Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 175989, 6 pages
doi:10.1155/2011/175989

Research Article

Relation between Fixed Point and Asymptotical Center of Nonexpansive Maps

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Received 19 October 2010; Accepted 22 November 2010

Academic Editor: Qamrul Hasan Ansari

We introduce the concept of asymptotic center of maps and consider relation between asymptotic center and fixed point of nonexpansive maps in a Banach space.

1. Introduction

Many topics and techniques regarding asymptotic centers and asymptotic radius were studied by Edelstein [1], Bose and Laskar [2], Downing and Kirk [3], Goebel and Kirk [4], and Lan and Webb [5]. Now, We recall that definitions of asymptotic center and asymptotic radius.

Let C be a nonempty subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Consider the functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n \to \infty} \|x_n - x\|, \quad x \in X. \tag{1.1}$$

The infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the asymptotic radius of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$. A point $z \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}. \tag{1.2}$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $Z_a(C, \{x_n\})$.

We present new definitions of asymptotic center and asymptotic radius that is for a mapping and obtain new results.

Definition 1.1. Let C be a bounded closed convex subset of X. A sequence $\{x_n\} \subseteq X$ is said to be an asymptotic center for a mapping $T : C \rightarrow X$ if, for each $x \in C$,

$$\limsup_{n\to\infty}\|Tx - x_n\| \leq \limsup_{n\to\infty}\|x_n - x\|. \tag{1.3}$$

Definition 1.2. Let C be a nonempty subset of X. We say that C has the fixed-point property for continuous mappings of C with asymptotic center if every continuous mapping $T : C \rightarrow C$ admitting an asymptotic center has a fixed point.

Definition 1.3. Let C be a nonempty subset of X. We say that C has Property (Z) if for every bounded sequence $\{x_n\} \subset X \setminus C$, the set $Z_a(C, \{x_n\})$ is a nonempty and compact subset of C.

Example 1.4. Let X be a normed space and C a nonempty subset of X. It is clear that

(i) if C is a compact set, then $Z_a(C, \{x_n\})$ in nonempty compact set and so has Property (Z);

(ii) if C is a open set, since $Z_a(C, \{x_n\}) \subset \partial C$, therefore $Z_a(C, \{x_n\})$ is empty and so fail to have Property (Z).

2. Main Results

Our new results are presented in this section.

Proposition 2.1. *Let X be a Banach space and let C be a nonempty closed bounded and convex subset of X. If C satisfies Property (Z), then every continuous mapping $T : C \rightarrow C$ asymptotically admitting a center in C has a fixed point.*

Proof. Assume that $T : C \rightarrow C$ is a continuous mapping and $\{x_n\}$ is a asymptotic center. Let $\{x_n\} \subset X \setminus C$ has set of asymptotic center $Z_a(C, \{x_n\})$. Since C has Property (Z), $Z_a(C, \{x_n\})$ is nonempty and compact and it is easy to see that it is also convex. In order to obtain the result, it will be enough to show that $Z_a(C, \{x_n\})$ is T-invariant since in this case we may apply Schauder's Fixed-Point Theorem [4, Theorem 18.10]. Indeed, let $y \in Z_a(C, \{x_n\})$. Since $\{x_n\}$ is a asymptotic center for T, we have

$$r_a(C, \{x_n\}) \leq \limsup_{n\to\infty}\|Ty - x_n\| \leq \limsup_{n\to\infty}\|x_n - y\| = r_a(C, \{x_n\}). \tag{2.1}$$

Therefore $Ty \in Z_a(C, \{x_n\})$. □

Theorem 2.2. *Let X be a Banach space and let C be a nonempty closed bounded and convex subset of X. If C has the fixed-point property for continuous mappings admitting an asymptotic center, then C has Property (Z).*

Proof. Suppose that C fails to have Property (Z). There exists $\{x_n\} \subset X$ such that either $Z_a(C, \{x_n\}) = \emptyset$ or $Z_a(C, \{x_n\})$ is noncompact. In the second case, by Klee's theorem in

[6] there exists a continuous function $S : Z_a(C, \{x_n\}) \to Z_a(C, \{x_n\})$ without fixed points ($Sx = x$). Since a closed convex subset of a normed space is a retract of the space, there exists a continuous mapping $r : C \to Z_a(C, \{x_n\})$ such that $r(x) = x$ for all $x \in Z_a(C, \{x_n\})$. Define $T : C \to Z_a(C, \{x_n\})$ by $T(x) = S(r(x))$. Clearly T is a continuous mapping. Moreover,

$$\begin{aligned}\limsup_{n\to\infty}\|T(x) - x_n\| &= \limsup_{n\to\infty}\|x_n - S(r(x))\| \\ &= \limsup_{n\to\infty}\|x_n - r(x)\| \\ &\le \limsup_{n\to\infty}\|x_n - x\|,\end{aligned} \tag{2.2}$$

that is, $\{x_n\}$ is an asymptotic center for T. Therefore, by Proposition 2.1, T has a fixed point in C, $T(x) = x \in Z_a(C, \{x_n\})$. Hence $x = S(r(x)) = S(x)$ sets a contradiction.

