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Research Article

Iterative Algorithms for Finding Common Solutions to Variational Inclusion Equilibrium and Fixed Point Problems

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The main purpose of this paper is to introduce an explicit iterative algorithm to study the existence problem and the approximation problem of solution to the quadratic minimization problem. Under suitable conditions, some strong convergence theorems for a family of nonexpansive mappings are proved. The results presented in the paper improve and extend the corresponding results announced by some authors.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C is a nonempty closed convex subset of H, and $F(T)=\{x\in H : Tx=x\}$ is the set of fixed points of mapping T.

A mapping $S : C\to C$ is called nonexpansive if

$$\|Sx-Sy\|\le\|x-y\|,\quad\forall x,y\in C. \tag{1.1}$$

Let $A : H\to H$ be a single-valued nonlenear mapping and $M : H\to 2^H$ be a multivalued mapping. The so-called quasivariational inclusion problem (see [1–3]) is to find $u\in H$ such that

$$\theta\in A(u)+M(u). \tag{1.2}$$

The set of solutions to quasivariational inclusion problem (1.2) is denoted by $\mathrm{VI}(H,A,M)$.

Special Cases

(I) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R}\cup\{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the subdifferential of ϕ, then the quasivariational inclusion problem (1.2) is equivalent to finding $u \in H$ such that

$$\langle A(u), y - u\rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H, \tag{1.3}$$

which is called the mixed quasivariational inequality (see [4]).

(II) If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, +\infty)$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.4}$$

then the quasivariational inclusion problem (1.2) is equivalent to finding $u \in C$ such that

$$\langle A(u), v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.5}$$

This problem is called the Hartman-Stampacchia variational inequality (see [5]). The set of solutions to variational inequality (1.5) is denoted by $\mathrm{VI}(A, C)$.

Let $B : C \to H$ be a nonlinear mapping and $F : C \times C \to \mathbb{R}$ be a bifunction. The so-called generalized equilibrium problem is to find a point $u \in C$ such that

$$F(u, y) + \langle B(u), y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions to (1.6) is denoted by GEP (see [5, 6]). If $B = 0$, then (1.6) reduces to the following equilibrium problem: to find $u \in C$ such that

$$F(u, y) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions to (1.7) is denoted by EP.

Iterative methods for nonexpansive mappings and equilibrium problems have been applied to solve convex minimization problems (see [7–9]). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in\mathcal{F}} \frac{1}{2}\|x\|^2, \tag{1.8}$$

where $\mathcal{F}$ is the fixed point set of a nonexpansive mapping T on H.

In 2010, Zhang et al. (see [10]) proposed the following iteration method for variational inclusion problem (1.5) and equilibrium problem (1.6) in a Hilbert space H:

$$x_t = SP_C\big((1-t)J_{M,\lambda}(I - \lambda A)T_\mu\big(I - \mu B\big)\big)x_t, \quad t \in (0, 1). \tag{1.9}$$

Under suitable conditions, they proved the sequence $\{x_n\}$ generated by (1.9) converges strongly to the fixed point x^*, which solves the quadratic minimization problem (1.8).

Motivated and inspired by the researches going on in this direction, especially inspired by Zhang et al. [10], the purpose of this paper is to introduce an explicit iterative algorithm to studying the existence problem and the approximation problem of the solution to the quadratic minimization problem (1.8) and prove some strong convergence theorems for a family of nonexpansive mappings in the setting of Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space, and C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

Such a mapping P_C from H onto C is called the metric projection. It is well-known that the metric projection $P_C : H \to C$ is nonexpansive.

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and the strong convergence of the sequence $\{x_n\}$, respectively.

Definition 2.1. A mapping $A : H \to H$ is called α-inverse strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in H. \tag{2.2}$$

A multivalued mapping $M : H \to 2^H$ is called monotone if $\forall x, y \in H, u \in Mx, v \in My$,

$$\langle u - v, x - y\rangle \geq 0. \tag{2.3}$$

A multivalued mapping $M : H \to 2^H$ is called maximal monotone if it is monotone and for any $x, u \in H \times H$, when

$$\langle u - v, x - y\rangle \geq 0 \quad \text{for every } (y, v) \in \operatorname{Graph}(M), \tag{2.4}$$

then $u \in Mx$.

