Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2010, Article ID 851857, 21 pages
doi:10.1155/2010/851857

Research Article

Classification of Base Sequences BS$(n+1, n)$

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Received 6 February 2010; Revised 8 April 2010; Accepted 12 April 2010

Academic Editor: Masaaki Harada

Base sequences BS$(n+1, n)$ are quadruples of $\{\pm 1\}$-sequences $(A; B; C; D)$, with A and B of length $n+1$ and C and D of length n, such that the sum of their nonperiodic autocor-relation functions is a δ-function. The base sequence conjecture, asserting that BS$(n+1, n)$ exist for all n, is stronger than the famous Hadamard matrix conjecture. We introduce a new definition of equivalence for base sequences BS$(n+1, n)$ and construct a canonical form. By using this canonical form, we have enumerated the equivalence classes of BS$(n+1, n)$ for $n \leq 30$. As the number of equivalence classes grows rapidly (but not monotonically) with n, the tables in the paper cover only the cases $n \leq 13$.

1. Introduction

Base sequences BS(m, n) are quadruples $(A; B; C; D)$ of binary sequences, with A and B of length m and C and D of length n, such that the sum of their nonperiodic autocorrelation functions is a δ-function. In this paper we take $m = n + 1$.

Sporadic examples of base sequences BS$(n+1, n)$ have been constructed by many authors during the last 30 years; see, for example, [1–4] and the survey paper [5] and its references. A more systematic approach has been taken by the author in [6, 7]. The BS$(n+1, n)$ are presently known to exist for all $n \leq 38$ (*ibid*) and for Golay numbers $n = 2^a 10^b 26^c$, where a, b, and c are arbitrary nonnegative integers. However the genuine classification of BS$(n+1, n)$ is still lacking. Due to the important role that these sequences play in various combinatorial constructions such as that for T-sequences, orthogonal designs, and Hadamard matrices [1, 5, 8], it is of interest to classify the base sequences of small length. Our main goal is to provide such classification for $n \leq 30$.

In Section 2 we recall the basic properties of base sequences of BS$(n+1, n)$. We also recall the quad decomposition and our encoding scheme for this particular type of base sequences.

Table 1: Number of equivalence classes of $BS(n+1, n)$.

n	Equ.	Nor.	n	Equ.	Nor.
0	1	1	16	1721	104
1	1	1	17	2241	0
2	1	1	18	1731	2
3	1	1	19	4552	2
4	3	2	20	3442	72
5	4	1	21	3677	0
6	5	0	22	15886	0
7	17	6	23	6139	0
8	27	14	24	10878	0
9	44	4	25	19516	4
10	98	10	26	10626	4
11	84	3	27	22895	0
12	175	8	28	31070	0
13	475	5	29	18831	2
14	331	0	30	19640	0
15	491	2	31	?	0

In Section 3 we enlarge the collection of standard elementary transformations of $BS(n+1, n)$ by introducing a new one. Thus we obtain new notion of equivalence and equivalence classes. Throughout the paper, the words "equivalence" and "equivalence class" are used in this new sense. We also introduce the canonical form for base sequences. By using it, we are able to compute the representatives of the equivalence classes.

In Section 4 we introduce an abstract group, G_{BS}, of order 2^{12} which acts naturally on all sets $BS(n+1, n)$. Its definition depends on the parity of n. The orbits of this group are just the equivalence classes of $BS(n+1, n)$.

In Section 5 we tabulate some of the results of our computations (those for $n \leq 13$) giving the list of representatives of the equivalence classes of $BS(n+1, n)$. The representatives are written in the encoded form which is explained in the next section. For $n \leq 8$ we also include the values of the nonperiodic autocorrelation functions of the four constituent sequences. We also raise the question of characterizing the binary sequences having the same nonperiodic autocorrelation function. A class of examples is constructed, showing that the question is interesting.

The column "Equ." in Table 1 gives the number of equivalence classes in $BS(n+1, n)$ for $n \leq 30$. The column "Nor." gives the number of normal equivalence classes (see Section 5 for their definition).