Concerning the first case we proceed as follows.

Let $d := r_a(C, \{x_n\}) > \circ$. We take $a > 0$ such that $a + d < \sup\{\|x - x_n\| : x \in C\}$. For each positive integer n, we consider the following nonempty sets:

$$B_m := B\left[\{x_n\}, d + \frac{a}{m}\right] \cap C, \tag{2.3}$$

where $B[\{x_n\}, r] := \{x \in X : \limsup_{n\to\infty}\|x_n - x\| < r\}$

$$\begin{aligned}A_m &:= B_m \setminus B_{m+1}, \\ S_m &:= \left\{x \in C : \limsup_{n\to\infty}\|x - x_n\| = d + \frac{a}{m}\right\}.\end{aligned} \tag{2.4}$$

Since $Z_a(C, \{x_n\}) = \emptyset$, we have that

$$B_1 = \bigcup_{m=1}^{\infty} A_m. \tag{2.5}$$

Fix an arbitrary $x_1 \in S_1$ and define, by induction, a sequence $\{y_m\}$ such that $\{y_m\} \in S_m$ and the segment $(y_{m+1}, y_m]$ does not meet B_{m+1}. Given $x \in B_1$, there exists a unique positive integer n such that $x \in A_n$. In this case we define

$$\begin{aligned}S(x) = {} & \frac{\limsup_{n\to\infty}\|x - x_n\| - (d + a/(m+1))}{a/m(m+1)} y_{m+1} \\ & + \left(1 - \frac{\limsup_{n\to\infty}\|x - x_n\| - (d + a/(m+1))}{a/m(m+1)}\right) y_{m+2}.\end{aligned} \tag{2.6}$$

It is a routine to check that S is a continuous mapping from B_1 to B_1. Furthermore, $S(A_m) \subset (y_{m+2}, y_{m+1}] \subset A_{m+1}$ for every $m \ge 1$.

Let r be a continuous retraction from C into the closed convex subset B_1. We can define $T : C \to C$ by $T(x) = S(r(x))$. It is clear that $\{x_n\}$ is a asymptotic center for T and that T is fixed-point free. □

Proposition 2.1 (Theorem 2.2) is a generalizations of Theorem 3.1 (Theorem 3.3) in [1]. It can be verified that definition of $L(\tau)$ space is not necessary here.

As an easy consequence of both Proposition 2.1 and Theorem 2.2, we deduce the following result.

Corollary 2.3. *Let C be a nonempty closed bounded and convex subset of a Banach space X. The following conditions are equivalent.*

(1) *C has the fixed-point property for continuous mappings admitting asymptotic center in C.*

(2) *C has Property (Z).*

Let C be a nonempty closed convex bounded subset of a Banach space X. By $KC(C)$ we denote the family of all nonempty compact convex subsets of C. On $KC(C)$ we consider the well-known Hausdorff metric H. Recall that a mapping $T : C \to KC(C)$ is said to be nonexpansive whenever

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in C. \tag{2.7}$$

Theorem 2.4. *Let X be a Banach space and let C be a nonempty closed convex and bounded subset of X satisfying Property (Z). If $T : C \to KC(C)$ is a nonexpansive mapping, then T has a fixed point.*

Proof. Let $T : C \to KC(C)$ be a nonexpansive mapping. The multivalued analog of Banach's Contraction Principle allows us to find a sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$.

For each $n \geq 1$, the compactness of Tx_n guarantees that there exists $y_n \in Tx_n$ satisfying $\|x_n - y_n\| = d(x_n, Tx_n)$.

Now we are going to show that for every $z \in Z_a(C, \{x_n\})$,

$$Z_a(C, \{x_n\}) \cap Tz \neq \emptyset. \tag{2.8}$$

Taking any $z \in Z_a(C, \{x_n\})$, from the compactness of Tz we can find $z_n \in Tz$ such that

$$\left\|y_n - z_n\right\| = d(y_n, Tz) \leq H(Tx_n, Tz) \leq \|x_n - z\|. \tag{2.9}$$

By compactness again we can assume that $\{z_n\}$ converges to a point $w_0 \in Tz$. From above it follows that

$$\limsup_{n\to\infty}\|x_n - w_0\| \leq \limsup_{n\to\infty}\left\|y_n - w_0\right\| \leq \limsup_{n\to\infty}\left\|y_n - z_n\right\| \leq \limsup_{n\to\infty}\|x_n - z\|. \tag{2.10}$$

Therefore $w_0 \in Z_a(C, \{x_n\})$.

