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Research Article

Classification of Triangle-Free 22_3 Configurations

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The 157, 211 triangle-free symmetric 22_3 configurations are classified and some of their properties are examined. We conclude that each such configuration has a blocking set. Further properties like transitivity on lines, self-duality, and self-polarity are discussed.

1. Introduction

A *finite incidence structure* $\mathcal{X}$ is a pair $(P, \mathcal{B})$, where P and $\mathcal{B}$ are finite sets. In particular, $P = \{p_1, p_2, \ldots, p_v\}$ is a set of v *points* and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is a set of b *blocks* (or *lines*) such that $B_i \subseteq P$ for $i = 1, 2, \ldots, b$.

The number of blocks containing a point $p \in P$ is called the *degree*, denoted by $[p]$. The number of points that are contained in a block B is called the *size* of B, denoted by $|B|$. A pair (p, B) with $p \in B \in \mathcal{B}$ is called a *flag*. In this case, we say that p lies on B, or that p and B are incident.

A (combinatorial) *configuration* (see [1]) of type (v_r, b_k) is an incidence structure $(P, \mathcal{B})$ with

(**C1**) $|B_j| = k$ for $j = 1, \ldots, b$,

(**C2**) $[p_i] = r$ for $i = 1, \ldots, v$,

(**C3**) any two distinct points being incident with at most one line.

The last axiom implies that two points determine at most one line and that two lines intersect in at most one point. We say that two points are *collinear* if they lie on a line (and two lines are *concurrent* if they intersect). Note that these structures are defined purely combinatorially (and hence sometimes called combinatorial configurations). It is a different question whether or not a given configuration can be embedded in projective space (such that the blocks arise from lines in that space). This question leads to the notion of geometric configurations. In this paper, we are concerned with combinatorial configurations only. We do not discuss the problem of whether or not they can be embedded in projective space. Clearly, every geometric configuration is also combinatorial. Therefore, the results in this paper may be seen as a starting point to classify the corresponding geometric configurations. For the sake of simplicity, henceforth we will simply talk about configurations. In all cases, this will mean combinatorial configurations.

A (v_r, b_k) configuration with $v = b$ (and hence $r = k$) is called *symmetric* (see for instance [2, 3]). A symmetric configuration is denoted by v_r.

Configurations are closely related to graphs. Let $\mathcal{C} = (P, \mathcal{B})$ be a v_3 configuration. The *Levi graph* (or incidence graph), denoted $L(\mathcal{C})$, associated with $\mathcal{C}$, is the cubic bipartite graph with vertex set $P \cup \mathcal{B}$ with $p \in P$ and $B \in \mathcal{B}$ adjacent if and only if $p \in B$; see [3–7]. Alternatively, it is the cubic bipartite graph with black vertices representing the points, with white vertices representing the lines, and with an edge joining two vertices if and only if the corresponding point and line are incident. According to Coxeter [4], configurations can be characterized in the following way. Recall that the girth in a graph is the length of the shortest cycle.

Proposition 1.1. *An incidence structure $\mathcal{C}$ is a v_3 configuration if and only if its Levi graph is cubic and has girth at least* 6.

An *isomorphism* between two incidence structures $\mathcal{C}_1 = (P_1, \mathcal{B}_1)$ and $\mathcal{C}_2 = (P_2, \mathcal{B}_2)$ is a bijection $\alpha : P_1 \to P_2$ which takes $\mathcal{B}_1$ to $\mathcal{B}_2$ (where $\alpha(B) = \{\alpha(p) \mid p \in B\}$ for $B \in \mathcal{B}_1$). If such an isomorphism exists, the incidence structures are *isomorphic*. It is well known that isomorphism of incidence structures is an equivalence relation. The equivalence classes are known as *isomorphism types*. The problem of classifying a class of incidence structures is determining the classes of isomorphic objects.

Furthermore, for an incidence structure $\mathcal{C} = (P, \mathcal{B})$, the isomorphisms from $\mathcal{C}$ to $\mathcal{C}$ are known as *automorphisms.* They form a group, the *automorphism group* of $\mathcal{C}$.

If G is any subgroup of the automorphism group of a configuration $\mathcal{C}$, then G may be seen as acting on the set of points, the set of blocks, and the set of flags; see [8] for more details. An orbit of G on points, blocks, flags (resp.) is known as a point-orbit, block-orbit, flag-orbit (resp.). It is well known that a flag transitive automorphism group G is also transitive on points and blocks (but not conversely).

A *blocking set* in a configuration is a subset H of points such that each block contains at least one element from H and one element not from H. Not every configuration has a blocking set. An example of a *blocking set free* configuration is the unique 7_3 configuration (or *Fano* plane) shown in Figure 1. The figure also shows the Levi graph of this configuration. This graph is also known as the *Heawood* graph.