Proposition 2.2 (see [11]). *Let $A : H \to H$ be an α-inverse strongly monotone mapping. Then, the following statements hold:*

(i) *A is an $1/\alpha$-Lipschitz continuous and monotone mapping;*

(ii) *if λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda A$ is nonexpansive, where I is the identity mapping on H.*

Lemma 2.3 (see [12]). *Let X be a strictly convex Banach space, C be a closed convex subset of X, and $\{T_n : C \to C\}$ be a sequence of nonexpansive mappings. Suppose $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\Sigma_{n=1}^{\infty}\lambda_n = 1$. Then the mapping $S : C \to C$ defined by*

$$Sx = \Sigma_{n=1}^{\infty}\lambda_n T_n x, \quad x \in C \tag{2.5}$$

is well defined. And it is nonexpansive and

$$F(S) = \bigcap_{n=1}^{\infty} F(T_n). \tag{2.6}$$

Definition 2.4. Let H be a Hilbert space and $M : H \to 2^H$ be a multivalued maximal monotone mapping. Then, the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H \tag{2.7}$$

is called the *resolvent operator associated with* M, where λ is any positive number and I is the identity mapping.

Proposition 2.5 (see [11]). (i) *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$, that is,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H, \ \forall \lambda > 0. \tag{2.8}$$

(ii) *The resolvent operator $J_{M,\lambda}$ is 1-inverse strongly monotone, that is,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \quad \forall x, y \in H. \tag{2.9}$$

Definition 2.6. A single-valued mapping $A : H \to H$ is said to be hemicontinuous if for any $x, y, z \in H$, function $t \mapsto \langle A(x + ty), z\rangle$ is continuous at 0.

It is well-known that every continuous mapping must be hemicontinuous.

Lemma 2.7 (see [13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X. Let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{2.10}$$

Suppose that

$$\begin{gathered} x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \quad \forall n \geq 0, \\ \limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \end{gathered} \tag{2.11}$$

Then,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.12}$$

Lemma 2.8 (see [14]). *Let X be a real Banach space, X^* be the dual space of X, $T : X \to 2^{X^*}$ be a maximal monotone mapping, and $P : X \to X^*$ be a hemicontinuous bound monotone mapping with $D(P) = X$. Then, the mapping $S = T + P : X \to 2^{X^*}$ is a maximal monotone mapping.*

Lemma 2.9 (see [15]). *Let X be a uniformly convex Banach space, let C be a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping with a fixed point. Then, $I - T$ is demiclosed in the sense that if $\{x_n\}$ is a sequence in C satisfying*

$$x_n \rightharpoonup x, \quad (I - T)_n \longrightarrow 0, \tag{2.13}$$

then

$$(I - T)x = 0. \tag{2.14}$$

Throughout this paper, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(H_1) $F(x, x) = 0$ for all $x \in C$;

(H_2) F is monotone, that is,

$$F(x, y) + F(y, x) \le 0, \quad \forall x, y \in C, \tag{2.15}$$

(H_3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \le F(x, y), \tag{2.16}$$

(H_4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.10 (see [16]). *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (H_1)–(H_4). Let $\mu > 0$ and $x \in H$. Then, there exists a point $z \in C$ such that*

$$F(z, y) + \frac{1}{\mu}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C. \tag{2.17}$$

Moreover, if $T_\mu : H \to C$ is a mapping defined by

$$T_\mu(x) = \left\{ z \in C : F(z, y) + \frac{1}{\mu}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\}, \qquad x \in H, \tag{2.18}$$

then the following results hold:

(i) T_μ *is single-valued and firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_\mu x - T_\mu y\|^2 \leq \langle T_\mu x - T_\mu y, x - y\rangle, \tag{2.19}$$

(ii) EP *is closed and convex, and* $\mathrm{EP} = F(T_\mu)$.

Lemma 2.11. (i) (see [11]) $u \in H$ *is a solution of variational inclusion* (1.2) *if and only if*

$$u = J_{M,\lambda}(u - \lambda Au), \quad \forall\lambda > 0, \tag{2.20}$$

that is,

$$\mathrm{VI}(H, A, M) = F(J_{M,\lambda}(u - \lambda Au)), \quad \forall\lambda > 0. \tag{2.21}$$

(ii) (see [10]) $u \in C$ *is a solution of generalized equilibrium problem* (1.6) *if and only if*

$$u = T_\mu(u - \mu Bu), \quad \forall\mu > 0, \tag{2.22}$$

that is,

$$\mathrm{GEP} = F(T_\mu(u - \mu Bu)), \quad \forall\mu > 0. \tag{2.23}$$

(iii) (see [10]) *Let* $A : H \to H$ *be an* α*-inverse strongly monotone mapping and* $B : C \to H$ *be a* β*-inverse strongly monotone mapping. If* $\lambda \in (0, 2\alpha]$ *and* $\mu \in (0, 2\beta]$, *then* $\mathrm{VI}(H, A, M)$ *is a closed convex subset in* H *and GEP is a closed convex subset in* C.

