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*Research Article*

# Fixed Point Theorems for Generalized Weakly Contractive Condition in Ordered Metric Spaces

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Fixed point results with the concept of generalized weakly contractive conditions in complete ordered metric spaces are derived. These results generalize the existing fixed point results in the literature.

## 1. Introduction and Preliminaries

There are a lot of generalizations of the Banach contraction mapping principle in the literature. One of the most interesting of them is the result of Khan et al. [1]. They addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

A function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance function if $\varphi$ is continuous, nondecreasing, and $\varphi(0) = 0$ holds.

Khan et al. [1] given the following result.

**Theorem 1.1.** *Let $(\mathcal{X}, d)$ be a complete metric space, let $\varphi$ be an altering distance function, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping which satisfies the following inequality:*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \le c\varphi(d(x, y)), \tag{1.1}$$

*for all $x, y \in \mathcal{X}$ and for some $0 < c < 1$. Then $\mathcal{T}$ has a unique fixed point.*

In fact, Khan et al. [1] proved a more general theorem of which the above result is a corollary. Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [2] in Hilbert Spaces by introducing the concept of weakly contractive mappings.

A self-mapping $\mathcal{T}$ on a metric space $\mathcal{X}$ is called weakly contractive if for each $x, y \in \mathcal{X}$,

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y) - \phi(d(x, y)), \tag{1.2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is positive on $(0, \infty)$ and $\phi(0) = 0$.

Rhoades [3] showed that most results of [2] are still valid for any Banach space. Also, Rhoades [3] proved the following very interesting fixed point theorem which contains contractions as special case $\phi(t) = (1 - k)t$.

**Theorem 1.2.** *Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a weakly contractive mapping, and in addition, $\phi$ is continuous and nondecreasing function, then $\mathcal{T}$ has a unique fixed point.*

In fact, Alber and Guerre-Delabriere [2] assumed an additional condition on $\phi$ which is $\lim_{t\to\infty}\phi(t) = \infty$. But Rhoades [3] obtained the result noted in Theorem 1.2 without using this particular assumption. Also, the weak contractions are closely related to maps of Boyd and Wong [4] and Reich type [5]. Namely, if $\phi$ is a lower semicontinuous function from the right, then $\psi(t) = t - \phi(t)$ is an upper semicontinuous function from the right, and moreover, (1.2) turns into $d(\mathcal{T}x, \mathcal{T}y) \leq \psi(d(x, y))$. Therefore, the weak contraction is of Boyd and Wong type. And if we define $\beta(t) = 1 - \phi(t)/t$ for $t > 0$ and $\beta(0) = 0$, then (1.2) is replaced by $d(\mathcal{T}x, \mathcal{T}y) \leq \beta(d(x, y))d(x, y)$. Therefore, the weak contraction becomes a Reich-type one.

Recently, the following generalized result was given by Dutta and Choudhury [6] combining Theorem 1.1 and Theorem 1.2.

**Theorem 1.3.** *Let $(\mathcal{X}, d)$ be a complete metric space, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping satisfying the inequality*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(x, y)) - \phi(d(x, y)), \tag{1.3}$$

*for all $x, y \in \mathcal{X}$, where $\varphi$, $\phi : [0, \infty) \to [0, \infty)$ are both continuous and nondecreasing functions with $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then, $\mathcal{T}$ has a unique fixed point.*

Also, Zhang and Song [7] given the following generalized version of Theorem 1.2.

**Theorem 1.4.** *Let $(\mathcal{X}, d)$ be a complete metric space, and let $\mathcal{T}$, $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be two mappings such that for each $x, y \in \mathcal{X}$,*

$$d(\mathcal{T}x, \mathcal{S}y) \leq \Phi(x, y) - \phi(\Phi(x, y)), \tag{1.4}$$

*where $\phi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$,*

$$\Phi(x, y) = \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{S}y), \frac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{S}y)]\right\}. \tag{1.5}$$

*Then, there exists a unique point $z \in \mathcal{X}$ such that $z = \mathcal{T}z = \mathcal{S}z$.*

Very recently, Abbas and Doric [8] and Abbas and Ali Khan [9] have obtained common fixed points of four and two mappings, respectively, which satisfy generalized weak contractive condition.

In recent years, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering $\preceq$ in the literature [10–25]. Most of them are a hybrid of two fundamental principle: Banach contraction theorem and the weakly contractive condition. Indeed, they deal with a monotone (either order-preserving or order-reversing) mapping satisfying, with some restriction, a classical contractive condition, and such that for some $x_0 \in \mathcal{X}$, either $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, where $\mathcal{T}$ is a self-map on metric space. The first result in this direction was given by Ran and Reurings [22, Theorem 2.1] who presented its applications to matrix equation. Subsequently, Nieto and Rodŕiguez-López [18] extended the result of Ran and Reurings [22] for nondecreasing mappings and applied to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions.

Further, Harjani and Sadarangani [26] proved the ordered version of Theorem 1.2, Amini-Harandi and Emami [12] proved the ordered version of Rich type fixed point theorem, and Harjani and Sadarangani [27] proved ordered version of Theorem 1.3.

The aim of this paper is to give a generalized ordered version of Theorem 1.4. We will do this using the concept of weakly increasing mapping mentioned by Altun and Simsek [11] (also see [28, 29]).

## 2. Main Results

We will begin with a single map. The following theorem is a generalized version of Theorems 2.1 and 2.2 of Harjani and Sadarangani [27].