To each (v_r, b_k) configuration $\mathcal{C} = (P, \mathcal{B})$, we may associate another configuration known as the *dual* configuration. The dual configuration is $\mathcal{C}^* = (\mathcal{B}, P)$, with the roles of points and blocks reversed, but with the same incidence. That is, a "point" B is on a "block" p in the dual configuration if $p \in B$ in $\mathcal{C}$. Clearly, $\mathcal{C}^*$ is a (b_k, v_r) configuration. Also, $\mathcal{C}$ and $\mathcal{C}^*$ have the same Levi graph, except that the color classes are reversed. Applying duality twice in a row, we obtain a configuration $\mathfrak{D}$ that is isomorphic to the original configuration $\mathcal{C}$.

If $\mathcal{C}$ is isomorphic to its dual ($\mathcal{C}^*$), we say that $\mathcal{C}$ is *self-dual* and a corresponding isomorphism is called *duality*. Moreover, a duality of order 2 is called a *polarity*. A *self-polar*

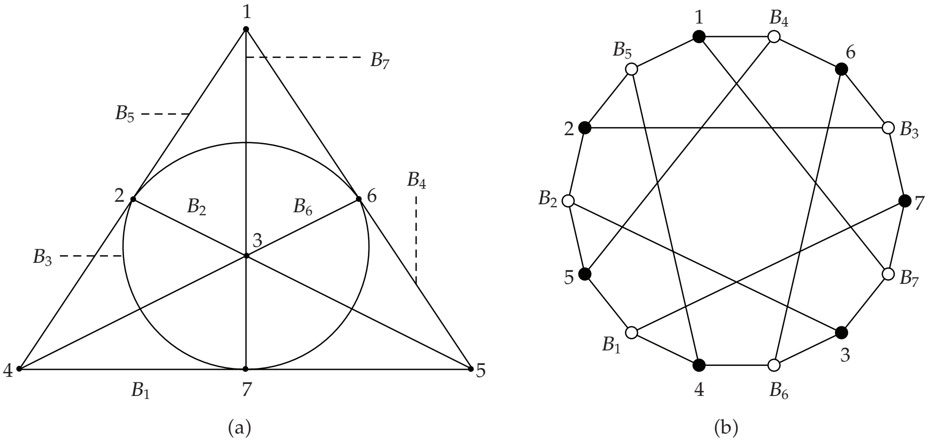

Figure 1: The Fano plane and its Levi graph (Heawood graph).

Table 1: Triangle-free configurations v_3 for $15 \le v \le 22$.

v	A	B	C	D	E	F
15	1	1	1	1	1	0
16	0	0	0	0	0	0
17	1	1	1	0	0	0
18	4	2	2	0	0	0
19	14	6	6	0	0	0
20	162	40	40	1	1	0
21	4,713	307	303	1	0	0
22	157,211	1,999	1,992	0	0	0

Note: A is the number of configurations v_3, B is the number of self-dual configurations v_3, C is the number of self-polar configurations v_3, D is the number of point-transitive configurations v_3, E is the number of flag-transitive configurations v_3, and F is the number of blocking set free configurations v_3.

configuration is a configuration that admits a polarity. The Fano plane of Figure 1 is both self-dual and self-polar. To see this, notice that the bijective mapping $p_i \leftrightarrow B_i$ for $i = 1, 2, \ldots, 7$ preserves the Levi graph of the Fano plane.

In this paper, we consider v_3 configurations that contain no *triangles*. A *triangle* in a configuration is a triple of points that are pairwise collinear but not with the same line. A configuration is triangle-free if and only if the girth of the incidence graph is at least 8 (configurations in general have girth at least 6). In terms of points and lines, we have that a point p that is not on a line B is collinear to at most one point of B. This is a weakening of the axiom of a *generalized quadrangle*: in a generalized quadrangle, a point p not on a line B is collinear to exactly one point of B.

So far, triangle-free v_3 configurations have been classified for $v \le 21$. In the current work, a classification for $v = 22$ is carried out. Table 1 shows the known results of the nonisomorphic triangle-free v_3 configurations for $v \le 22$. The entries for $v \le 21$ were determined previously in [8, 9]. The entries in the last row of Table 1 are new and were constructed in our search.

2. Search and Results

Necessary existence conditions for a (v_r, b_k) configuration are

$$vr = bk, \qquad v - 1 \geq r(k-1), \tag{2.1}$$

obtained by counting in two ways the incidences of points with lines and the incidences of pairs of points containing a given point with lines, respectively. For $k = 3$, these conditions are also sufficient; see Gropp [10].

In this paper, we use computer search to classify the triangle-free 22_3 configurations. We find exactly 157,211 nonisomorphic configurations of this type. Moreover, we also verified the results in [8] for triangle-free configurations v_3 with at most 21 points.