2. Quad Decomposition and the Encoding Scheme

We denote finite sequences of integers by capital letters. If, say, A is such a sequence of length n then we denote its elements by the corresponding lower case letters. Thus

$$A = a_1, a_2, \ldots, a_n. \tag{2.1}$$

Table 2: Equivalence classes of $\mathrm{BS}(n+1, n)$, $n \leq 8$.

	$ABCD$	N_A & N_B	N_C & N_D
		$n = 1$	
1	0	2, 1	1
	0	2, −1	1
		$n = 2$	
1	03	3, −2, 1	2, 1
	1	3, 0, −1	2, 1
		$n = 3$	
1	06	4, −1, 0, 1	3, 2, 1
	11	4, 1, −2, −1	3, −2, 1
		$n = 4$	
1	060	5, 0, 1, 0, 1	4, −1, 0, 1
	16	5, 2, −1, −2, −1	4, −1, 0, 1
2	082	5, 0, −1, −2, 1	4, 3, 2, 1
	12	5, −2, 1, 0, −1	4, −1, −2, 1
3	083	5, 0, −1, −2, 1	4, −1, 0, 1
	16	5, 2, 1, 0, −1	4, −1, 0, 1
		$n = 5$	
1	016	6, 1, 0, 1, 2, 1	5, 2, −1, −2, −1
	640	6, −1, 2, −1, 0, −1	5, −2, −1, 2, −1
2	017	6, 1, −4, −1, 2, 1	5, −2, 1, 0, −1
	613	6, 3, 2, 1, 0, −1	5, −2, 1, 0, −1
3	064	6, 1, −2, −1, 0, 1	5, 0, 1, 0, 1
	160	6, −1, 0, 1, −2, −1	5, 0, 1, 0, 1
4	065	6, −3, 2, −1, 0, 1	5, 0, −1, 2, 1
	113	6, 3, 0, −3, −2, −1	5, 0, −1, 2, 1
		$n = 6$	
1	0612	7, −2, 3, −2, 1, 0, 1	6, 1, −4, −1, 2, 1
	127	7, 4, 1, 0, −1, −2, −1	6, −3, 0, 3, −2, 1
2	0820	7, −2, 1, 0, 3, −2, 1	6, 1, 0, −1, −2, 1
	188	7, 0, −1, 2, 1, 0, −1	6, 1, 0, −1, −2, 1
3	0861	7, −2, −3, 2, 1, −2, 1	6, 1, 2, 1, 0, 1
	162	7, 0, 3, 0, −1, 0, −1	6, 1, −2, −3, 0, 1
4	0872	7, 2, 1, 0, −1, −2, 1	6, 1, 0, 1, 2, 1
	126	7, 0, −1, −2, 1, 0, −1	6, −3, 0, 1, −2, 1
5	0882	7, 2, 1, 0, −1, −2, 1	6, 1, −2, −1, 0, 1
	164	7, 0, −1, 2, 1, 0, −1	6, −3, 2, −1, 0, 1
		$n = 7$	
1	0165	8, −1, 2, −1, 0, 1, 2, 1	7, 0, −1, 2, 1, 0, −1
	6123	8, 1, 0, 1, −2, −1, 0, −1	7, 0, −1, −2, 1, 0, −1
2	0165	8, −1, 2, −1, 0, 1, 2, 1	7, 0, 3, 0, −1, 0, −1
	6141	8, 1, 0, 1, −2, −1, 0, −1	7, 0, −5, 0, 3, 0, −1
3	0166	8, 3, −2, −1, 0, 1, 2, 1	7, 0, −1, 2, 1, 0, −1
	6122	8, 1, 0, 1, −2, −1, 0, −1	7, −4, 3, −2, 1, 0, −1
4	0173	8, −1, −2, 1, −2, −1, 2, 1	7, 0, 3, 0, −1, 0, −1
	6161	8, 1, 0, −1, 4, 1, 0, −1	7, 0, −1, 0, −1, 0, −1

Table 2: Continued.