**Theorem 2.1.** *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing mapping such that*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \quad \text{for } y \preceq x, \tag{2.1}$$

*where*

$$\Theta(x, y) = ad(x, y) + bd(x, \mathcal{T}x) + cd(y, \mathcal{T}y) + e[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)], \tag{2.2}$$

*$a > 0$, $b, c, e \geq 0$, $a + b + c + 2e \leq 1$, $\varphi, \phi : [0, \infty) \to [0, \infty)$, $\varphi$ is continuous, nondecreasing, $\phi$ is lower semicontinuous functions, and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Also, suppose that there exists $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$. If*

$$\mathcal{T} \text{ is continuous}, \tag{2.3}$$

*or*

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \text{ then } x_n \preceq z \ \forall n \tag{2.4}$$

*holds. Then, $\mathcal{T}$ has a fixed point.*

*Proof.* If $\mathcal{T}x_0 = x_0$, then the proof is completed. Suppose that $\mathcal{T}x_0 \neq x_0$. Now, since $x_0 \preceq \mathcal{T}x_0$, and $\mathcal{T}$ is nondecreasing, we have

$$x_0 \preceq \mathcal{T}x_0 \preceq \mathcal{T}^2x_0 \preceq \cdots \preceq \mathcal{T}^n x_0 \preceq \mathcal{T}^{n+1}x_0 \cdots. \tag{2.5}$$

Put $x_n = \mathcal{T}^n x_0$, and so $x_{n+1} = \mathcal{T}x_n$. If there exists $n_0 \in \{1, 2, \ldots\}$ such that $\Theta(x_{n_0}, x_{n_0-1}) = 0$, then it is clear that $x_{n_0-1} = x_{n_0} = \mathcal{T}x_{n_0-1}$, and so we are finished. Now, we can suppose that

$$\Theta(x_n, x_{n-1}) > 0, \tag{2.6}$$

for all $n \geq 1$.

First, we will prove that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

From (2.2), we have for $n \geq 1$

$$\begin{aligned}\Theta(x_n, x_{n-1}) &= ad(x_n, x_{n-1}) + bd(x_n, \mathcal{T}x_n) + cd(x_{n-1}, \mathcal{T}x_{n-1}) \\ &\quad + e[d(x_{n-1}, \mathcal{T}x_n) + d(x_n, \mathcal{T}x_{n-1})] \\ &= (a+c)d(x_n, x_{n-1}) + bd(x_n, x_{n+1}) + ed(x_{n-1}, x_{n+1}) \\ &\leq (a+c+e)d(x_n, x_{n-1}) + (b+e)d(x_n, x_{n+1}).\end{aligned} \tag{2.7}$$

Now, we claim that

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}), \tag{2.8}$$

for all $n \geq 1$. Suppose that this is not true; that is, there exists $n_0 \geq 1$ such that $d(x_{n_0+1}, x_{n_0}) > d(x_{n_0}, x_{n_0-1})$. Now, since $x_{n_0} \preceq x_{n_0+1}$, we can use the (2.1) for these elements, then we have

$$\begin{aligned}\varphi(d(x_{n_0+1}, x_{n_0})) &= \varphi(d(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0-1})) \\ &\leq \varphi(\Theta(x_{n_0}, x_{n_0-1})) - \phi(\Theta(x_{n_0}, x_{n_0-1})) \\ &\leq \varphi((a+c+e)d(x_{n_0}, x_{n_0-1}) + (b+e)d(x_{n_0}, x_{n_0+1})) \\ &\quad - \phi(\Theta(x_{n_0}, x_{n_0-1})) \\ &\leq \varphi((a+b+c+2e)d(x_{n_0}, x_{n_0+1})) - \phi(\Theta(x_{n_0}, x_{n_0-1})) \\ &\leq \varphi(d(x_{n_0}, x_{n_0+1})) - \phi(\Theta(x_{n_0}, x_{n_0-1})).\end{aligned} \tag{2.9}$$

This implies $\phi(\Theta(x_{n_0}, x_{n_0-1})) = 0$, by the property of $\phi$, we have $\Theta(x_{n_0}, x_{n_0-1}) = 0$, which this contradict to (2.6). Therefore, (2.8) is true, and so the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus there exists $\rho \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = \rho$. Now suppose that $\rho > 0$. Therefore from (2.2)

$$\begin{aligned}\lim_{n\to\infty} ad(x_n, x_{n-1}) &\leq \limsup_{n\to\infty} \Theta(x_n, x_{n-1}) \\ &= \limsup_{n\to\infty} [(a+c)d(x_n, x_{n-1}) + bd(x_n, x_{n+1}) + ed(x_{n-1}, x_{n+1})] \\ &\leq \limsup_{n\to\infty} [(a+c+e)d(x_n, x_{n-1}) + (b+e)d(x_n, x_{n+1})].\end{aligned} \tag{2.10}$$

This implies

$$0 < a\rho \leq \limsup_{n\to\infty} \Theta(x_n, x_{n-1}) \leq (a+b+c+2e)\rho \leq \rho \tag{2.11}$$

and so there exist $\rho_1 > 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \Theta(x_{n(k)}, x_{n(k)-1}) = \rho_1 \leq \rho$.

By the lower semicontinuity of $\phi$ we have

$$\phi(\rho_1) \leq \liminf_{k\to\infty} \phi(\Theta(x_{n(k)}, x_{n(k)+1})). \tag{2.12}$$

From (2.1), we have

$$\begin{aligned}\varphi(d(x_{n(k)+1}, x_{n(k)})) &= \varphi(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)-1})) \\ &\leq \varphi(\Theta(x_{n(k)}, x_{n(k)-1})) - \phi(\Theta(x_{n(k)}, x_{n(k)-1})),\end{aligned} \tag{2.13}$$

and taking upper limit as $k \to \infty$, we have

$$\begin{aligned}\varphi(\rho) &\leq \varphi(\rho_1) - \liminf_{k\to\infty} \phi(\Theta(x_{n(k)}, x_{n(k)+1})) \\ &\leq \varphi(\rho_1) - \phi(\rho_1) \\ &\leq \varphi(\rho) - \phi(\rho_1),\end{aligned} \tag{2.14}$$

that is, $\phi(\rho_1) = 0$. Thus, by the property of $\phi$, we have $\rho_1 = 0$, which is a contradiction. Therefore, we have $\rho = 0$.

