Let X be a real uniformly convex Banach space and K a nonempty closed and convex subset of X. Let $T: K \to X$ be a nonself asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \ge 1$ and $\sum_{n=0}^{\infty}(k_n^2-1)<\infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0,1]$, such that $\{b_n+c_n\}$ and $\{\alpha_n+\beta_n+\gamma_n\}$ in $[0,1]$ for all $n \ge 1$. Let $\{x_n\}$ be a sequence in K defined by (1.12), then one has the following conclusions.*

(1) *If $0<\liminf_n \alpha_n \le \limsup_n(\alpha_n+\beta_n+\gamma_n)<1$, then $\lim_n\|T(PT)^{n-1}y_n-x_n\|=0$.*

(2) *If either $0<\liminf_n \beta_n \le \limsup_n(\alpha_n+\beta_n+\gamma_n)<1$ or $0<\liminf_n \alpha_n$ and $0\le \limsup_n b_n \le \limsup_n(b_n+c_n)<1$, then $\lim_n\|T(PT)^{n-1}z_n-x_n\|=0$.*

(3) *If the following conditions*

(i) $0<\liminf_n \gamma_n \le \limsup_n(\alpha_n+\beta_n+\gamma_n)<1$,

(ii) *either $0<\liminf_n \alpha_n$ and $0\le\limsup_n b_n \le \limsup_n(b_n+c_n)<1$ or $0<\liminf_n \beta_n \le \limsup_n(\alpha_n+\beta_n+\gamma_n)<1$ and $\limsup_n a_n<1$ are satisfied, then $\lim_n\|T(PT)^{n-1}x_n-x_n\|=0$.*

Proof. Let $M=\sup\{k_n, n\ge 1\}$. By Lemma 3.2, we know that $\lim_{n\to\infty}\|x_n-q\|$ exits for any $q\in F(T)$. Then the sequence $\{x_n-q\}$ is bounded. It follows that the sequences $\{y_n-q\}$ and $\{z_n-q\}$ are also bounded. Since $T:K\to X$ is a nonself asymptotically nonexpansive mapping, then the sequences $\{T(PT)^{n-1}x_n-q\}$, $\{T(PT)^{n-1}y_n-q\}$, and $\{T(PT)^{n-1}z_n-q\}$ are also bounded. Therefore, there exists $R>0$ such that $\{\|x_n-q\|\}$, $\{T(PT)^{n-1}x_n-q\}$, $\{y_n-q\}$, $\{T(PT)^{n-1}y_n-q\}$, $\{z_n-q\}$, $\{T(PT)^{n-1}z_n-q\}\subset B_R$. By Lemma 2.4 and (1.12), we have

$$\begin{aligned}
\|z_n-q\|^2 &= \Big\|P\Big(a_nT(PT)^{n-1}x_n+(1-a_n)x_n\Big)-Pq\Big\|^2\\
&\le \Big\|\Big(a_nT(PT)^{n-1}x_n+(1-a_n)x_n-q\Big)\Big\|^2\\
&\le \Big\|a_n\Big(T(PT)^{n-1}x_n-q\Big)+(1-a_n)(x_n-q)\Big\|^2\\
&\le a_n\Big\|T(PT)^{n-1}x_n-q\Big\|^2+(1-a_n)\|x_n-q\|^2-a_n\Big(g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big)\\
&\le a_nk_n^2\|x_n-q\|^2+(1-a_n)\|x_n-q\|^2-a_n\Big(g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big)\\
&\le \Big(1+a_nk_n^2-a_n\Big)\|x_n-q\|^2+(1-a_n)\|x_n-q\|^2\\
&= \Big(1+a_n\Big(k_n^2-1\Big)\Big)\|x_n-q\|^2
\end{aligned}$$