Gropp [3] has stated that there is a blocking set free 22_3 configuration. Our search shows that there is no blocking set free 22_3 configuration which is also triangle-free. Thus, Gropp's configuration must contain triangles.

A v_3 configuration $(P, \mathcal{B})$ with $P = \{p_1, \ldots, p_v\}$ and $\mathcal{B} = \{B_1, \ldots, B_v\}$ can be represented by a $\{0,1\}$ incidence matrix $A = (a_{i,j})$, say, with v rows and v columns. The entry $a_{i,j}$ is one if and only if $p_i \in B_j$. Clearly, there are three ones in each row and column. Also, the *dot product* of any two distinct rows is at most 1. Those properties are equivalent to those that were mentioned earlier in Section 1, namely, (**C1**), (**C2**), and (**C3**). Moreover, two matrices are isomorphic if one can be obtained from the other by permuting the rows and the columns. In the incidence matrices that are displayed below, we write "**x**" if $a_{i,j} = 1$ and "empty square" if $a_{i,j} = 0$.

Let us now describe the algorithm that we use to classify the triangle free v_3 configurations. The algorithm is an instance of the method of *orderly generation* [11].

We carry out a row-by-row (or point-by-point) backtrack search over all incidence matrices of triangle free v_3 configurations. We start with the all-zero matrix and augment it one row at a time, subject to the properties (**C1**), (**C2**), and (**C3**). Augmenting means deciding on the positions of the three ones in one particular row. The rows are augmented in order. Moreover, we use an algorithm from [12, 13] for testing the girth condition (of the partially filled incidence matrix). In this way, we ensure that all of the considered structures are triangle-free configurations. Once a row has been completed, one of two actions is taken. If the number of completed rows is between 17 and 21, no further action is taken. In the other cases, we perform a test whether or not the lexicographically least form of the incidence matrix agrees with the matrix that was created. If yes, we keep the row that was just added and proceed with the search. If no, the row is rejected and we backtrack. This test is our way to solve the isomorphism problem. Namely, each isomorphism class is represented by its lexicographically least representative. Since we perform this test after 22 rows, the resulting objects are pairwise nonisomorphic. The reason for not testing after rows 18, 19, and 20 is the following. Computing the lexicographically least incidence matrix is expensive. Therefore, the isomorphism test described above slows down the search quite a bit. The benefits of using the isomorphism test are worth the effort in the early rows (up to row 17). Namely, the isomorphism test keeps the number of possibilities down and thereby reduces the size of the search space. On the other hand, for rows 18–20, the number of possibilities of partial incidence matrices increases dramatically. We observe, however, that many of these matrices do not complete and hence do not contribute to the classification. Therefore, switching off the isomorphism test for rows 18–20 means that we do not spend time on classifying partial incidence matrices that do not complete anyway. This simple trick saves us a lot of time.

Table 2

Order	Types of groups
4	$C_2 \times C_2$ (160 ×), C_4 (27 ×)
6	$C_3 \times C_2$ (once), D_3 (13 ×)
8	E_8 (35 ×), $C_4 \times C_2$ (2 ×), D_4 (2 ×)
12	$E_4 : C_3$ (once), $C_2 \times D_3$ (5 ×)
16	$C_2 \times D_4$ (4 ×), $D_4 : C_2$ (2 ×)
22	C_{22} (once)
24	$D_3 \times E_4$ (3 ×)

In fact, we can classify the triangle free 22_3 configurations in about 19 hours CPU time (on a single CPU machine).

We remark that we use our own algorithm to compute the lexicographically least representative of the isomorphism class of a matrix. The complexity of this algorithm is exponential in the size of the input. No fast algorithm to solve this problem is known.

Alternately, the idea of *canonical augmentation* due to McKay [14] can be used. In this method, the lexicographically least representative is replaced by a *canonical representative* (that in almost all cases is different from the lexicographically least representative). We also tried this method, using *nauty* [15] to compute the canonical representative. We found that orderly generation using lexicographically least representatives worked better for us. This may not be seen as a critique of "canonical augmentation". We simply did not try very hard, so a comparison is unfair. We remark that while canonical representatives of an isomorphism class may be computed faster than lexicographically least representatives, the procedure is still exponential. Again, no fast (i.e., polynomial) algorithm to solve this problem is known.

Appendix

A. Some Selected Configurations

In what follows, we will present some examples and discuss some properties of the configurations that were found in our search. We start by listing the distribution of automorphism group orders

$$\begin{array}{ccc} 153,430 \times 1 & 14 \times 6 & 6 \times 16 \\ 3,485 \times 2 & 39 \times 8 & 1 \times 22 \\ 39 \times 3 & 1 \times 11 & 3 \times 24 \\ 187 \times 4 & 6 \times 12 & \end{array} \tag{A.1}$$

Here, $x \times y$ means that there are x configurations with an automorphism group of order y.