Now we define the mapping $S : Z_a(C, \{x_n\}) \to KC(Z_a(C, \{x_n\}))$ by $S(z) = Z_a(C, \{x_n\}) \cap T(z)$. Since the mapping S is upper semicontinuous and $S(z)$ for every $z \in Z_a(C, \{x_n\})$ is a compact convex set we can apply the Kakutani-Bohnenblust-Karlin Theorem in [5] to obtain a fixed point for $S(z)$ and hence for T. □

Let X be a metric space and $T : X \to X$ a mapping. Then a sequence $\{x_n\}$ in X is said to be an approximating fixed-point sequence of T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let C be a bounded closed and convex subset of a Banach space X, $T : C \to C$ a nonexpansive mapping and $\alpha \in (0,1)$. Then a mappings $T_\alpha : C \to C$ define by $T_\alpha(x) = \alpha x + (1-\alpha)Tx$ is always asymptotically regular, that is, for every $x \in C$, $\lim_{n\to\infty}\|T_\alpha^{n+1}x - T_\alpha^n x\| = 0$.

Proposition 2.5. *Let X be a Banach space and C a closed bounded convex subset of X, $x_0 \in C$ and $\alpha \in (0,1)$. If $T : C \to C$ is a nonexpansive mapping, then the sequence $\{T_\alpha^n x_0\}$ is an asymptotic center for T.*

Proof. The above comments guarantee that $\{T_\alpha^n x_0\}$ is an approximated fixed-point sequence for T_α^n. Let us see that the sequence $\{T_\alpha^n x_0\}$ an asymptotic center for T. Given $x \in C$ we have

$$\begin{aligned}\limsup_{n\to\infty}\|Tx - T_\alpha^n x_0\| &\le \limsup_{n\to\infty}\|Tx - T(T_\alpha^n x_0)\| + \limsup_{n\to\infty}\|T(T_\alpha^n x_0) - T_\alpha^n x_0\| \\ &= \limsup_{n\to\infty}\|Tx - T(T_\alpha^n x_0)\| \\ &\le \limsup_{n\to\infty}\|x - T_\alpha^n x_0\|.\end{aligned} \tag{2.11}$$

Therefore $\{T_\alpha^n x_0\}$ is asymptotic center for T. □

Theorem 2.6. *Let X be a normed space, $T : X \to X$ a nonexpansive mapping with an approximating fixed point sequence $\{x_n\} \subseteq X$ and C be a nonempty subset of X such that $Z_a(C,\{x_n\})$ is a nonempty star-shaped subset of X. Then T has an approximating fixed-point sequence in $Z_a(C,\{x_n\})$.*

Proof. Suppose $y \in Z_a(C,\{x_n\})$. Therefore

$$\begin{aligned}\limsup_{n\to\infty}\|Ty - x_n\| &\le \limsup_{n\to\infty}\|Ty - Tx_n\| + \limsup_{n\to\infty}\|Tx_n - x_n\| \\ &= \limsup_{n\to\infty}\|Ty - Tx_n\| \\ &\le \limsup_{n\to\infty}\|y - x_n\| = r_a(C,\{x_n\}),\end{aligned} \tag{2.12}$$

and so $Ty \in Z_a(C,\{x_n\})$.

Now, let p be the star center of $Z_a(C,\{x_n\})$. For every $n \in \mathbb{N}$ define $T_n : Z_a(C,\{x_n\}) \to Z_a(C,\{x_n\})$ by

$$T_n(x) = \left(1 - \frac{1}{n}\right)Tx + \frac{1}{n}p. \tag{2.13}$$

For every $n \in \mathbb{N}$, T_n is a contraction, so there exists exactly one fixed point y_n of T_n. Now

$$\|y_n - Ty_n\| = \left(1 - \frac{1}{n}\right)\|Ty_n - p\| = \left(1 - \frac{1}{n}\right)k \longrightarrow 0. \tag{2.14}$$

Therefore $\{y_n\}$ is the approximating fixed-point sequence in $Z_a(C,\{x_n\})$ of T. □

Corollary 2.7. *Let X be a normed space, $T : X \to X$ a nonexpansive mapping with an approximating fixed-point sequence $\{x_n\} \subseteq X$ and C be a nonempty subset of X such that $Z_a(C, \{x_n\}) \neq \emptyset$. Suppose $Z_a(C, \{x_n\})$ is a nonempty weakly compact star-shaped subset of K. If $I - T$ is demiclosed, then T has a fixed point in $Z_a(C, \{x_n\})$.*

Proof. By the last theorem, T has an approximating fixed-point sequence $\{y_n\} \in Z_a(C, \{x_n\})$. Because $Z_a(C, \{x_n\})$ is weakly compact, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \to z \in Z_a(C, \{x_n\})$. Since $I - T$ is demiclosed on $Z_a(C, \{x_n\})$ and $y_{n_i} - Ty_{n_i} \to 0$, it follows that $z \in F(T)$. Therefore, $Z_a(C, \{x_n\}) \cap F(T) \neq \emptyset$. □

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