$$
\begin{aligned}
\|y_n - q\|^2 &= \left\| P\left(b_n T(PT)^{n-1} z_n + c_n T(PT)^{n-1} x_n + (1-b_n-c_n)x_n\right) - Pq \right\|^2 \\
&\le \left\| b_n T(PT)^{n-1} z_n + c_n T(PT)^{n-1} x_n + (1-b_n-c_n)x_n - q \right\|^2 \\
&\le b_n \left\| T(PT)^{n-1} z_n - q \right\|^2 + c_n \left\| T(PT)^{n-1} x_n - q \right\|^2 + (1-b_n-c_n)\|x_n - q\|^2 \\
&\quad - \frac{1}{3}(1-b_n-c_n)\left(b_n g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right) + c_n g\left(\left\| T(PT)^{n-1} x_n - x_n \right\|\right)\right) \\
&\le b_n k_n^2 \|z_n - q\|^2 + c_n k_n^2 \|x_n - q\|^2 + (1-b_n-c_n)\|x_n - q\|^2 \\
&\quad - \frac{1}{3} b_n (1-b_n-c_n)\left(g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right)\right) \\
&\le b_n k_n^2 \left(1 + a_n\left(k_n^2 - 1\right)\right)\|x_n - q\|^2 + \left(c_n k_n^2 + (1-b_n-c_n)\right)\|x_n - q\|^2 \\
&\quad - \frac{1}{3} b_n (1-b_n-c_n)\left(g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right)\right) \\
&= \left(1 + \left(k_n^2 - 1\right)\left(b_n + c_n + a_n b_n k_n^2\right)\right)\|x_n - q\|^2 \\
&\quad - \frac{1}{3} b_n (1-b_n-c_n)\left(g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right)\right) \\
\|x_{n+1} - q\|^2 &= \Big\| P\Big(\alpha_n T(PT)^{n-1} y_n + \beta_n T(PT)^{n-1} z_n + \gamma_n T(PT)^{n-1} x_n \\
&\qquad + (1-\alpha_n-\beta_n-\gamma_n)x_n\Big) - Pq \Big\| \\
&\le \alpha_n \left\| T(PT)^{n-1} y_n - q \right\|^2 + \beta_n \left\| T(PT)^{n-1} z_n - q \right\|^2 + \gamma_n \left\| T(PT)^{n-1} x_n - q \right\|^2 \\
&\quad + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - q\|^2 \\
&\quad - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\left(\left\| T(PT)^{n-1} y_n - x_n \right\|\right) + \beta_n g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right) \\
&\qquad + \gamma_n g\left(\left\| T(PT)^{n-1} x_n - x_n \right\|\right)\Big) \\
&\le \alpha_n k_n^2 \|y_n - q\|^2 + \beta_n k_n^2 \|z_n - q\|^2 + \gamma_n k_n^2 \|x_n - q\|^2 + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - q\|^2 \\
&\quad - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\left(\left\| T(PT)^{n-1} y_n - x_n \right\|\right) + \beta_n g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right) \\
&\qquad + \gamma_n g\left(\left\| T(PT)^{n-1} x_n - x_n \right\|\right)\Big) \\
&\le \left(\alpha_n k_n^2\right)\left(1 + \left(k_n^2 - 1\right)\left(\left(b_n + c_n + a_n b_n k_n^2\right)\right)\right)\|x_n - q\|^2 \\
&\quad - \frac{1}{3}\left(\alpha_n k_n^2\right) b_n (1-b_n-c_n)\left(g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right)\right) \\
&\quad + \beta_n k_n^2 \left(\left(1 + a_n\left(k_n^2 - 1\right)\right)\|x_n - q\|^2\right) \\
&\quad + \gamma_n k_n^2 \|x_n - q\|^2 + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - q\|^2 \\
&\quad - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\left(\left\| T(PT)^{n-1} y_n - x_n \right\|\right) + \beta_n g\left(\left\| T(PT)^{n-1} z_n - x_n \right\|\right) \\
&\qquad + \gamma_n g\left(\left\| T(PT)^{n-1} x_n - x_n \right\|\right)\Big) \\
&\le \Big[\left(\alpha_n k_n^2\right)\left[b_n k_n^2 + \beta_n k_n^4 a_n - \beta_n k_n^2 a_n + c_n k_n^2 + 1 - b_n - c_n\right] \\