	$ABCD$	N_A & N_B	N_C & N_D
5	0173	8, −1, −2, 1, −2, −1, 2, 1	7, 4, 1, 0, −1, −2, −1
	6411	8, 1, 0, −1, 4, 1, 0, −1	7, −4, 1, 0, −1, 2, −1
6	0183	8, −1, −2, 1, −2, −1, 2, 1	7, 4, 3, 2, 1, 0, −1
	6121	8, −3, 0, −1, 0, 1, 0, −1	7, 0, −1, −2, 1, 0, −1
7	0613	8, −1, 0, 3, 0, 1, 0, 1	7, −2, 3, −2, 1, 0, 1
	1623	8, 1, −2, 1, 2, −1, −2, −1	7, 2, −1, −2, −3, 0, 1
8	0614	8, −1, 4, −1, 0, 1, 0, 1	7, 2, −1, 0, −1, 0, 1
	1641	8, 1, −2, 1, 2, −1, −2, −1	7, −2, −1, 0, −1, 0, 1
9	0615	8, −1, 0, 3, 0, 1, 0, 1	7, 2, −3, −2, 1, 2, 1
	1263	8, 1, 2, 1, −2, −1, −2, −1	7, −2, 1, −2, 1, −2, 1
10	0615	8, −1, 0, 3, 0, 1, 0, 1	7, 2, −3, −4, −1, 2, 1
	1272	8, 1, 2, 1, −2, −1, −2, −1	7, −2, 1, 0, 3, −2, 1
11	0616	8, −1, 4, −1, 0, 1, 0, 1	7, 2, −3, −2, 1, 2, 1
	1262	8, 1, 2, 1, −2, −1, −2, −1	7, −2, −3, 2, 1, −2, 1
12	0618	8, −1, 0, −1, −2, 1, 0, 1	7, 2, 1, 2, 1, 2, 1
	1261	8, 1, −2, 1, 0, −1, −2, −1	7, −2, 1, −2, 1, −2, 1
13	0635	8, −1, −4, 1, 2, −1, 0, 1	7, 2, 3, 2, 1, 0, 1
	1621	8, −3, 2, −1, 0, 1, −2, −1	7, 2, −1, −2, −3, 0, 1
14	0638	8, −1, 0, −3, 0, −1, 0, 1	7, 2, 3, 2, 1, 0, 1
	1620	8, 1, −2, −1, 2, 1, −2, −1	7, −2, −1, 2, −3, 0, 1
15	0641	8, 3, 0, 1, 0, −1, 0, 1	7, −2, 3, −2, 1, 0, 1
	1622	8, 1, −2, −1, 2, 1, −2, −1	7, −2, −1, 2, −3, 0, 1
16	0646	8, 3, 0, −3, −2, −1, 0, 1	7, 2, 1, 0, 3, 2, 1
	1222	8, −3, 2, −1, 0, 1, −2, −1	7, −2, −3, 4, −1, −2, 1
17	0646	8, 3, 0, −3, −2, −1, 0, 1	7, 2, 1, 2, 1, 2, 1
	1260	8, −3, 2, −1, 0, 1, −2, −1	7, −2, −3, 2, 1, −2, 1
		$n = 8$	
1	06113	9, 0, 1, 4, −1, 2, 1, 0, 1	8, −1, 0, −3, 0, −1, 0, 1
	1638	9, 2, −1, 2, 1, 0, −1, −2, −1	8, −1, 0, −3, 0, −1, 0, 1
2	06122	9, 0, 1, 4, −1, 2, 1, 0, 1	8, 3, 0, −3, −2, −1, 0, 1
	1644	9, −2, 3, −2, 1, 0, −1, −2, −1	8, −1, −4, 1, 2, −1, 0, 1
3	06141	9, 0, 5, 0, 1, 0, 1, 0, 1	8, −1, 0, 1, −2, −1, 0, 1
	1663	9, 2, −5, −2, 3, 2, −1, −2, −1	8, −1, 0, 1, −2, −1, 0, 1
4	06142	9, 0, 1, 0, −3, 0, 1, 0, 1	8, −1, 4, −1, 0, 1, 0, 1
	1624	9, 2, −1, −2, 3, 2, −1, −2, −1	8, −1, −4, 3, 0, −3, 0, 1
5	06151	9, 0, 1, 0, 5, 0, 1, 0, 1	8, −1, 0, −1, −2, 1, 0, 1
	1618	9, 2, −1, 2, −1, −2, −1, −2, −1	8, −1, 0, −1, −2, 1, 0, 1
6	06152	9, 0, −3, 0, 1, 0, 1, 0, 1	8, 3, −2, −1, 0, 1, 2, 1
	1264	9, 2, 3, 2, −1, −2, −1, −2, −1	8, −5, 2, −1, 0, 1, −2, 1
7	06183	9, 0, 1, −4, −1, −2, 1, 0, 1	8, −1, −2, 5, 0, −1, 2, 1
	1271	9, 2, −1, −2, 1, 0, −1, −2, −1	8, −1, 2, 1, 0, 3, −2, 1
8	06183	9, 0, 1, −4, −1, −2, 1, 0, 1	8, −1, 0, 3, 0, 1, 0, 1
	1613	9, 2, −1, −2, 1, 0, −1, −2, −1	8, −1, 0, 3, 0, 1, 0, 1
9	06310	9, 0, 1, 2, 1, 4, −1, 0, 1	8, −1, 0 − 1, 0, −3, 0, 1
	1686	9, 2, −1, 0, −1, 2, 1, −2, −1	8, −1, 0 − 1, 0, −3, 0, 1
10	06380	9, −4, 1, −2, 1, 0, −1, 0, 1	8, 3, 0, 1, 0, −1, 0, 1
	1661	9, −2, −1, 0, −1, 2, 1, −2, −1	8, 3, 0, 1, 0, −1, 0, 1