Lemma 2.12 (see [17]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq 1 - \gamma_n a_n + \delta_n, \quad \forall n \geq 1, \tag{2.24}$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence such that:*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H, $A : H \to H$ *be an* α*-inverse strongly monotone mapping and* $B : C \to H$ *be a* β*-inverse strongly monotone* *mapping. Let* $M : H \to 2^H$ *be a maximal monotone mapping,* $\{T_n : C \to C\}$ *be a sequence of nonexpansive mappings with* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$*,* $S : C \to C$ *be the nonexpansive mapping defined by* (2.5)*, and* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions* (H_1)–(H_4)*. Let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(SP_C\big((1 - t_n)J_{M,\lambda}(I - \lambda A)T_\mu\big(I - \mu B\big)\big)x_n\big), \tag{3.1}$$

where the mapping $T_\mu : H \to C$ *is defined by* (2.18)*, and* λ, μ *are two constants with* $\lambda \in (0, 2\alpha]$*,* $\mu \in (0, 2\beta]$*, and*

$$t_n \in (0,1), \quad t_n \longrightarrow 0(n \longrightarrow \infty), \quad \sum_{n=1}^{\infty} t_n = \infty, \quad 0 < a < \alpha_n < b < 1. \tag{3.2}$$

If

$$\Omega := F(S) \cap \mathrm{VI}(H, A, M) \cap \mathrm{GEP} \neq \emptyset, \tag{3.3}$$

where VI (H, A, M) *and GEP is the set of solutions of variational inclusion* (1.2) *and generalized equilibrium problem* (1.6)*, respectively, then the sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to* $x^* \in \Omega$*, which is the unique solution of the following quadratic minimization problem:*

$$\|x^*\|^2 = \min_{x \in \Omega} \|x\|^2. \tag{3.4}$$

Proof. We divide the proof of Theorem 3.1 into four steps.

Step 1 (The sequence $\{x_n\}$ is bounded). Set

$$u_n = T_\mu\big(I - \mu B\big)x_n, \qquad y_n = J_{M,\lambda}(I - \lambda A)u_n, \qquad z_n = SP_C\big((1 - t_n)y_n\big). \tag{3.5}$$

Taking $z \in \Omega$, then it follows from Lemma 2.11 that

$$z = T_\mu\big(z - \mu Bz\big) = J_{M,\lambda}(z - \lambda Az) = SP_C z. \tag{3.6}$$

Since both T_μ and $J_{M,\lambda}$ are nonexpansive, A and B are α-inverse strongly monotone and β-inverse strongly monotone, respectively, from Proposition 2.2, we have

$$\begin{aligned}\|u_n - z\|^2 &= \left\|T_\mu(I-\mu B)x_n - T_\mu(z-\mu Bz)\right\|^2 \\ &\le \left\|(I-\mu B)x_n - (z-\mu Bz)\right\|^2 \\ &\le \|x_n - z\|^2 + \mu(\mu-2\beta)\|Bx_n - Bz)\|^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\left\|y_n - z\right\|^2 &= \|J_{M,\lambda}(I-\lambda A)u_n - J_{M,\lambda}(z-\lambda Az)\|^2 \\ &\le \|(I-\lambda A)u_n - (z-\lambda Az)\|^2 \\ &\le \|u_n - z\|^2 + \lambda(\lambda-2\alpha)\|Au_n - Az)\|^2 \\ &\le \|x_n - z\|^2 + \lambda(\lambda-2\alpha)\|Au_n - Az)\|^2 + \mu(\mu-2\beta)\|Bx_n - Bz)\|^2.\end{aligned} \tag{3.8}$$