&\qquad + \beta_n k_n^2\left[1 + a_n k_n^2 - a_n\right] + \left[\gamma_n k_n^2 + 1 - \alpha_n - \beta_n - \gamma_n\right]\Big]\|x_n - q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{3}b_n\alpha_n k_n^2(1-b_n-c_n)\Big(g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\Big)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\Big(\Big\|T(PT)^{n-1}y_n-x_n\Big\|\Big)+\beta_n g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\\
&\qquad\qquad\qquad\qquad +\gamma_n g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big)\\
&=\|x_n-q\|^2+\Big[\alpha_n k_n^4 b_n+\alpha_n k_n^6 b_n a_n-\alpha_n k_n^4 b_n a_n\\
&\qquad\qquad +\alpha_n k_n^4 c_n+\alpha_n k_n^2-\alpha_n k_n^2 b_n-\alpha_n k_n^2 c_n+\beta_n k_n^2\\
&\qquad\qquad +\beta_n k_n^4 a_n-\beta_n k_n^2 a_n+\gamma_n k_n^2-\alpha_n-\beta_n-\gamma_n\Big]\|x_n-q\|^2\\
&-\frac{1}{3}b_n\alpha_n k_n^2(1-b_n-c_n)\Big(g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\Big)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\Big(\Big\|T(PT)^{n-1}y_n-x_n\Big\|\Big)+\beta_n g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\\
&\qquad\qquad\qquad\qquad +\gamma_n g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big)\\
&=\|x_n-q\|^2+\Big[\alpha_n\Big(k_n^2-1\Big)+\beta_n\Big(k_n^2-1\Big)+\gamma_n\Big(k_n^2-1\Big)\\
&\qquad\qquad +\Big(\alpha_n k_n^2 b_n\Big)\Big(k_n^2-1\Big)+\Big(\alpha_n a_n b_n k_n^4\Big)\Big(k_n^2-1\Big)\\
&\qquad\qquad +\Big(\beta_n k_n^2 a_n\Big)\Big(k_n^2-1\Big)+\Big(\alpha_n k_n^2 c_n\Big)\Big(k_n^2-1\Big)\Big]\|x_n-q\|^2\\
&-\frac{1}{3}b_n\alpha_n k_n^2(1-b_n-c_n)\Big(g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\Big)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\Big(\Big\|T(PT)^{n-1}y_n-x_n\Big\|\Big)+\beta_n g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\\
&\qquad\qquad\qquad\qquad +\gamma_n g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big)\\
&=\|x_n-q\|^2+\Big(k_n^2-1\Big)\Big[\alpha_n+\beta_n+\gamma_n+\Big(\alpha_n k_n^2 b_n\Big)+\Big(\alpha_n a_n b_n k_n^4\Big)\\
&\qquad\qquad\qquad\qquad +\Big(\beta_n k_n^2 a_n\Big)+\Big(\alpha_n k_n^2 c_n\Big)\Big]\|x_n-q\|^2\\
&-\frac{1}{3}b_n\alpha_n k_n^2(1-b_n-c_n)\Big(g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\Big)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\Big(\Big\|T(PT)^{n-1}y_n-x_n\Big\|\Big)+\beta_n g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\\
&\qquad\qquad\qquad\qquad +\gamma_n g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big).\\
&=\|x_n-q\|^2+\Big(k_n^2-1\Big)\Big[\alpha_n+\beta_n+\gamma_n+\Big(\alpha_n k_n^2 b_n\Big)+\Big(\alpha_n a_n b_n k_n^4\Big)\\
&\qquad\qquad\qquad\qquad +\Big(\beta_n k_n^2 a_n\Big)+\Big(\alpha_n k_n^2 c_n\Big)\Big]\|x_n-q\|^2\\
&-\frac{1}{3}b_n\alpha_n k_n^2(1-b_n-c_n)\Big(g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\Big)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g\Big(\Big\|T(PT)^{n-1}y_n-x_n\Big\|\Big)+\beta_n g\Big(\Big\|T(PT)^{n-1}z_n-x_n\Big\|\Big)\\
&\qquad\qquad\qquad\qquad +\gamma_n g\Big(\Big\|T(PT)^{n-1}x_n-x_n\Big\|\Big)\Big)\\
&\le\|x_n-q\|^2+\Big(k_n^2-1\Big)\Big(M^4+3M^2+3\Big)R^2
\end{aligned}
$$