In the following, we will look at some of the configurations with nontrivial automorphism groups.

At first, we determine the type of the automorphism group. We use C_n, and D_n, and E_n to denote the cyclic group of order n, the Dihedral group of order $2n$, and the elementary abelian group of order n, respectively. For groups N and H, let $N : H$ be a split extension of N by H (with normal subgroup N). For groups of order other than a prime, we find the following types in Table 2.

It is often convenient to identify the blocks of a configuration v_3 with the triple of points that are incident with it. In what follows, we write $1, 2, \ldots, v$ for points and we write $B_1, B_2, \ldots, B_v$ for blocks. Also, we give the corresponding incidence matrix with row indices $1, 2, \ldots, v$ and column indices $v+1, v+2, \ldots, 2v$ corresponding to the set of v points and v blocks, respectively. The orbits of the automorphism group acting on points and blocks (resp.) form a partition (of P and of $\mathcal{B}$, resp.). In the figures, we group points and blocks according to this partition. The boundaries of the classes of the partition are indicated by **boldface** lines.

A.1. Triangle-Free Configuration with Automorphism Group of Order 4

As mentioned in Table 1, there are exactly seven configurations that are self-dual but are **not** self-polar. Six configurations have a group of order 2 while one has a group of order 4. The latter configuration has the following blocks:

$$\begin{array}{ccccccccccccccc}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9 & B_{10} & B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\
1 & 1 & 1 & 4 & 3 & 12 & 9 & 8 & 10 & 5 & 2 & 2 & 2 & 3 & 5 \\
17 & 20 & 19 & 6 & 7 & 15 & 13 & 14 & 11 & 13 & 9 & 8 & 12 & 4 & 6 \\
18 & 22 & 21 & 16 & 15 & 16 & 15 & 16 & 12 & 14 & 22 & 21 & 17 & 17 & 19
\end{array}$$
$$\begin{array}{ccccccc}
B_{16} & B_{17} & B_{18} & B_{19} & B_{20} & B_{21} & B_{22} \\
5 & 6 & 7 & 4 & 3 & 11 & 10 \\
7 & 9 & 8 & 10 & 11 & 14 & 13 \\
20 & 18 & 18 & 20 & 19 & 22 & 21
\end{array} \quad \text{(A.2)}$$

Its partitioned incidence matrix is presented in Figure 2. The automorphism group is generated by a and b where

$$\begin{aligned}
a &= (3,4)(6,7)(8,9)(10,11)(13,14)(15,16)(19,20)(21,22), \\
b &= (1,18)(6,20)(7,19)(8,21)(9,22)(10,16)(11,15)(13,14).
\end{aligned} \quad \text{(A.3)}$$

It has ten point-orbits of three different sizes:

$$\{1,18\}, \{2\}, \{3,4\}, \{5\}, \{6,7,19,20\}, \{8,9,21,22\}, \{10,11,15,16\}, \{12\}, \{13,14\}, \{17\}. \quad \text{(A.4)}$$

	23	25	24	39	40	42	41	26	27	31	28	29	30	43	44	32	33	34	35	36	37	38
1	x	x	x																			
18	x			x	x																	
2																	x	x	x			
3						x			x											x		
4							x	x												x		
5																x					x	x
19		x				x															x	
20			x				x															x
6				x				x													x	
7					x				x													x
21		x													x			x				
22			x											x			x					
9				x								x					x					
8					x								x					x				
11						x				x				x								
10							x			x					x							
16								x			x		x									
15									x		x	x										
12										x	x								x			
13												x			x	x						
14													x	x		x						
17	x																		x	x		

Figure 2: The incidence matrix of the unique self-dual configuration with automorphism group of order 4.

A.2. Triangle-Free Configuration with Automorphism Group of Order 11

The *unique* triangle-free 22_3 configuration with automorphism group of order 11 has the following blocks:

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	3	8	9	7	6	11	10	2	2	2	6	6
17	19	20	4	11	13	10	12	15	12	9	8	10	9	7
18	22	21	5	14	16	14	15	16	13	22	21	17	18	20

(A.5)

B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
5	3	4	5	4	3	8
7	11	14	16	15	13	12
19	17	18	21	22	20	19

Its partitioned incidence matrix is presented in Figure 3. Its Levi graph is shown in Figure 4. This configuration is self-polar by the correspondence

	23	34	38	43	31	40	35	44	37	28	42	24	25	41	26	39	27	29	32	30	36	33
1	x											x	x									
21		x											x	x								
5			x											x	x							
3				x											x	x						
11					x											x	x					
14						x											x	x				
10							x											x	x			
12								x											x	x		
6									x											x	x	
9										x											x	x
22											x	x										x
19			x					x				x										
20				x					x				x									
16					x					x				x								
4						x					x				x							
17	x						x									x						
8		x						x									x					
7			x						x									x				
13				x						x									x			
15					x						x									x		
18	x					x															x	
2		x					x															x

Figure 3: The incidence matrix of the unique configuration with automorphism group of order 11.