Table 2: Continued.

	$ABCD$	N_A & N_B	N_C & N_D
11	06382	$9, 0, 1, -2, -3, 0, -1, 0, 1$	$8, 3, 0, 1, 0, -1, 0, 1$
	1641	$9, -2, -1, 0, -1, 2, 1, -2, -1$	$8, -1, 0, 1, 4, -1, 0, 1$
12	06412	$9, 0, 1, 2, -3, 0, -1, 0, 1$	$8, -1, 0, 1, 4, -1, 0, 1$
	1632	$9, 2, -1, 0, -1, 2, 1, -2, -1$	$8, -1, 0, -3, 0, -1, 0, 1$
13	06580	$9, -4, 1, -2, 1, 0, -1, 0, 1$	$8, 3, -2, -3, 0, 3, 2, 1$
	1127	$9, 2, 3, 0, -1, -2, -3, -2, -1$	$8, -1, -2, 5, 0, -1, 2, 1$
14	06633	$9, 0, -3, 2, -1, -2, -1, 0, 1$	$8, -1, 2, -1, 0, 1, 2, 1$
	1163	$9, 2, -1, 0, 1, 0, -3, -2, -1$	$8, -1, 2, -1, 0, 1, 2, 1$
15	06852	$9, 0, -3, 0, 1, 0, -3, 0, 1$	$8, -1, 2, -1, 0, 1, 2, 1$
	1163	$9, 2, -1, 2, -1, -2, -1, -2, -1$	$8, -1, 2, -1, 0, 1, 2, 1$
16	06860	$9, 0, -3, 0, 1, 0, -3, 0, 1$	$8, -1, 2, -1, 0, 1, 2, 1$
	1163	$9, 2, -1, 2, -1, -2, -1, -2, -1$	$8, -1, 2, -1, 0, 1, 2, 1$
17	08110	$9, 0, 3, 2, 1, 0, 3, -2, 1$	$8, -1, -2, -1, 0, 1, -2, 1$
	1866	$9, 2, 1, 0, -1, -2, 1, 0, -1$	$8, -1, -2, -1, 0, 1, -2, 1$
18	08350	$9, 0, -1, -4, 1, 0, 1, -2, 1$	$8, -1, 2, 1, 0, 3, -2, 1$
	1822	$9, -2, -3, 2, -1, -2, 3, 0, -1$	$8, 3, 2, 1, 0, -1, -2, 1$
19	08383	$9, 0, 3, 0, -1, -2, 1, -2, 1$	$8, -1, -2, -1, 0, 1, -2, 1$
	1866	$9, 2, 1, 2, 1, 0, 3, 0, -1$	$8, -1, -2, -1, 0, 1, -2, 1$
20	08630	$9, -4, -1, 0, 1, 0, 1, -2, 1$	$8, 3, 0, 1, 0, -1, 0, 1$
	1661	$9, -2, 1, -2, -1, 2, -1, 0, -1$	$8, 3, 0, 1, 0, -1, 0, 1$
21	08640	$9, 0, -1, -4, 1, 0, 1, -2, 1$	$8, -1, -2, 5, 0, -1, 2, 1$
	1282	$9, -2, 1, -2, -1, 2, -1, 0, -1$	$8, 3, 2, 1, 0, -1, -2, 1$
22	08642	$9, 0, -1, 0, -3, 0, 1, -2, 1$	$8, 3, 0, 1, 0, -1, 0, 1$
	1641	$9, -2, 1, -2, -1, 2, -1, 0, -1$	$8, -1, 0, 1, 4, -1, 0, 1$
23	08660	$9, 0, -1, -4, 1, 0, 1, -2, 1$	$8, -1, -2, 5, 0, -1, 2, 1$
	1271	$9, 2, 1, -2, -1, -2, -1, 0, -1$	$8, -1, 2, 1, 0, 3, -2, 1$
24	08660	$9, 0, -1, -4, 1, 0, 1, -2, 1$	$8, -1, 0, 3, 0, 1, 0, 1$
	1613	$9, 2, 1, -2, -1, -2, -1, 0, -1$	$8, -1, 0, 3, 0, 1, 0, 1$
25	08833	$9, 0, -1, 2, -1, 2, -1, -2, 1$	$8, -1, 0, -1, 0, -3, 0, 1$
	1686	$9, 2, 1, 0, 1, 4, 1, 0, -1$	$8, -1, 0, -1, 0, -3, 0, 1$
26	08862	$9, 0, -1, 2, -1, 2, -1, -2, 1$	$8, -1, 4, -1, 0, 1, 0, 1$
	1626	$9, 2, -3, 0, 1, 0, 1, 0, -1$	$8, -1, 0, -1, 0, -3, 0, 1$
27	08863	$9, 0, -1, 2, -1, 2, -1, -2, 1$	$8, -1, 0, -3, 0, -1, 0, 1$
	1638	$9, 2, 1, 4, 1, 0, 1, 0, -1$	$8, -1, 0, -3, 0, -1, 0, 1$