This implies that

$$\left\|y_n - z\right\| \le \|u_n - z\| \le \|x_n - z\|. \tag{3.9}$$

It follows from (3.1) and (3.9) that

$$\begin{aligned}\|x_{n+1} - z\| &= \left\|\alpha_n x_n + (1-\alpha_n)SP_C\left((1-t_n)y_n\right) - z\right\| \\ &= \left\|\alpha_n(x_n - z) + (1-\alpha_n)\left(SP_C\left((1-t_n)y_n\right) - SP_C z\right)\right\| \\ &\le \alpha_n\|x_n - z\| + (1-\alpha_n)\left\|SP_C\left((1-t_n)y_n\right) - SP_C z\right\| \\ &\le \alpha_n\|x_n - z\| + (1-\alpha_n)\left\|(1-t_n)y_n - z\right\| \\ &\le \alpha_n\|x_n - z\| + (1-\alpha_n)\left((1-t_n)\left\|y_n - z\right\| + t_n\|z\|\right) \\ &\le \alpha_n\|x_n - z\| + (1-\alpha_n)((1-t_n)\|x_n - z\| + t_n\|z\|) \\ &\le (1-t_n(1-\alpha_n))\|x_n - z\| + t_n(1-\alpha_n)\|z\| \\ &\le \max\{\|x_n - z\|, \|z\|\} \\ &\le \max\{\|x_{n-1} - z\|, \|z\|\} \\ &\le \cdots \le \max\{\|x_1 - z\|, \|z\|\} = M,\end{aligned} \tag{3.10}$$

where $M = \max\{\|x_1 - z\|, \|z\|\}$. This shows that $\{x_n\}$ is bounded. Hence, it follows from (3.9) that the sequence $\{u_n\}$ and $\{y_n\}$ are also bounded.

It follows from (3.5), (3.6), and (3.9) that

$$\begin{aligned}\|z_n - z\| &= \left\|SP_C(1-t_n)y_n - SP_C z\right\| \le \left\|(1-t_n)y_n - z\right\| \\ &\le (1-t_n)\left\|y_n - z\right\| + t_n\|z\| \le (1-t_n)\|x_n - z\| + t_n\|z\| \le M.\end{aligned} \tag{3.11}$$

This shows that $\{z_n\}$ is bounded.

Step 2. Now, we prove that

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|u_n - y_n\| = \lim_{n\to\infty}\|x_n - y_n\| = 0,$$
$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.12}$$

Since SP_C is nonexpansive, from (3.5) and (3.9), we have that

$$\|y_{n+1} - y_n\| \le \|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\|, \tag{3.13}$$

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|SP_C((1-t_{n+1})y_{n+1}) - SP_C((1-t_n)y_n)\| \\ &\le \|(1-t_{n+1})y_{n+1} - (1-t_n)y_n\| \\ &= \|(1-t_{n+1})(y_{n+1} - y_n) + (1-t_{n+1} - (1-t_n))y_n\| \\ &\le (1-t_{n+1})\|y_{n+1} - y_n\| + |t_{n+1} - t_n|\|y_n\| \\ &\le \|y_{n+1} - y_n\| + |t_{n+1} - t_n|\|y_n\| \\ &\le \|u_{n+1} - u_n\| + |t_{n+1} - t_n|\|y_n\| \\ &\le \|x_{n+1} - x_n\| + |t_{n+1} - t_n|\|y_n\|.\end{aligned} \tag{3.14}$$

Let $n \to \infty$ in (3.14), in view of condition $t_n \to 0(n \to \infty)$, we have

$$\lim_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) = 0. \tag{3.15}$$

By virtue of Lemma 2.7, we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.16}$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\alpha_n)\|z_n - x_n\| = 0. \tag{3.17}$$

We derive from (3.17) that

$$\begin{aligned}\lim_{n\to\infty}\left(\|x_n - z\|^2 - \|x_{n+1} - z\|^2\right) &= \lim_{n\to\infty}\left(\|x_n - x_{n+1}\|^2 + 2\langle x_n - x_{n+1}, x_{n+1} - z\rangle\right)\\ &\le \lim_{n\to\infty}\left(\|x_n - x_{n+1}\|^2 + 2\|x_n - x_{n+1}\|\cdot\|x_{n+1} - z\|\right) = 0.\end{aligned} \tag{3.18}$$