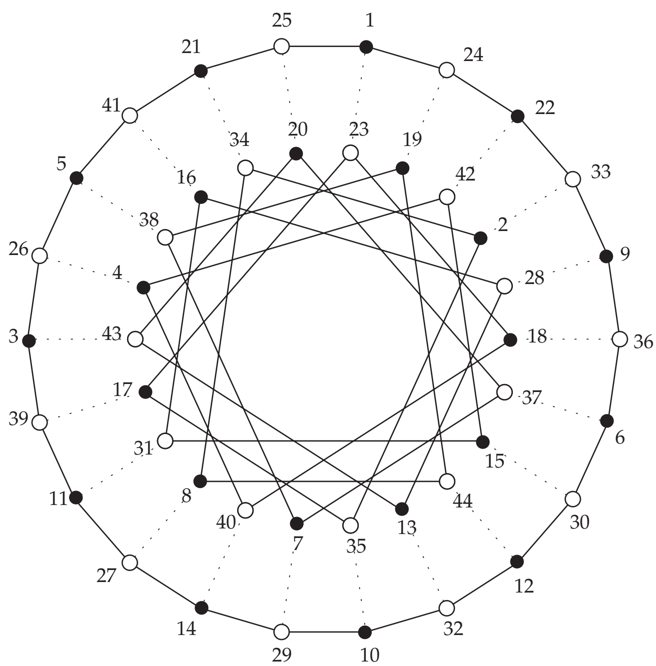

Figure 4: The Levi graph of the configuration with automorphism group of order 11.

$$\begin{gathered}(1, B_2)(2, B_{12})(3, B_8)(4, B_{15})(5, B_{14})(6, B_4)(7, B_{18})(8, B_{13})\\ (9, B_{19})(10, B_5)(11, B_{10})(12, B_{17})(13, B_9)(14, B_7)(15, B_{21})\\ (16, B_6)(17, B_{22})(18, B_{16})(19, B_1)(20, B_{20})(21, B_{11})(22, B_3).\end{gathered} \tag{A.6}$$

Its automorphism group is generated by

$$\alpha = (1, 11, 6, 21, 14, 9, 5, 10, 22, 3, 12)(2, 4, 13, 19, 17, 15, 20, 8, 18, 16, 7). \tag{A.7}$$

It has two point-orbits of size 11, shown in Figure 3.

A.3. Triangle-Free Configuration with Automorphism Group of Order 16

There are six triangle-free 22_3 configurations with an automorphism group of order 16. Four of those structures are self-dual and self-polar. Here we present only one of those four configurations. It has the following blocks:

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	2	5	6	10	9	12	11	8	7	2	2	4
17	20	19	3	8	7	14	13	14	13	10	9	5	6	10
18	22	21	4	11	12	15	16	16	15	22	21	21	22	18

(A.8)

B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
3	5	6	7	8	3	4
9	16	15	11	12	14	13
17	18	17	20	19	20	19

Its partitioned incidence matrix is shown in Figure 5. The Levi graph is shown in Figure 6. The configuration is self-polar under the correspondence

$$\begin{gathered}(1, B_4)(2, B_1)(3, B_2)(4, B_3)(5, B_{17})(6, B_{18})(7, B_7)(8, B_8)\\ (9, B_{11})(10, B_{12})(11, B_9)(12, B_{10})(13, B_{20})(14, B_{19})(15, B_6)\\ (16, B_5)(17, B_{14})(18, B_{13})(19, B_{22})(20, B_{21})(21, B_{15})(22, B_{16})\end{gathered} \tag{A.9}$$

The automorphism group is generated by the following three elements:

$$\begin{aligned} a &= (3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22),\\ b &= (1, 2)(3, 17)(4, 18)(5, 19)(6, 20)(11, 12)(13, 16)(14, 15),\\ c &= (1, 5)(2, 19)(3, 13)(6, 12)(8, 22)(11, 20)(14, 15)(16, 17).\end{aligned} \tag{A.10}$$

	43	26	44	32	40	23	39	31	24	36	35	25	42	27	41	28	29	30	37	38	33	34
20	x								x						x							
2		x								x	x											
19			x									x	x									
11				x										x	x							
6					x					x						x						
1						x			x			x										
5							x				x			x								
12								x					x			x						
3	x	x																		x		
4		x	x																x			
13			x	x														x				
15				x	x												x					
17					x	x														x		
18						x	x												x			
16							x	x										x				
14	x							x									x					
7															x	x						x
8													x	x							x	
21											x	x										x
22									x	x											x	
9																		x		x		x
10																	x		x		x	

Figure 5: The incidence matrix of a self-dual and self-polar configuration with automorphism group of order 16.