To this sequence we associate the polynomial

$$A(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}, \tag{2.2}$$

viewed as an element of the Laurent polynomial ring $\mathbf{Z}[x, x^{-1}]$. (As usual, $\mathbf{Z}$ denotes the ring of integers.) The *nonperiodic autocorrelation function* N_A of A is defined by

$$N_A(i) = \sum_{j \in \mathbf{Z}} a_j a_{i+j}, \quad i \in \mathbf{Z}, \tag{2.3}$$

Table 3: Equivalence classes of BS(10, 9).

	AB	*CD*		*AB*	*CD*		*AB*	*CD*		*AB*	*CD*
1	01235	66450	2	01324	66181	3	01618	64150	4	01624	64183
5	01627	64130	6	01633	64140	7	01642	64560	8	01652	61453
9	01652	64313	10	01654	61163	11	01655	61180	12	01672	61281
13	01675	61430	14	01682	61180	15	01684	61122	16	01734	64160
17	01735	61640	18	01764	61821	19	01765	61281	20	01767	61831
21	01783	61411	22	01867	61311	23	06124	16282	24	06136	16640
25	06147	16450	26	06152	12763	27	06164	16133	28	06172	12681
29	06175	12670	30	06175	16143	31	06187	16131	32	06351	16460
33	06382	16460	34	06388	16340	35	06412	16273	36	06412	16381
37	06413	16460	38	06451	16163	39	06458	12612	40	06481	12623
41	06481	16161	42	06581	11622	43	06583	11631	44	06875	11622

where $a_k = 0$ for $k < 1$ and for $k > n$. Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbf{Z}$ and $N_A(i) = 0$ for $i \geq n$. The *norm* of A is the Laurent polynomial $N(A) = A(x)A(x^{-1})$. We have

$$N(A) = \sum_{i \in \mathbf{Z}} N_A(i) x^i. \tag{2.4}$$

The negation, $-A$, of A is the sequence

$$-A = -a_1, -a_2, \ldots, -a_n. \tag{2.5}$$

The *reversed* sequence A' and the *alternated* sequence A^* of the sequence A are defined by

$$\begin{aligned} A' &= a_n, a_{n-1}, \ldots, a_1, \\ A^* &= a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1} a_n. \end{aligned} \tag{2.6}$$

Observe that $N(-A) = N(A') = N(A)$ and $N_{A^*}(i) = (-1)^i N_A(i)$ for all $i \in \mathbf{Z}$. By A, B we denote the concatenation of the sequences A and B.