Next, we show that $\{x_n\}$ is Cauchy.

Suppose that this is not true. Then, there is an $\varepsilon > 0$ such that for an integer $k$, there exist integers $m(k) > n(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{2.15}$$

For every integer $k$, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.15) and such that

$$d(x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{2.16}$$

Now,

$$\begin{aligned} \varepsilon &< d(x_{n(k)}, x_{m(k)}) \\ &\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}). \end{aligned} \tag{2.17}$$

Then, by (2.15) and (2.16), it follows that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.18}$$

Also, by the triangle inequality, we have

$$\left| d(x_{n(k)}, x_{m(k)-1}) - d(x_{n(k)}, x_{m(k)}) \right| < d(x_{m(k)-1}, x_{m(k)}). \tag{2.19}$$

By using (2.18), we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.20}$$

Now, by (2.2), we get

$$\begin{aligned} ad(x_{n(k)}, x_{m(k)-1}) &\leq \Theta(x_{n(k)}, x_{m(k)-1}) \\ &= ad(x_{n(k)}, x_{m(k)-1}) + bd(x_{n(k)}, \mathcal{T}x_{n(k)}) + cd(x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), \\ &\quad e\left[d(x_{m(k)-1}, \mathcal{T}x_{n(k)}) + d(x_{n(k)}, \mathcal{T}x_{m(k)-1})\right] \\ &= ad(x_{n(k)}, x_{m(k)-1}) + bd(x_{n(k)}, x_{n(k)+1}) + cd(x_{m(k)-1}, x_{m(k)}), \\ &\quad e\left[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})\right] \\ &\leq ad(x_{n(k)}, x_{m(k)-1}) + bd(x_{n(k)}, x_{n(k)+1}) + cd(x_{m(k)-1}, x_{m(k)}), \\ &\quad e\left[d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})\right], \end{aligned} \tag{2.21}$$

and taking upper limit as $k \to \infty$ and using (2.18) and (2.20), we have

$$0 < a\varepsilon \leq \limsup_{k\to\infty} \Theta(x_{n(k)}, x_{m(k)-1}) \leq (a + 2e)\varepsilon \leq \varepsilon. \tag{2.22}$$

This implies that there exist $\varepsilon_1 > 0$ and a subsequence $\{x_{n(k(p))}\}$ of $\{x_{n(k)}\}$ such that

$$\lim_{p\to\infty} \Theta(x_{n(k(p))}, x_{m(k(p))-1}) = \varepsilon_1 \leq \varepsilon. \tag{2.23}$$

By the lower semicontinuity of $\phi$, we have

$$\phi(\varepsilon) \leq \liminf_{k\to\infty} \phi(\Theta(x_{n(k)}, x_{m(k)-1})). \tag{2.24}$$

Now, by (2.1), we get

$$\begin{aligned}
\varphi(\varepsilon) &= \limsup_{p\to\infty} \varphi\left(d\left(x_{n(k(p))}, x_{m(k(p))}\right)\right) \\
&\le \limsup_{p\to\infty} \varphi\left(d\left(x_{n(k(p))}, x_{n(k(p))+1}\right) + d\left(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p))-1}\right)\right) \\
&= \limsup_{p\to\infty} \varphi\left(d\left(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p))-1}\right)\right) \\
&\le \limsup_{p\to\infty} \left[\varphi\left(\Theta\left(x_{n(k(p))}, x_{m(k(p))-1}\right)\right) - \phi\left(\Theta\left(x_{n(k(p))}, x_{m(k(p))-1}\right)\right)\right] \\
&= \varphi(\varepsilon_1) - \liminf_{p\to\infty} \phi\left(\Theta\left(x_{n(k(p))}, x_{m(k(p))-1}\right)\right) \\
&\le \varphi(\varepsilon_1) - \phi(\varepsilon_1) \\
&\le \varphi(\varepsilon) - \phi(\varepsilon_1),
\end{aligned} \tag{2.25}$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$, there exists $z \in \mathcal{X}$ such that $x_n \to z$ as $n \to \infty$. If $\mathcal{T}$ is continuous, then it is clear that $\mathcal{T}z = z$. If (2.4) holds, then we have $x_n \preceq z$ for all $n$. Therefore, for all $n$, we can use (2.1) for $x_n$ and $z$. Since

$$\begin{aligned}
\Theta(z, x_n) &= ad(z, x_n) + bd(z, \mathcal{T}z) + cd(x_n, \mathcal{T}x_n) + e[d(x_n, \mathcal{T}z) + d(z, \mathcal{T}x_n)] \\
&= ad(z, x_n) + bd(z, \mathcal{T}z) + cd(x_n, x_{n+1}) + e[d(x_n, \mathcal{T}z) + d(z, x_{n+1})],
\end{aligned} \tag{2.26}$$

and so $\lim_{n\to\infty}\Theta(z, x_n) = (b+e)d(z, \mathcal{T}z)$, we have

$$\begin{aligned}
\varphi(d(\mathcal{T}z, z)) &= \limsup_{n\to\infty} \varphi(d(\mathcal{T}z, x_{n+1})) \\
&= \limsup_{n\to\infty} \varphi(d(\mathcal{T}z, \mathcal{T}x_n)) \\
&\le \limsup_{n\to\infty} \left[\varphi(\Theta(z, x_n)) - \phi(\Theta(z, x_n))\right] \\
&\le \varphi((b+e)d(\mathcal{T}z, z)) - \phi((b+e)d(\mathcal{T}z, z)) \\
&\le \varphi(d(\mathcal{T}z, z)) - \phi((b+e)d(\mathcal{T}z, z)).
\end{aligned} \tag{2.27}$$

By the property of $\phi$, we have $\mathcal{T}z = z$. Thus, the proof is complete. □

The following corollary is a generalized version of Theorems 1.2 and 1.3 of Harjani and Sadarangani [26].