$$\begin{aligned}
&-\frac{1}{3}\left(b_n\alpha_n k_n^2\right)(1-b_n-c_n)\left(g\left(\left\|T(PT)^{n-1}z_n-x_n\right\|\right)\right)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\left(\alpha_n g\left(\left\|T(PT)^{n-1}y_n-x_n\right\|\right)+\beta_n g\left(\left\|T(PT)^{n-1}z_n-x_n\right\|\right)\right.\\
&\qquad\qquad\left.+\gamma_n g\left(\left\|T(PT)^{n-1}x_n-x_n\right\|\right)\right),
\end{aligned}\tag{3.5}$$

Let $\kappa_n=(k_n^2-1)(M^4+3M^2+3)R^2$.
Therefore, the assumption $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$ implies that $\sum_{n=1}^{\infty}\kappa_n<\infty$.
Thus, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2\le{}&\|x_n-q\|^2+\kappa_n-\frac{1}{3}(1-b_n-c_n)\left(b_n\alpha_n k_n^2\right)\left(g\left(\left\|T(PT)^{n-1}z_n-x_n\right\|\right)\right)\\
&-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\left(\alpha_n g\left(\left\|T(PT)^{n-1}y_n-x_n\right\|\right)+\beta_n g\left(\left\|T(PT)^{n-1}z_n-x_n\right\|\right)\right.\\
&\qquad\qquad\left.+\gamma_n g\left(\left\|T(PT)^{n-1}x_n-x_n\right\|\right)\right).
\end{aligned}\tag{3.6}$$

From the last inequality, we have

$$\alpha_n(1-\alpha_n-\beta_n-\gamma_n)g\left(\left\|T(PT)^{n-1}y_n-x_n\right\|\right)\le 3\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2+\kappa_n\right),\tag{3.7}$$

$$\beta_n(1-\alpha_n-\beta_n-\gamma_n)g\left(\left\|T(PT)^{n-1}z_n-x_n\right\|\right)\le 3\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2+\kappa_n\right),\tag{3.8}$$

$$\gamma_n(1-\alpha_n-\beta_n-\gamma_n)g\left(\left\|T(PT)^{n-1}x_n-x_n\right\|\right)\le 3\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2+\kappa_n\right),\tag{3.9}$$

$$(1-b_n-c_n)\left(b_n\alpha_n k_n^2\right)g\left(\left\|T(PT)^{n-1}z_n-x_n\right\|\right)\le 3\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2+\kappa_n\right).\tag{3.10}$$

By condition

$$0<\liminf_n\alpha_n\le\limsup_n(\alpha_n+\beta_n+\gamma_n)<1,\tag{3.11}$$

there exists a positive integer n_0 and $\delta,\delta'\in(0,1)$ such that $0<\delta<\alpha_n$ and $\alpha_n+\beta_n+\gamma_n<\delta'<1$ for all $n\ge n_0$, then it follows from (3.7) that

$$\begin{aligned}
&(\delta(1-\delta'))\lim_{n\to\infty}\alpha_n(1-\alpha_n-\beta_n-\gamma_n)g\left(\left\|T(PT)^{n-1}y_n-x_n\right\|\right)\\
&\quad\le 3\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2+\kappa_n\right),
\end{aligned}\tag{3.12}$$

for all $n \geq n_0$. Thus, for $m \geq n_0$, we write

$$\begin{aligned}\sum_{n=n_0}^{m} g\left(\left\|T(PT)^{n-1}y_n - x_n\right\|\right) &\leq \frac{3}{(\delta(1-\delta'))}\sum_{n=n_0}^{m}\left(\|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \kappa_n\right) \\ &\leq \frac{3}{(\delta(1-\delta'))}\left(\|x_{n_0} - q\|^2 + \sum_{n=n_0}^{m}(\kappa_n)\right).\end{aligned} \tag{3.13}$$