From (3.1) and (3.8), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le (\alpha_n\|x_n - z\| + (1-\alpha_n)\|(1-t_n)y_n - z\|)^2\\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|(1-t_n)(y_n - z) - t_n z\|^2\\ &= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\left((1-t_n)^2\|y_n - z\|^2 - 2t_n(1-t_n)\langle z, y_n - z\rangle + t_n^2\|z\|^2\right)\\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\left(\|y_n - z\|^2 + t_n M_1\right)\\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\\ &\quad\times\left(\|x_n - z\|^2 + \lambda(\lambda - 2\alpha)\|Au_n - Az\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bz\|^2 + t_n M_1\right)\\ &= \|x_n - z\|^2 + (1-\alpha_n)\left(\lambda(\lambda - 2\alpha)\|Au_n - Az\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bz\|^2 + t_n M_1\right),\end{aligned} \tag{3.19}$$

where

$$M_1 = \sup_n\left\{\|z\|^2 + 2(\lambda\|u_n - y_n\| + \mu\|x_n - y_n\|)\right\} < \infty, \tag{3.20}$$

that is,

$$\begin{aligned}&(1-\alpha_n)\left(\lambda(2\alpha - \lambda)\|Au_n - Az\|^2 + \mu(2\beta - \mu)\|Bx_n - Bz\|^2\right)\\ &\quad\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\alpha_n)t_n M_1.\end{aligned} \tag{3.21}$$

Let $n \to \infty$, noting the assumptions that $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, from (3.2) and (3.18), we have

$$\lim_{n\to\infty}\|Au_n - Az\| = \lim_{n\to\infty}\|Bx_n - Bz\| = 0. \tag{3.22}$$

By virtue of Lemma 2.10(i) and (3.1), we have

$$\begin{aligned}
\|u_n - z\|^2 &= \left\|T_\mu(x_n - \mu Bx_n) - T_\mu(z - \mu Bz)\right\|^2 \\
&\le \langle (x_n - \mu Bx_n) - (z - \mu Bz), u_n - z\rangle \\
&= \frac{1}{2}\Big(\left\|(x_n - \mu Bx_n) - (z - \mu Bz)\right\|^2 + \|u_n - z\|^2 \\
&\quad - \left\|(x_n - z) - \mu(Bx_n - Bz) - (u_n - z)\right\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \left\|(x_n - u_n) - \mu(Bx_n - Bz)\right\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2\mu\langle x_n - u_n, Bx_n - Bz\rangle - \mu^2\|Bx_n - Bz\|^2\Big).
\end{aligned} \tag{3.23}$$

Simplifying it, we have

$$\begin{aligned}
\|u_n - z\|^2 &\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu\langle x_n - u_n, Bx_n - Bz\rangle - \mu^2\|Bx_n - Bz\|^2 \\
&\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu\|x_n - u_n\| \cdot \|Bx_n - Bz\| \\
&\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + M_1\|Bx_n - Bz\|.
\end{aligned} \tag{3.24}$$

Similarly, in view of Proposition 2.5(ii) and (3.1), we have

$$\begin{aligned}
\left\|y_n - z\right\|^2 &= \|J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(z - \lambda Az)\|^2 \\
&\le \langle (u_n - \lambda Au_n) - (z - \lambda Az), y_n - z\rangle \\
&= \frac{1}{2}\Big(\|(u_n - \lambda Au_n) - (z - \lambda Az)\|^2 + \left\|y_n - z\right\|^2 \\
&\quad - \left\|(u_n - \lambda Au_n) - (z - \lambda Az) - (y_n - z)\right\|^2\Big) \\
&\le \frac{1}{2}\Big(\|u_n - z\|^2 + \left\|y_n - z\right\|^2 - \left\|(u_n - y_n) - \lambda(Au_n - Az)\right\|^2\Big) \\
&= \frac{1}{2}\Big(\|u_n - z\|^2 + \left\|y_n - z\right\|^2 - \left\|u_n - y_n\right\|^2 \\
&\quad + 2\lambda\langle u_n - y_n, Au_n - Az\rangle - \lambda^2\|Au_n - Az\|^2\Big).
\end{aligned} \tag{3.25}$$

Simplifying it, from (3.24), we have

$$\begin{aligned}\|y_n - z\|^2 &\le \|u_n - z\|^2 - \|u_n - y_n\|^2 + 2\lambda\langle u_n - y_n, Au_n - Az\rangle - \lambda^2\|Au_n - Az\|^2 \\ &\le \|u_n - z\|^2 - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\| \cdot \|Au_n - Az\| \\ &\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + M_1\|Bx_n - Bz\| - \|u_n - y_n\|^2 \\ &\quad + M_1\|Au_n - Az\|.\end{aligned} \tag{3.26}$$