**Corollary 2.2.** *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing mapping such that*

$$d(\mathcal{T}x, \mathcal{T}y) \le \Theta(x, y) - \phi(\Theta(x, y)) \quad \text{for } y \preceq x, \tag{2.28}$$

*where*

$$\Theta(x,y) = ad(x,y) + bd(x,\mathcal{T}x) + cd(y,\mathcal{T}y) + e\left[d(y,\mathcal{T}x) + d(x,\mathcal{T}y)\right], \tag{2.29}$$

$a > 0,\ b,c,e \geq 0,\ a+b+c+2e \leq 1$, $\phi : [0,\infty) \to [0,\infty)$ *is a lower semicontinuous functions, and* $\phi(t) = 0$ *if and only if* $t = 0$. *Also, suppose that there exists* $x_0 \in \mathcal{X}$ *with* $x_0 \preceq \mathcal{T}x_0$. *If*

$$\mathcal{T} \text{ is continuous,} \tag{2.30}$$

*or*

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X},\ \text{ then } x_n \preceq z\ \forall n \tag{2.31}$$

*holds. Then,* $\mathcal{T}$ *has a fixed point.*

*Remark 2.3.* In Theorem 1.1 [22], it is proved that if

$$\text{every pair of elements has a lower bound and upper bound,} \tag{2.32}$$

then for every $x \in \mathcal{X}$,

$$\lim_{n\to\infty} \mathcal{T}^n x = y, \tag{2.33}$$

where $y$ is the fixed point of $\mathcal{T}$ such that

$$y = \lim_{n\to\infty} \mathcal{T}^n x_0, \tag{2.34}$$

and hence, $\mathcal{T}$ has a unique fixed point. If condition (2.32) fails, it is possible to find examples of functions $\mathcal{T}$ with more than one fixed point. There exist some examples to illustrate this fact in [18].

*Example 2.4.* Let $\mathcal{X} = \mathbb{R}$, and consider a relation on $\mathcal{X}$ as follows:

$$x \preceq y \Longleftrightarrow \{(x = y) \text{ or } (x,y \in [0,1] \text{ with } x \leq y). \tag{2.35}$$

It is easy to see that $\preceq$ is a partial order on $\mathcal{X}$. Let $d$ be Euclidean metric on $\mathcal{X}$. Now, define a self map of $\mathcal{X}$ as follows:

$$\mathcal{T}x = \begin{cases} 2x - \dfrac{3}{2}, & x > 1, \\ \dfrac{x}{4}, & 0 \leq x \leq 1, \\ 0, & x < 0. \end{cases} \tag{2.36}$$

Now, we claim that the condition (2.1) of Theorem 2.1 is satisfied with $\varphi(t) = t,\ \phi(t) = t/2$. Indeed, if $x, y \notin [0,1]$, then $x \preceq y \Leftrightarrow x = y$. Therefore, since $d(\mathcal{T}x, \mathcal{T}y) = 0$, then the condition (2.1) is satisfied. Again, if $x \in [0,1]$ and $y \notin [0,1]$, then $x$ and $y$ are not comparative. Now, if $x, y \in [0,1]$, then $x \preceq y \Leftrightarrow x \le y$ and

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= d\left(\frac{x}{4}, \frac{y}{4}\right) \\ &= \frac{1}{4} d(x,y) \\ &= \frac{1}{2}\Theta(x,y), \quad \left(a = \frac{1}{2},\ b = c = e = 0\right) \\ &= \Theta(x,y) - \frac{1}{2}\Theta(x,y) \\ &= \Theta(x,y) - \phi(\Theta(x,y)). \end{aligned} \tag{2.37}$$

Also, it is easy to see that the other conditions of Theorem 2.1 are satisfied, and so $\mathcal{T}$ has a fixed point in $\mathcal{X}$. Also, note that the weak contractive condition of Theorem 1.3 of this paper and Corollary 2.2 of [7] is not satisfied.

Now, we will give a common fixed point theorem for two maps. For this, we need the following definition, which is given in [28].

*Definition 2.5.* Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Two mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are said to be weakly increasing if $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$.

Note that two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [11].

**Theorem 2.6.** *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are two weakly increasing mappings such that*

$$\varphi(d(\mathcal{T}x, \mathcal{S}y)) \le \varphi(\Phi(x,y)) - \phi(\Phi(x,y)), \tag{2.38}$$

*for all comparable $x, y \in \mathcal{X}$, where*

$$\Phi(x,y) = ad(x,y) + bd(x,\mathcal{T}x) + cd(y,\mathcal{S}y) + e[d(y,\mathcal{T}x) + d(x,\mathcal{S}y)], \tag{2.39}$$

$a > 0,\ b, c, e \ge 0,\ a + b + c + 2e \le 1, \varphi,\ \phi : [0,\infty) \to [0,\infty)$, *$\varphi$ is continuous, nondecreasing, $\phi$ is lower semicontinuous functions, and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. If*

$$\mathcal{S} \text{ is continuous}, \tag{2.40}$$

*or*

$$\mathcal{T} \text{ is continuous,} \tag{2.41}$$

*or*

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \text{ then } x_n \preceq z \ \forall n \tag{2.42}$$

*holds. Then,* $\mathcal{S}$ *and* $\mathcal{T}$ *have a common fixed point.*

*Remark 2.7.* Note that in this theorem, we remove the condition "there exists an $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$" of Theorem 2.1. Again, we can consider the result of Remark 2.3 for this theorem.