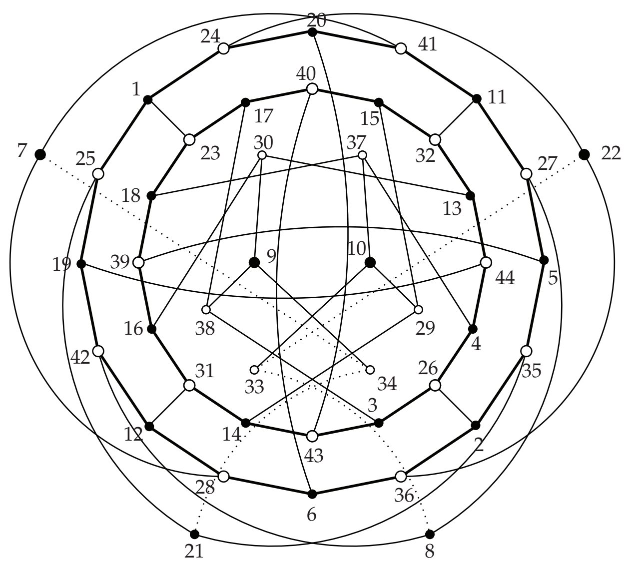

Figure 6: The Levi graph of a configuration with automorphism group of order 16.

	24	23	35	27	30	42	26	41	29	28	36	25	43	37	31	34	39	40	33	32	38	44
1	x	x										x										
18		x	x										x									
12			x	x										x								
16				x	x										x							
6					x	x										x						
3						x	x										x					
4							x	x										x				
5								x	x										x			
15									x	x										x		
11										x	x										x	
17	x										x											x
7											x				x		x					
10										x				x		x						
14									x				x		x							
22								x				x		x								
2							x						x									x
21						x						x									x	
13					x															x		x
9				x															x		x	
8			x															x		x		
20		x															x		x			
19	x															x		x				

Figure 7: The incidence matrix of the unique self-dual configuration with automorphism group of order 22.

A.4. Triangle-Free Configuration with Automorphism Group of Order 22

The unique triangle-free 22_3 configuration with automorphism group of order 22 has the following blocks:

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	2	9	10	5	6	7	8	5	6	8	7	10
18	17	21	3	12	11	14	13	14	13	9	10	12	11	12
20	19	22	4	16	15	15	16	16	15	20	19	18	17	22

(A.11)

B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
9	3	4	4	3	2	2
11	7	8	5	6	14	13
21	20	19	22	21	18	17

Figure 7 shows its partitioned incidence matrix. It is self-polar via the correspondence

$$\begin{gathered}(1, B_1)(2, B_{18})(3, B_{19})(4, B_4)(5, B_{20})(6, B_7)(7, B_{15})(8, B_{22})\\ (9, B_{16})(10, B_9)(11, B_5)(12, B_{14})(13, B_{10})(14, B_{12})(15, B_8)\\ (16, B_6)(17, B_{13})(18, B_2)(19, B_{21})(20, B_3)(21, B_{11})(22, B_{17})\end{gathered} \tag{A.12}$$

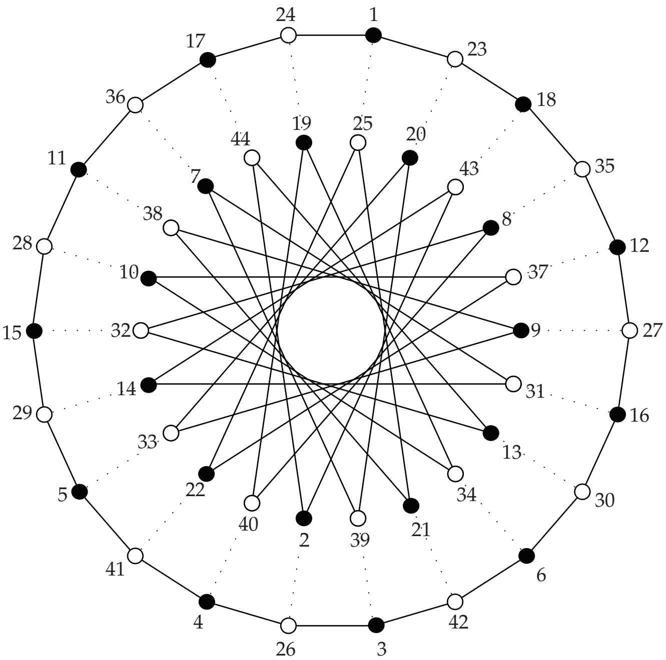

Figure 8: The unique self-dual configuration with automorphism group of order 22.