From (3.19) and (3.26), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\left(\|y_n - z\|^2 + t_n M_1\right) \\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n) \\ &\quad \times \left(\|x_n - z\|^2 - \|x_n - u_n\|^2 - \|u_n - y_n\|^2 + M_1(\|Bx_n - Bz\| + \|Au_n - Az\| + t_n)\right) \\ &= \|x_n - z\|^2 + (1-\alpha_n) \\ &\quad \times \left(M_1(\|Bx_n - Bz\| + \|Au_n - Az\| + t_n) - \|x_n - u_n\|^2 - \|u_n - y_n\|^2\right).\end{aligned} \tag{3.27}$$

Let $n \to \infty$ nd in view of (3.18) and (3.22), we have

$$\lim_{n\to\infty}\left(\|x_n - u_n\|^2 + \|u_n - y_n\|^2\right) = 0. \tag{3.28}$$

This shows that

$$\begin{gathered}\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|u_n - y_n\| = 0, \\ \lim_{n\to\infty}\|x_n - y_n\| = 0.\end{gathered} \tag{3.29}$$

Then, we have

$$\begin{aligned}\|x_{n+1} - Sx_{n+1}\| &= \|x_{n+1} - Sx_n + Sx_n - Sx_{n+1}\| \\ &\le \|x_{n+1} - Sx_n\| + \|Sx_n - Sx_{n+1}\| \\ &= \|SP_C((1-t_n)y_n) - SP_C x_n\| + \|Sx_n - Sx_{n+1}\| \\ &\le (1-t_n)\|y_n - x_n\| + t_n\|x_n\| + \|x_n - x_{n+1}\| \\ &\longrightarrow 0 (n \longrightarrow \infty).\end{aligned} \tag{3.30}$$

Step 3 (sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$). Because $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n \rightharpoonup x^* \in H$. In view of (3.12), it yields that $u_n \rightharpoonup x^*$ and $y_n \rightharpoonup x^*$. From Lemma 2.9 and (3.30), we know that $x^* \in F(S)$.

Next, we prove that $x^* \in \mathrm{GEP} \cap \mathrm{VI}(H, A, M)$.

Since $u_n = T_\mu(x_n - \mu B x_n)$, we have

$$F(u_n, y) + \frac{1}{\mu}\langle y - u_n, u_n - (x_n - \mu B x_n)\rangle \geq 0, \quad \forall y \in C. \tag{3.31}$$

It follows from condition (H_2) that

$$\frac{1}{\mu}\langle y - u_n, u_n - (x_n - \mu B x_n)\rangle \geq F(y, u_n), \quad \forall y \in C. \tag{3.32}$$

Therefore,

$$\left\langle y - u_n, \frac{u_n - x_n}{\mu} + B x_n \right\rangle \geq F(y, u_n), \quad \forall y \in C. \tag{3.33}$$

For any $t \in (0,1)$ and $y \in C$, then $y_t = ty + (1-t)x^* \in C$. From (3.33), we have

$$\begin{aligned}
\langle y_t - u_n,\ B y_t\rangle &\geq \langle y_t - u_n,\ B y_t\rangle - \left\langle y_t - u_n,\ \frac{u_n - x_n}{\mu} + B x_n \right\rangle + F(y_t, u_n) \\
&= \langle y_t - u_m,\ B y_t - B u_m\rangle + \langle y_t - u_n,\ B u_n - B x_n\rangle \\
&\quad - \left\langle y_t - u_n, \frac{u_n - x_n}{\mu} \right\rangle + F(y_t, u_n).
\end{aligned} \tag{3.34}$$

Since B is β-inverse strongly monotone, from Proposition 2.2(i) and (3.12), we have

$$\begin{gathered}
\|B u_n - B x_n\| \leq \frac{1}{\beta}\|u_n - x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty), \\
\langle y_t - u_n, B y_t - B u_n\rangle \geq \beta \|B y_t - B u_n\|^2 \geq 0.
\end{gathered} \tag{3.35}$$

Let $n \to \infty$ in (3.34), in view of condition (H_4) and $u_n \rightharpoonup x^*$, we have

$$\langle y_t - x^*, B y_t\rangle \geq F(y_t, x^*). \tag{3.36}$$

It follows from conditions (H_1), (H_4) and (3.36) that

$$\begin{aligned}
0 = F(y_t, y_t) &\leq t F(y_t, y) + (1-t) F(y_t, x^*) \\
&\leq t F(y_t, y) + (1-t)\langle y_t - x^*, B y_t\rangle \\
&= t F(y_t, y) + (1-t) t \langle y - x^*, B y_t\rangle,
\end{aligned} \tag{3.37}$$

that is,

$$0 \le F(y_t, y) + (1-t)\langle y - x^*, By_t \rangle. \tag{3.38}$$

Let t to 0 in (3.38), we have

$$F(x^*, y) + \langle y - x^*, Bx^* \rangle \ge 0, \quad \forall y \in C. \tag{3.39}$$

This shows that $x^* \in \mathrm{GEP}$.