*Proof of Theorem 2.6.* First of all we show that if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let $z$ be a fixed point of $\mathcal{S}$. Now, assume $d(z, \mathcal{T}z) > 0$. If we use (2.38) for $x = y = z$, we have

$$\begin{aligned} \varphi(d(\mathcal{T}z, z)) &= \varphi(d(\mathcal{T}z, \mathcal{S}z)) \\ &\le \varphi(\Phi(z, z)) - \phi(\Phi(z, z)) \\ &\le \varphi(d(\mathcal{T}z, z)) - \phi((b + e)d(\mathcal{T}z, z)), \end{aligned} \tag{2.43}$$

which is a contradiction. Thus, $d(z, \mathcal{T}z) = 0$, and so $z$ is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Similarly, if $z$ is a fixed point of $\mathcal{T}$, then it is also fixed point of $\mathcal{S}$. Now, let $x_0$ be an arbitrary point of $\mathcal{X}$. If $x_0 = \mathcal{S}x_0$, the proof is finished, so assume that $x_0 \neq \mathcal{S}x_0$. We can define a sequence $\{x_n\}$ in $\mathcal{X}$ as follows:

$$x_{2n+1} = \mathcal{S}x_{2n}, \qquad x_{2n+2} = \mathcal{T}x_{2n+1} \qquad \text{for } n \in \{0, 1, \ldots\}. \tag{2.44}$$

Without lost of generality, we can suppose that the successive term of $\{x_n\}$ are different. Otherwise, we are again finished. Note that since $\mathcal{S}$ and $\mathcal{T}$ are weakly increasing, we have

$$\begin{aligned} x_1 &= \mathcal{S}x_0 \preceq \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2, \\ x_2 &= \mathcal{T}x_1 \preceq \mathcal{S}\mathcal{T}x_1 = \mathcal{S}x_2 = x_3, \end{aligned} \tag{2.45}$$

and continuing this process, we have

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{2.46}$$

Now, since $x_{2n-1}$ and $x_{2n}$ are comparable, then we can use (2.38) for these points, then we have

$$\varphi(d(\mathcal{T}x_{2n-1}, \mathcal{S}x_{2n})) \le \varphi(\Phi(x_{2n-1}, x_{2n})) - \phi(\Phi(x_{2n-1}, x_{2n})), \tag{2.47}$$

where

$$\Phi(x_{2n-1}, x_{2n}) = ad(x_{2n-1}, x_{2n}) + bd(x_{2n-1}, \mathcal{T}x_{2n-1}) + cd(x_{2n}, \mathcal{S}x_{2n}) \\ + e[d(x_{2n}, \mathcal{T}x_{2n-1}) + d(x_{2n-1}, \mathcal{S}x_{2n})] \\ \leq (a+b+e)d(x_{2n-1}, x_{2n}) + (c+e)d(x_{2n}, x_{2n+1}). \tag{2.48}$$

Now, if $d(x_{2n+1}, x_{2n}) > d(x_{2n}, x_{2n-1})$ for some $n \in \{1, 2, \ldots\}$, then

$$\Phi(x_{2n-1}, x_{2n}) \leq (a+b+c+2e)d(x_{2n+1}, x_{2n}) \leq d(x_{2n+1}, x_{2n}), \tag{2.49}$$

and so, from (2.47) we have

$$\varphi(d(x_{2n}, x_{2n+1})) \leq \varphi(d(x_{2n+1}, x_{2n})) - \phi(\Phi(x_{2n-1}, x_{2n})), \tag{2.50}$$

which is a contradiction. So, we have $d(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n-1})$ for all $n \in \{1, 2, \ldots\}$. Similarly, we have $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$ for all $n \in \{0, 1, \ldots\}$. Therefore, we have for all $n \in \{1, 2, \ldots\}$

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}), \tag{2.51}$$

and so the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus, there exists $\rho \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = \rho$. This implies that $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = \lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = \rho$. Suppose that $\rho > 0$. Therefore, from (2.39),

$$\limsup_{n\to\infty} ad(x_{2n-1}, x_{2n}) \leq \limsup_{n\to\infty} \Phi(x_{2n-1}, x_{2n}) \\ \leq \limsup_{n\to\infty} \{(a+b+e)d(x_{2n-1}, x_{2n}) + (c+e)d(x_{2n}, x_{2n+1})\}. \tag{2.52}$$

This implies $0 < a\rho \leq \limsup_{n\to\infty} \Phi(x_{2n-1}, x_{2n}) \leq (a+b+c+2e)\rho \leq \rho$, and so there exist $\rho_1 > 0$ and a subsequence $\{\Phi(x_{2n(k)-1}, x_{2n(k)})\}$ of $\{\Phi(x_{2n-1}, x_{2n})\}$ such that $\lim_{k\to\infty} \Phi(x_{2n(k)-1}, x_{2n(k)}) = \rho_1 \leq \rho$.

By the lower semicontinuity of $\phi$, we have

$$\phi(\rho_1) \leq \liminf_{k\to\infty} \phi(\Phi(x_{2n(k)-1}, x_{2n(k)})). \tag{2.53}$$

Now, from (2.38), we have

$$\varphi(d(x_{2n(k)}, x_{2n(k)+1})) = \varphi(d(\mathcal{T}x_{2n(k)-1}, \mathcal{S}x_{2n(k)})) \\ \leq \varphi(\Phi(x_{2n(k)-1}, x_{2n(k)})) - \phi(\Phi(x_{2n(k)-1}, x_{2n(k)})), \tag{2.54}$$

and taking upper limit as $k \to \infty$, we have

$$\begin{aligned}\varphi(\rho) &\leq \varphi(\rho_1) - \liminf_{k\to\infty} \phi(\Phi(x_{2n(k)-1}, x_{2n(k)})) \\ &\leq \varphi(\rho_1) - \phi(\rho_1) \\ &\leq \varphi(\rho) - \phi(\rho_1),\end{aligned} \tag{2.55}$$

which is a contradiction. Therefore, we have

$$\rho = 0 = \lim_{n\to\infty} d(x_{n+1}, x_n). \tag{2.56}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. For this, it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose it is not true. Then, we can find an $\delta > 0$ such that for each even integer $2k$, there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(x_{2n(k)}, x_{2m(k)}) \geq \delta \quad \text{for } k \in \{1, 2, \ldots\}. \tag{2.57}$$