The automorphism group is generated by the following two elements:

$$a = (3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22),$$
$$b = (1,3,17,6,11,16,15,12,5,18,4)(2,19,21,7,13,10,9,14,8,22,20). \quad \text{(A.13)}$$

It has two point-orbits of equal sizes:

$$\{1,3,4,5,6,11,12,15,16,17,18\}, \{2,7,8,9,10,13,14,19,20,21,22\}. \quad \text{(A.14)}$$

Its Levi graph is shown in Figure 8.

A.5. Triangle-Free Configuration with Automorphism Group of Order 24

Altogether, there are three 22_3 configurations with an automorphism group of order 24. In this section, we present those three configurations, say A, B, and C. The blocks for configuration A are

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	12	13	8	4	5	6	7	4	6	4	6	7
17	18	19	14	15	9	11	10	11	10	5	7	8	8	9
22	20	21	16	16	16	14	12	15	13	17	17	18	19	21

B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
5	2	3	2	3	3	2
9	11	10	12	15	14	13
20	22	22	18	21	20	19

(A.15)

The blocks for configuration B are

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	12	13	8	4	5	6	7	4	5	4	6	7
17	18	19	14	15	9	11	10	11	10	7	6	8	8	9
22	21	20	16	16	16	14	12	15	13	17	17	18	19	21

(A.16)

B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
5	2	3	2	3	3	2
9	11	10	12	15	14	13
20	22	22	18	21	20	19

Finally, the blocks for configuration C are

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
1	1	1	4	11	10	6	9	8	7	5	4	5	4	6
19	20	17	5	14	14	13	12	13	12	6	9	8	7	9
22	21	18	14	15	16	15	16	16	15	22	19	20	21	18

(A.17)

B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
7	2	3	2	3	3	2
8	10	11	11	10	13	12
18	19	22	21	20	17	17

Partitioned incidence matrices for configurations A, B, and C (resp.) are shown in Figures 9, 10, and 11, respectively.

	23	40	39	33	30	41	24	43	29	34	32	44	25	42	31	35	38	26	36	37	27	28
17	x			x						x												
10		x			x						x											
2			x			x						x										
1	x						x						x									
3		x						x						x								
11			x						x						x							
4				x					x							x						
5				x	x												x					
12					x	x												x				
18						x	x									x						
20							x	x									x					
14								x	x									x				
6										x					x				x			
7										x	x									x		
13											x	x									x	
19												x	x						x			
21													x	x						x		
15														x	x						x	
8																x			x			x
9																	x			x		x
16																		x			x	x
22	x	x	x																			

Figure 9: The incidence matrix of the configuration A with automorphism group of order 24.

	23	40	39	33	30	41	24	43	29	34	32	44	25	42	31	35	38	26	36	37	27	28
17	x			x						x												
10		x			x						x											
2			x			x						x										
1	x						x						x									
3		x						x						x								
11			x						x						x							
4				x					x							x						
5					x					x							x					
12					x	x												x				
18						x	x									x						
20								x					x				x					
14								x	x									x				
6										x					x				x			
7				x							x									x		
13											x	x									x	
19												x	x						x			
21							x							x						x		
15														x	x						x	
8																x			x			x
9																	x			x		x
16																		x			x	x
22	x	x	x																			

Figure 10: The incidence matrix of the configuration B with automorphism group of order 24.

	44	25	43	39	23	40	41	24	42	37	30	31	38	32	29	28	34	33	27	36	35	26
2	x			x			x															
1		x			x			x														
3			x			x			x													
18		x								x			x									
12	x										x			x								
13			x									x			x							
10				x					x							x						
19				x	x												x					
22					x	x												x				
11						x	x												x			
21							x	x												x		
20								x	x												x	
6										x					x			x				
9										x	x						x					
16											x	x				x						
8												x	x								x	
7													x	x						x		
15														x	x				x			
14																x			x			x
4																	x			x		x
5																		x			x	x
17	x	x	x																			

Figure 11: The incidence matrix of the configuration C with automorphism group of order 24.