Step 4 (now, we prove that $x^* \in \mathrm{VI}(H, A, M)$). Since A is α-inverse strongly monotone, from Proposition 2.2 (i), we know that A is an $1/\alpha$-Lipschitz continuous and monotone mapping and $D(A) = H$, where $D(A)$ is the domain of A. It follows from Lemma 2.8 that $M + A$ is maximal monotone. Let $(v, f) \in \mathrm{Graph}(M + A)$, that is, $f - Av \in M(v)$. Since $y_n = J_{M,\lambda}(u_n - \lambda Au_n)$, we have $u_n - \lambda Au_n \in (I + \lambda M)(y_n)$, that is, $1/\lambda(u_n - y_n - \lambda Au_n) \in M(y_n)$. By virtue of the maximal monotonicity of M, we have

$$\left\langle v - y_n, f - Av - \frac{1}{\lambda}(u_n - y_n - \lambda Au_n) \right\rangle \ge 0. \tag{3.40}$$

Therefore we have

$$\begin{aligned} \langle v - y_n, f \rangle &\ge \left\langle v - y_n, Av + \frac{1}{\lambda}(u_n - y_n - \lambda Au_n) \right\rangle \\ &= \left\langle v - y_n, Av - Ay_n + Ay_n - Au_n + \frac{1}{\lambda}(u_n - y_n) \right\rangle. \end{aligned} \tag{3.41}$$

Since A is monotone, this implies that

$$\langle v - y_n, f \rangle \ge 0 + \langle v - y_n, Ay_n - Au_n \rangle + \left\langle v - y_n, \frac{1}{\lambda}(u_n - y_n) \right\rangle. \tag{3.42}$$

Since

$$\|u_n - y_n\| \longrightarrow 0, \quad \|Au_n - Ay_n\| \longrightarrow 0, \quad y_n \rightharpoonup x^*, \quad (n \longrightarrow \infty), \tag{3.43}$$

from (3.42), we have

$$\lim_{n \to \infty} \langle v - y_n, f \rangle = \langle v - x^*, f \rangle \ge 0. \tag{3.44}$$

Since $A + M$ is maximal monotone, $\theta \in (M + A)(x^*)$, that is, $x^* \in \mathrm{VI}(H, M, A)$.

Summing up the above arguments, we have proved that

$$x^* \in \Omega := F(S) \cap \mathrm{VI}(H, M, A) \cap \mathrm{GEP}. \tag{3.45}$$

On the other hand, for any $z \in \Omega$, we have

$$\begin{aligned}
\|z_n - z\|^2 &= \left\|SP_C((1-t_n)y_n) - SP_C z\right\|^2 \\
&\le \left\|(1-t_n)y_n - z\right\|^2 = \left\|y_n - z - t_n y_n\right\|^2 \\
&= \left\|y_n - z\right\|^2 - 2t_n\langle y_n, y_n - z\rangle + t_n^2\left\|y_n\right\|^2 \\
&= \left\|y_n - z\right\|^2 - 2t_n\langle y_n - z, y_n - z\rangle - 2t_n\langle z, y_n - z\rangle + t_n^2\left\|y_n\right\|^2 \\
&= (1-2t_n)\left\|y_n - z\right\|^2 + 2t_n\langle z, z - y_n\rangle + t_n^2\left\|y_n\right\|^2 \\
&\le (1-2t_n)\|x_n - z\|^2 + 2t_n\langle z, z - y_n\rangle + t_n^2\left\|y_n\right\|^2,
\end{aligned} \tag{3.46}$$

and so we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(x_n - z) + (1-\alpha_n)(z_n - z)\|^2 \\
&\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\Big((1-2t_n)\|x_n - z\|^2 + 2t_n\langle z, z - y_n\rangle\Big) + t_n^2\left\|y_n\right\|^2 \\
&= [1 - 2t_n(1-\alpha_n)]\|x_n - z\|^2 + 2(1-\alpha_n)t_n\langle z, z - y_n\rangle + t_n^2\left\|y_n\right\|^2 \\
&\le [1 - 2t_n(1-b)]\|x_n - z\|^2 + 2(1-\alpha_n)t_n\langle z, z - y_n\rangle + t_n^2\left\|y_n\right\|^2.
\end{aligned} \tag{3.47}$$