We may also assume that

$$d(x_{2m(k)-2}, x_{2n(k)}) < \delta, \tag{2.58}$$

by choosing $2m(k)$ to be smallest number exceeding $2n(k)$ for which (2.57) holds. Now, (2.56), (2.57), and (2.58) imply

$$\begin{aligned}0 < \delta &\leq d(x_{2n(k)}, x_{2m(k)}) \\ &\leq d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \\ &\leq \delta + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}),\end{aligned} \tag{2.59}$$

and so

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \delta. \tag{2.60}$$

Also, by the triangular inequality,

$$\begin{aligned}\left|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| &\leq d(x_{2m(k)-1}, x_{2m(k)}), \\ \left|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| &\leq d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}).\end{aligned} \tag{2.61}$$

Therefore, we get

$$\begin{aligned}\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) &= \delta, \\ \lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)-1}) &= \delta.\end{aligned} \tag{2.62}$$

On the other hand, since $x_{2n(k)}$ and $x_{2m(k)-1}$ are comparable, we can use the condition (2.38) for these points. Since

$$\begin{aligned}\lim_{k\to\infty}\Phi\left(x_{2m(k)-1},x_{2n(k)}\right)&=\lim_{k\to\infty}\Big\{ad\left(x_{2m(k)-1},x_{2n(k)}\right)+bd\left(x_{2m(k)-1},\mathcal{T}x_{2m(k)-1}\right)\\&\qquad+cd\left(x_{2n(k)},\mathcal{S}x_{2n(k)}\right)\\&\qquad+e\left[d\left(x_{2n(k)},\mathcal{T}x_{2m(k)-1}\right)+d\left(x_{2m(k)-1},\mathcal{S}x_{2n(k)}\right)\right]\Big\}\\&=\lim_{k\to\infty}\Big\{ad\left(x_{2m(k)-1},x_{2n(k)}\right)+b\mathrm{d}\left(x_{2m(k)-1},x_{2m(k)}\right)\\&\qquad+cd\left(x_{2n(k)},x_{2n(k)+1}\right)\\&\qquad+e\left[d\left(x_{2n(k)},x_{2m(k)}\right)+d\left(x_{2m(k)-1},x_{2n(k)+1}\right)\right]\Big\}\\&=(a+2e)\delta,\end{aligned}\tag{2.63}$$

we have

$$\begin{aligned}\varphi(\delta)&\leq\limsup_{k\to\infty}\varphi\left(d\left(x_{2n(k)},x_{2m(k)}\right)\right)\\&\leq\limsup_{k\to\infty}\varphi\left(d\left(x_{2n(k)},x_{2n(k)+1}\right)+d\left(x_{2n(k)+1},x_{2m(k)}\right)\right)\\&\leq\limsup_{k\to\infty}\varphi\left(d\left(x_{2n(k)},x_{2n(k)+1}\right)+d\left(\mathcal{S}x_{2n(k)},\mathcal{T}x_{2m(k)-1}\right)\right)\\&=\limsup_{k\to\infty}\varphi\left(d\left(\mathcal{S}x_{2n(k)},\mathcal{T}x_{2m(k)-1}\right)\right)\\&\leq\limsup_{k\to\infty}\left[\varphi\left(\Phi\left(x_{2m(k)-1},x_{2n(k)}\right)\right)-\phi\left(\Phi\left(x_{2m(k)-1},x_{2n(k)}\right)\right)\right]\\&=\varphi((a+2e)\delta)-\liminf_{k\to\infty}\phi\left(\Phi\left(x_{2m(k)-1},x_{2n(k)}\right)\right)\\&\leq\varphi((a+2e)\delta)-\phi((a+2e)\delta)\\&\leq\varphi(\delta)-\phi((a+2e)\delta).\end{aligned}\tag{2.64}$$

This is a contradiction. Thus, $\{x_{2n}\}$ is a Cauchy sequence in $\mathcal{X}$, so $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a $z\in\mathcal{X}$ with $\lim_{n\to\infty}x_n=z$.

If $\mathcal{S}$ or $\mathcal{T}$ is continuous hold, then clearly, $z=\mathcal{S}z=\mathcal{T}z$. Now, suppose that (2.42) holds and $d(\mathcal{S}z,z)>0$. Since $\lim_{n\to\infty}x_n=z$, then from (2.42), $x_{2n-1}\preceq z$ for all $n$. Using (2.38), we have

$$\varphi(d(\mathcal{T}x_{2n-1},\mathcal{S}z))\leq\varphi(\Phi(x_{2n-1},z))-\phi(\Phi(x_{2n-1},z)),\tag{2.65}$$

or

$$\varphi(d(x_{2n},\mathcal{S}z))\leq\varphi(\Phi(x_{2n-1},z))-\phi(\Phi(x_{2n-1},z)),\tag{2.66}$$

so taking upper limit from the last inequality, we have

$$\varphi(d(z,\mathcal{S}z)) \leq \varphi((c+e)d(z,\mathcal{S}z)) - \phi((c+e)d(z,\mathcal{S}z)), \tag{2.67}$$

which is a contradiction. Thus, $d(z,\mathcal{S}z)=0$, and so $z=\mathcal{S}z=\mathcal{T}z$. □

**Corollary 2.8.** *Let $(\mathcal{X},\preceq)$ be a partially ordered set, and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X},d)$ is a complete metric space. Let $\mathcal{S},\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be two weakly increasing mappings such that*

$$\varphi(d(\mathcal{T}x,\mathcal{S}y)) \leq \varphi(d(x,y)) - \phi(d(x,y)), \tag{2.68}$$