Letting $m \to \infty$, we have $\sum_{n=n_0}^{m} g(\|T(PT)^{n-1}y_n - x_n\|) < \infty$, so that

$$\lim_{n\to\infty} g\left(\left\|T(PT)^{n-1}y_n - x_n\right\|\right) = 0. \tag{3.14}$$

From g is continuous strictly increasing with $g(0) = 0$ and (1), then we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}y_n - x_n\right\| = 0. \tag{3.15}$$

By using a similar method for inequalities (3.8) and (3.10), we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}z_n - x_n\right\| = 0. \tag{3.16}$$

Next, to prove

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n - x_n\right\| = 0, \tag{3.17}$$

we assume that $0 < \liminf_n \alpha_n$ and $0 \leq \limsup_n(b_n) \leq \limsup_n(b_n + c_n) < 1$,

$$\begin{aligned}\left\|T(PT)^{n-1}x_n - x_n\right\| &\leq \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\right\| + \left\|T(PT)^{n-1}y_n - x_n\right\| \\ &\leq k_n\|x_n - y_n\| + \left\|T(PT)^{n-1}y_n - x_n\right\| \\ &\leq k_n b_n\left\|T(PT)^{n-1}z_n - x_n\right\| + k_n c_n\left\|T(PT)^{n-1}x_n - x_n\right\| \\ &\quad + \left\|T(PT)^{n-1}y_n - x_n\right\|,\end{aligned} \tag{3.18}$$

$$(1 - k_n c_n)\left\|T(PT)^{n-1}x_n - x_n\right\| \leq k_n b_n\left\|T(PT)^{n-1}z_n - x_n\right\| + \left\|T(PT)^{n-1}y_n - x_n\right\|. \tag{3.19}$$

By Lemma 3.1, there exists a positive integer N_1 and $\gamma \in (0,1)$ such that $c_n k_n < \gamma$ for all $n \geq N_1$. This together with (3.18) implies that for $n \geq N_1$,

$$\begin{aligned}(1-\gamma)\left\|(PT)^{n-1}x_n - x_n\right\| &< (1 - k_n c_n)\left\|T(PT)^{n-1}x_n - x_n\right\| \\ &\leq k_n b_n\left\|T(PT)^{n-1}z_n - x_n\right\| + \left\|T(PT)^{n-1}y_n - x_n\right\|.\end{aligned} \tag{3.20}$$

It follows from (3.15) and (3.16) that

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n - x_n\right\| = 0. \tag{3.21}$$

This completes the proof. □

Next, we show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Lemma 3.4. *Let X be a real uniformly convex Banach space and K a nonempty closed convex subset of X. Let $T : K \longrightarrow X$ be a nonself asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=0}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}$ be a sequence in K defined by (1.12) with the following restrictions:*

(1) $0 < \min\{\liminf_n \alpha_n, \liminf_n \beta_n, \liminf_n \gamma_n\} \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n a_n < 1$,

(2) $0 \leq \limsup_n b_n \leq \limsup_n(b_n + c_n) < 1$,

then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. We first consider

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad\leq \left\|\left(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n + \gamma_n T(PT)^{n-1}x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n\right) - x_n\right\| \\
&\quad\leq \alpha_n\left\|T(PT)^{n-1}y_n - x_n\right\| + \beta_n\left\|T(PT)^{n-1}z_n - x_n\right\| \\
&\qquad + \gamma_n\left\|T(PT)^{n-1}x_n - x_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.22}$$

We note that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian. Also note that

$$\begin{aligned}
&\left\|x_{n+1} - T(PT)^{n-1}x_{n+1}\right\| \\
&\quad\leq \|x_{n+1} - x_n\| + \left\|x_n - T(PT)^{n-1}x_n\right\| + \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}x_{n+1}\right\| \\
&\quad= \|x_{n+1} - x_n\| + \left\|T(PT)^{n-1}x_{n+1} - T(PT)^{n-1}x_n\right\| + \left\|T(PT)^{n-1}x_n - x_n\right\| \\
&\quad\leq \|x_{n+1} - x_n\| + L\|x_{n+1} - x_n\| + \left\|T(PT)^{n-1}x_n - x_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.23}$$