Put $z = x^*$ in (3.47), we have

$$\|x_{n+1} - x^*\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + \delta_n, \tag{3.48}$$

where $\gamma_n = 2t_n(1-b)$ and $\delta_n = 2(1-\alpha_n)t_n\langle x^*, x^* - y_n\rangle + t_n^2\|y_n\|^2$. Since $y_n \rightharpoonup x^*$, it is easy to see that $\sum_{n=1}^{+\infty}\gamma_n = \infty$ and $\lim_{n\to\infty}(\delta_n/\gamma_n) = 0$. By Lemma 2.12, we conclude that $x_n \to x^*$ as $n \to \infty$, where x^* is the unique solution of the following quadratic minimization problem:

$$\|x^*\|^2 = \min_{x\in\Omega}\|x\|^2. \tag{3.49}$$

This completes the proof of Theorem 3.1. □

In Theorem 3.1, if $T = T_n$ $(\forall n \ge 1)$, then the following corollary can be obtained immediately.

Corollary 3.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $A : H \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $M : H \to 2^H$ be a maximal monotone mapping, $\{T : C \to C\}$ be a nonexpansive* *mappings with* $F(T) \neq \emptyset$*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions (*H_1*)–(*H_4*). Let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(TP_C\big((1-t_n)J_{M,\lambda}(I-\lambda A)T_\mu(I-\mu B)\big)\big)x_n \tag{3.50}$$

where the mapping $T_\mu : H \to C$ *is defined by* (2.18)*, and* λ, μ *are two constants with* $\lambda \in (0, 2\alpha]$*,* $\mu \in (0, 2\beta]$*, and*

$$t_n \in (0,1), \quad t_n \longrightarrow 0 (n \longrightarrow \infty), \quad \sum_{n=1}^{\infty} t_n = \infty, \quad 0 < a < \alpha_n < b < 1. \tag{3.51}$$

If

$$\Omega_1 := F(T) \cap \mathrm{VI}(H, A, M) \cap \mathrm{GEP} \neq \emptyset, \tag{3.52}$$

where VI (H, A, M) *and GEP are the sets of solutions of variational inclusion* (1.2) *and generalized equilibrium problem* (1.6)*, then the sequence* $\{x_n\}$ *defined by* (3.50) *converges strongly to* $x^* \in \Omega_1$*, which is the unique solution of the following quadratic minimization problem:*

$$\|x^*\|^2 = \min_{x \in \Omega_1} \|x\|^2. \tag{3.53}$$

In Theorem 3.1, if $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0, \infty]$ is the indicator function of C, then the variational inclusion problem (1.2) is equivalent to variational inequality (1.5), that is, to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$, for all $v \in C$. Since $M = \partial\delta_C$, $J_{M,\lambda} = P_C$. Consequently, we have the following corollary.

Corollary 3.3. *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H*,* $A : H \to H$ *be an* α*-inverse strongly monotone mapping and* $B : C \to H$ *be a* β*-inverse strongly monotone mapping. Let* $M = \partial\delta_C : H \to 2^H$ *and* $\{T : C \to C\}$ *be a nonexpansive mappings with* $F(T) \neq \emptyset$*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions (*H_1*)–(*H_4*). Let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(T\big((1-t_n)P_C(I-\lambda A)T_\mu(I-\mu B)\big)\big)x_n, \tag{3.54}$$

where the mapping $T_\mu : H \to C$ *is defined by* (2.18)*, and* λ, μ *are two constants with* $\lambda \in (0, 2\alpha]$*,* $\mu \in (0, 2\beta]$*, and*

$$t_n \in (0,1), \quad t_n \longrightarrow 0 (n \longrightarrow \infty), \quad \sum_{n=1}^{\infty} t_n = \infty, \quad 0 < a < \alpha_n < b < 1. \tag{3.55}$$

If

$$\Omega_2 := F(T) \cap \mathrm{VI}(A, C) \cap \mathrm{GEP} \neq \emptyset, \tag{3.56}$$

where VI (A, C) *and GEP are the sets of solutions of variational inclusion* (1.5) *and generalized equilibrium problem* (1.6)*, then the sequence* $\{x_n\}$ *defined by* (3.54) *converges strongly to* $x^* \in \Omega_2$*, which is the unique solution of the following quadratic minimization problem:*

$$\|x^*\|^2 = \min_{x \in \Omega_2} \|x\|^2. \tag{3.57}$$

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