*for all comparable $x,y\in\mathcal{X}$, where $\varphi,\phi:[0,\infty)\to[0,\infty)$, $\varphi$ is a continuous, nondecreasing, $\phi$ is lower semicontinuous functions, and $\varphi(t)=0=\phi(t)$ if and only if $t=0$. If*

$$\mathcal{S} \text{ is continuous,} \tag{2.69}$$

*or*

$$\mathcal{T} \text{ is continuous,} \tag{2.70}$$

*or*

$$\{x_n\}\subset\mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \text{ then } x_n \preceq z \ \forall n \tag{2.71}$$

*holds. Then, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.*

**Corollary 2.9.** *Let $(\mathcal{X},\preceq)$ be a partially ordered set, and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X},d)$ is a complete metric space. Let $\mathcal{S},\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be two weakly increasing mappings such that*

$$d(\mathcal{T}x,\mathcal{S}y) \leq \Phi(x,y) - \phi(\Phi(x,y)), \tag{2.72}$$

*for all comparable $x,y\in\mathcal{X}$, where*

$$\Phi(x,y) = ad(x,y) + bd(x,\mathcal{T}x) + cd(y,\mathcal{S}y) + e\left[d(y,\mathcal{T}x) + d(x,\mathcal{S}y)\right], \tag{2.73}$$

*$a>0$, $b,c,e\geq 0$, $a+b+c+2e\leq 1$, $\phi:[0,\infty)\to[0,\infty)$ is a lower semicontinuous functions, and $\phi(t)=0$ if and only if $t=0$. If*

$$\mathcal{S} \text{ is continuous,} \tag{2.74}$$

*or*

$$\mathcal{T} \text{ is continuous,} \tag{2.75}$$

*or*

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \text{ then } x_n \preceq z \ \forall n \tag{2.76}$$

*holds. Then, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.*

## 3. Some Applications

In this section, we present some applications of previous sections, first we obtain some fixed point theorems for single mapping and pair of mappings satisfying a general contractive condition of integral type in complete partially ordered metric spaces. Second, we give an existence theorem for common solution of two integral equations.

Set $\Upsilon = \{\Psi : \mathbb{R}^+ \to \mathbb{R}^+ : \Psi$ is a Lebesgue integrable mapping which is summable and nonnegative and satisfies $\int_0^{\epsilon} \Psi(t)dt > 0$, for each $\epsilon > 0\}$.

**Theorem 3.1.** *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing mapping such that*

$$\int_0^{\varphi(d(\mathcal{T}x,\mathcal{T}y))} \Psi(t)dt \le \int_0^{\varphi(\Theta(x,y))} \Psi(t)dt - \int_0^{\phi(\Theta(x,y))} \Psi(t)dt \quad \text{for } y \preceq x \tag{3.1}$$

*where*

$$\Theta(x,y) = ad(x,y) + bd(x,\mathcal{T}x) + cd(y,\mathcal{T}y) + e\left[d(y,\mathcal{T}x) + d(x,\mathcal{T}y)\right] \tag{3.2}$$

$a > 0,\ b,c,e \ge 0,\ a+b+c+2e \le 1,\ \varphi,\phi : [0,\infty) \to [0,\infty),\ \varphi$ *is continuous, nondecreasing,* $\phi$ *is lower semicontinuous functions, and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Also, suppose that there exists $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$. If*

$$\mathcal{T} \text{ is continuous,} \tag{3.3}$$

*or*

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \text{ then } x_n \preceq z \ \forall n \tag{3.4}$$

*holds. Then, $\mathcal{T}$ has a fixed point.*

*Proof.* Define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x \Psi(t)dt$, then $\Lambda$ is continuous and nondecreasing with $\Lambda(0) = 0$. Thus, (3.1) becomes

$$\Lambda(\varphi(d(\mathcal{T}x,\mathcal{T}y))) \le \Lambda(\varphi(\Theta(x,y))) - \Lambda(\phi(\Theta(x,y))), \tag{3.5}$$

which further can be written as

$$\varphi_1(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi_1(\Theta(x, y)) - \phi_1(\Theta(x, y)), \tag{3.6}$$

where $\varphi_1 = \Lambda \circ \varphi$ and $\phi_1 = \Lambda \circ \phi$. Hence by Theorem 2.1 has unique fixed fixed point. □

**Theorem 3.2.** *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, and suppose that there exists a metric $d$ in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be two weakly increasing mappings such that*

$$\int_0^{\varphi(d(\mathcal{T}x, \mathcal{S}y))} \Psi(t)dt \leq \int_0^{\varphi(\Theta(x,y))} \Psi(t)dt - \int_0^{\phi(\Theta(x,y))} \Psi(t)dt \quad \text{for } y \preceq x, \tag{3.7}$$

*for all comparable $x, y \in \mathcal{X}$, where*

$$\Phi(x, y) = ad(x, y) + bd(x, \mathcal{T}x) + cd(y, \mathcal{S}y) + e\left[d(y, \mathcal{T}x) + d(x, \mathcal{S}y)\right], \tag{3.8}$$

*$a > 0$, $b, c, e \geq 0$, $a + b + c + 2e \leq 1$, $\varphi$, $\phi : [0, \infty) \to [0, \infty)$, $\varphi$ is continuous, nondecreasing, $\phi$ is lower semicontinuous functions, and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. If*

$$\mathcal{S} \text{ is continuous}, \tag{3.9}$$

*or*

$$\mathcal{T} \text{ is continuous}, \tag{3.10}$$

*or*

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \text{ then } x_n \preceq z \ \forall n \tag{3.11}$$

*holds. Then, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.*

*Proof.* Define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x \Psi(t)dt$, then $\Lambda$ is continuous and nondecreasing with $\Lambda(0) = 0$. Thus, (3.7) becomes

$$\Lambda(\varphi(d(\mathcal{T}x, \mathcal{S}y))) \leq \Lambda(\varphi(\Theta(x, y))) - \Lambda(\phi(\Theta(x, y))), \tag{3.12}$$

which further can be written as

$$\varphi_1(d(\mathcal{T}x, \mathcal{S}y)) \leq \varphi_1(\Theta(x, y)) - \varphi_1(\Theta(x, y)), \tag{3.13}$$

where $\phi_1 = \Lambda \circ \phi$ and $\varphi_1 = \Lambda \circ \varphi$. Hence, Theorem 2.6 has unique fixed fixed point. □

Now, consider the integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + g(t), \quad t \in [a,b],$$
$$x(t) = \int_a^b K_2(t,s,x(s))ds + g(t), \quad t \in [a,b]. \tag{3.14}$$

Let $\ll$ be a partial order relation on $\mathbb{R}^n$.