In addition,

$$\begin{aligned}
&\left\|x_{n+1} - T(PT)^{n-2}x_{n+1}\right\| \\
&\quad \le \|x_{n+1} - x_n\| + \left\|x_n - T(PT)^{n-2}x_n\right\| + \left\|T(PT)^{n-2}x_n - T(PT)^{n-2}x_{n+1}\right\| \\
&\quad \le \|x_{n+1} - x_n\| + \left\|T(PT)^{n-2}x_n - x_n\right\| + L\|x_{n+1} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.24}$$

We denote as $(PT)^{1-1}$ the identity maps from K into itself. Thus, by above inequality, we write

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \left\|x_{n+1} - T(PT)^{n-1}x_{n+1}\right\| + \left\|T(PT)^{n-1}x_{n+1} - Tx_{n+1}\right\| \\
&= \left\|x_{n+1} - T(PT)^{n-1}x_{n+1}\right\| + L\left\|T(PT)^{n-2}x_{n+1} - x_{n+1}\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.25}$$

which implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.26}$$

□

In the next result, we prove our first strong convergence theorem as follows.

Theorem 3.5. *Let X be a real uniformly convex Banach space and K a nonempty closed convex subset of X. Let $T : K \to X$ be a nonself asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \ge 1$ and $\sum_{n=0}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \ge 1$. Let $\{x_n\}$ be a sequence in K defined by (1.12) with the following restrictions:*

(1) $0 < \min\{\liminf_n \alpha_n, \liminf_n \beta_n, \liminf_n \gamma_n\} \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n a_n < 1$,

(2) $0 \le \limsup_n b_n \le \limsup_n (b_n + c_n) < 1$.

If, in addition, T is either completely continuous or demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.2, $\{x_n\}$ is bounded. It follows by our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to q*$ as $k \to \infty$. Therefore, by Lemma 3.4, we have $\|Tx_{n_k} - q*\| \to 0$ which implies that $x_{n_k} \to q*$ as $k \to \infty$. Again by Lemma 3.4, we have

$$\|q * - Tq*\| = \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{3.27}$$

It follows that $q* \in F(T)$. Moreover, since $\lim_{k\to\infty}\|x_n - q*\| = 0$ exists, then $\lim_{k\to\infty}\|x_n - q*\| = 0$, that is, $\{x_n\}$ converges strongly to a fixed point q^* of T.

We assume that T is demicompact. Then, using the same ideas and argument, we also prove that $\{x_n\}$ converges strongly to a fixed point of T. □

Finally, we prove the weak convergence of the iterative scheme (1.12) for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.6. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K a nonempty closed convex subset of X. Let $T : K \to X$ be a nonself asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=0}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}$ be a sequence in K defined by* (1.12) *with the following restrictions:*

(1) $0 < \min\{\liminf_n \alpha_n, \liminf_n \beta_n, \liminf_n \gamma_n\} \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n a_n < 1$,

(2) $0 \leq \limsup_n b_n \leq \limsup_n(b_n + c_n) < 1$,

then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Let $q \in F(T)$. Then as in Lemma 3.2, $\lim_{n\to\infty}\|x_n - q\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. We assume that q_1 and q_2 are weak limits of the subsequences $\{x_{n_k}\}$, $\{x_{n_j}\}$, or $\{x_n\}$, respectively. By Lemma 3.4, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed by Lemma 2.1, $Tq_1 = q_1$ and in the same way, $Tq_2 = q_2$. Therefore, we have $q_1, q_2 \in F(T)$. It follows from Lemma 2.5 that $q_1 = q_2$. Thus, $\{x_n\}$ converges weakly to an element of $F(T)$. This completes the proof. □

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