The automorphism group of configuration A is generated by

$$\begin{aligned}
a_1 &= (1,2)(3,10)(5,14)(7,15)(9,16)(11,17)(12,20)(13,21),\\
a_2 &= (2,3)(4,5)(6,7)(8,9)(10,11)(12,14)(13,15)(18,20)(19,21),\\
a_3 &= (4,6)(5,7)(12,13)(14,15)(18,19)(20,21),
\end{aligned} \tag{A.18}$$

and for B is generated by

$$\begin{aligned}
b_1 &= (1,2)(3,10)(5,15)(7,14)(9,16)(11,17)(12,21)(13,20),\\
b_2 &= (2,3)(4,5)(6,7)(8,9)(10,11)(12,14)(13,15)(18,20)(19,21),\\
b_3 &= (4,6)(5,7)(12,13)(14,15)(18,19)(20,21),
\end{aligned} \tag{A.19}$$

and for C is generated by

$$\begin{aligned}
c_1 &= (1,3)(4,14)(7,16)(9,15)(10,21)(11,19)(13,18),\\
c_2 &= (2,3)(4,5)(6,7)(8,9)(12,13)(19,20)(21,22),\\
c_3 &= (1,3)(4,14)(6,8)(7,15)(9,16)(10,19)(11,21)(13,18)(20,22),\\
c_4 &= (1,12)(2,18)(3,13)(5,14)(6,10)(7,21)(8,11)(9,19)(15,20)(16,22).
\end{aligned} \tag{A.20}$$

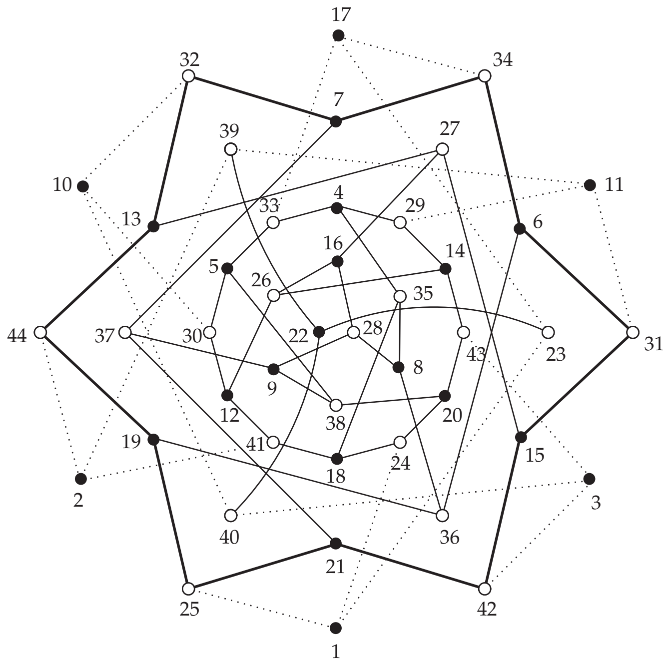

Figure 12: The configuration A with automorphism group of order 24.

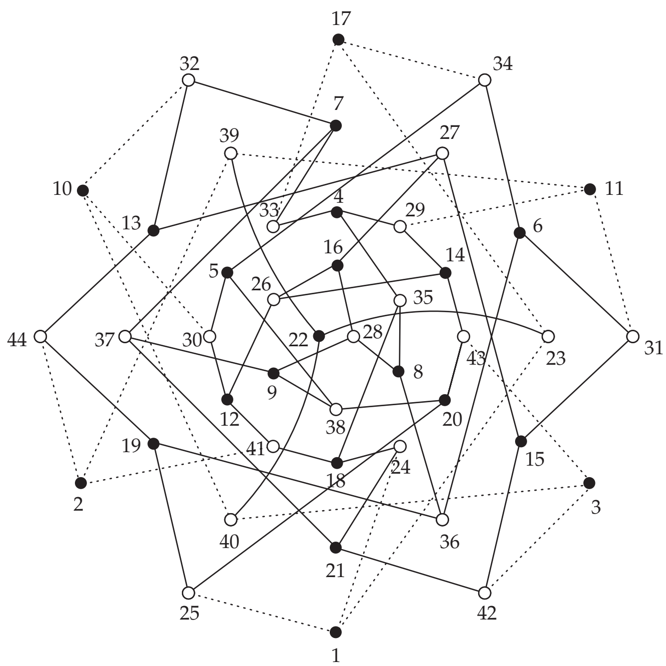

Figure 13: The configuration B with automorphism group of order 24.

Configurations A, B, and C have four point-orbits of four different sizes. This can be seen in the corresponding partitioned incidence matrices in Figures 9, 10, and 11.

Configuration B is self-polar under the correspondence

$$(1,B_5)(2,B_{14})(3,B_{15})(4,B_7)(5,B_8)(6,B_{19})(7,B_{21})(8,B_{17})$$
$$(9,B_{18})(10,B_{16})(11,B_{13})(12,B_{12})(13,B_3)(14,B_{11})(15,B_2) \quad \text{(A.21)}$$
$$(16,B_1)(17,B_4)(18,B_9)(19,B_{22})(20,B_{10})(21,B_{20})(22,B_6).$$

Configurations A and C are not self-dual. Figures 12 and 13 show the Levi graph of configurations A and B, respectively.

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