**Theorem 3.3.** *Consider the integral equations* (3.14)*.*

(i) $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and* $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *are continuous,*

(ii) *for each* $t, s \in [a,b]$,

$$K_1(t,s,x(s)) \ll K_2\left(t,s,\int_a^b K_1(s,\tau,x(\tau))d\tau + g(s)\right),$$
$$K_2(t,s,x(s)) \ll K_1\left(t,s,\int_a^b K_2(s,\tau,x(\tau))d\tau + g(s)\right), \tag{3.15}$$

(iii) *there exist a continuous function* $p : [a,b] \times [a,b] \longrightarrow \mathbb{R}_+$ *such that*

$$|K_1(t,s,u) - K_2(t,s,v)| \le p(t,s)\sqrt{\ln\left(|u-v|^2 + 1\right)} \tag{3.16}$$

*for each* $t, s \in [a,b]$ *and comparable* $u, v \in \mathbb{R}^n$,

(iv) $\sup_{t\in[a,b]} \int_a^b p(t,s)^2 ds \le 1/(b-a)$.

*Then, the integral equations* (3.14) *have a unique common solution* $x^*$ *in* $C([a,b],\mathbb{R}^n)$.

*Proof.* Let $\mathcal{X} := C([a,b],\mathbb{R}^n)$ with the usual supremum norm; that is, $\|x\| = \max_{t\in[a,b]}|x(t)|$, for $x \in C([a,b],\mathbb{R}^n)$. Consider on X the partial order defined by

$$x, y \in C([a,b],\mathbb{R}^n), \quad x \preceq y \text{ iff } x(t) \ll y(t) \text{ for any } t \in [a,b]. \tag{3.17}$$

Then, $(\mathcal{X}, \preceq)$ is a partially ordered set. Also, $(\mathcal{X}, \|\cdot\|)$ is a complete metric space. Moreover, for any increasing sequence $\{x_n\}$ in X converging to $x^* \in \mathcal{X}$, we have $x_n(t) \ll x^*(t)$ for any $t \in [a,b]$. Also, for every $x, y \in \mathcal{X}$, there exists $c(x,y) \in \mathcal{X}$ which is comparable to $x$ and $y$ [21].

Define $\mathcal{T}, S : \mathcal{X} \to \mathcal{X}$, by

$$\begin{aligned}\mathcal{T}x(t) &= \int_a^b K_1(t,s,x(s))ds + g(t), \quad t \in [a,b],\\ \mathcal{S}x(t) &= \int_a^b K_2(t,s,x(s))ds + g(t), \quad t \in [a,b].\end{aligned} \tag{3.18}$$

Now, from (ii), we have for all $t \in [a,b]$,

$$\begin{aligned}\mathcal{T}x(t) &= \int_a^b K_1(t,s,x(s))ds + g(t)\\ &\ll \int_a^b K_2\left(t,s,\int_a^b K_1(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t)\\ &= \int_a^b K_2(t,s,Tx(s))ds + g(t)\\ &= \mathcal{S}\mathcal{T}x(t),\\ \mathcal{S}x(t) &= \int_a^b K_2(t,s,x(s))ds + g(t)\\ &\ll \int_a^b K_1\left(t,s,\int_a^b K_2(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t)\\ &= \int_a^b K_1(t,s,Sx(s))ds + g(t)\\ &= \mathcal{T}\mathcal{S}x(t).\end{aligned} \tag{3.19}$$

Thus, we have $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ and $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ for all $x \in \mathcal{X}$. This shows that $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing. Also, for each comparable $x, y \in \mathcal{X}$, we have

$$\begin{aligned}\left|\mathcal{T}x(t) - \mathcal{S}y(t)\right| &= \left|\int_a^b K_1(t,s,x(s))ds - \int_a^b K_2(t,s,y(s))ds\right|\\ &\le \int_a^b \left|K_1(t,s,x(s)) - K_2(t,s,y(s))\right|ds\\ &\le \int_a^b p(t,s)\sqrt{\ln\left(\left|x(s)-y(s)\right|^2+1\right)}ds\\ &\le \left(\int_a^b p(t,s)^2 ds\right)^{1/2}\left(\int_a^b \ln\left(\left|x(s)-y(s)\right|^2+1\right)ds\right)^{1/2}.\end{aligned} \tag{3.20}$$

Hence,

$$\begin{aligned}\|\mathcal{T}x-\mathcal{S}y\|^2 &\le \sup_{t\in[a,b]}\int_a^b p(t,s)^2 ds\left(\int_a^b \ln\left(|x(s)-y(s)|^2+1\right)ds\right)\\ &\le \ln\left(\|x-y\|^2+1\right)\\ &= \|x-y\|^2-\left(\|x-y\|^2-\ln\left(\|x-y\|^2+1\right)\right).\end{aligned} \tag{3.21}$$

Put $\varphi(x)=x^2,\ \phi(x)=x^2-\ln(x^2+1)$. Therefore,

$$\varphi(\|\mathcal{T}x-\mathcal{S}y\|)\le\varphi(\|x-y\|)-\phi(\|x-y\|), \tag{3.22}$$

for each comparable $x,y\in\mathcal{X}$. Therefore, all conditions of Corollary 2.8 are satisfied. Thus, the conclusion follows. □

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