A *binary sequence* is a sequence whose terms belong to $\{\pm 1\}$. When displaying such sequences, we will often write + for +1 and − for −1. The *base sequences* consist of four binary sequences $(A; B; C; D)$, with A and B of length m and C and D of length n, such that

$$N(A) + N(B) + N(C) + N(D) = 2(m + n). \tag{2.7}$$

Thus, for $i \neq 0$, we have

$$N_A(i) + N_B(i) + N_C(i) + N_D(i) = 0. \tag{2.8}$$

We denote by $\mathrm{BS}(m, n)$ the set of such base sequences with m and n fixed. From now on we will consider only the case $m = n + 1$.

Table 4: Equivalence classes of BS(11, 10).

	AB	CD		AB	CD		AB	CD
1	061173	16456	2	061253	16246	3	061350	16645
4	061450	16267	5	061450	16443	6	061460	16434
7	061463	12826	8	061463	16271	9	061553	12716
10	061563	16134	11	061582	12631	12	061633	12671
13	061740	12684	14	061870	12286	15	061870	16144
16	063140	16862	17	063413	16822	18	063510	16382
19	063513	16441	20	063550	16414	21	063583	16341
22	063810	16616	23	063821	16445	24	063833	16613
25	063840	16262	26	063842	16242	27	063870	16322
28	063873	16217	29	063881	16342	30	064122	16277
31	064130	16465	32	064141	16423	33	064141	16452
34	064170	16344	35	064313	16282	36	064413	12826
37	064413	16271	38	064480	16213	39	064480	16321
40	064510	12638	41	064510	12676	42	064870	12616
43	065843	11276	44	068110	11863	45	068383	11863
46	068560	11263	47	068571	11632	48	068580	11627
49	068580	11643	50	068611	11634	51	068632	11632
52	068641	11634	53	068752	11276	54	068771	11645
55	082661	18642	56	083110	18863	57	083383	18863
58	083510	18226	59	083521	18642	60	083850	18622
61	086231	16248	62	086243	16277	63	086263	16332
64	086310	16613	65	086333	16616	66	086343	16228
67	086421	16427	68	086432	16242	69	086463	16217
70	086473	16217	71	086483	16344	72	086532	16142
73	086640	12631	74	086643	16134	75	086740	12642
76	086840	12682	77	086860	12671	78	086870	12671
79	087110	12863	80	087110	16273	81	087120	16461
82	087130	16461	83	087131	16262	84	087221	16284
85	087323	16282	86	087343	16282	87	087361	16422
88	087372	12864	89	087383	16382	90	087663	16146
91	087683	12637	92	087732	12684	93	088651	16264
94	088651	16424	95	088673	16434	96	088762	16246
97	088771	16264	98	088771	16424			

Let $(A; B; C; D) \in \mathrm{BS}(n+1, n)$. For convenience we fix the following notation. For n even (odd) we set $n = 2m$ ($n = 2m+1$). We decompose the pair $(A; B)$ into quads

$$\begin{bmatrix} a_i & a_{n+2-i} \\ b_i & b_{n+2-i} \end{bmatrix}, \quad i = 1, 2, \ldots, \left[\frac{n+1}{2}\right], \tag{2.9}$$

and, if n is even, the central column $\left[\begin{smallmatrix} a_{m+1} \\ b_{m+1} \end{smallmatrix}\right]$. Similar decomposition is valid for the pair $(C; D)$.

Recall the following basic and well-known property [3, Theorem 1].

Table 5: Equivalence classes of BS(12, 11).

	AB	CD		AB	CD		AB	CD
1	011823	661422	2	012356	661422	3	013682	663120
4	013753	663120	5	016426	641272	6	016445	616230
7	016472	645121	8	016525	614232	9	016525	643511
10	016535	612770	11	016542	612463	12	016542	612843
13	016542	614242	14	016546	612440	15	016572	614320
16	016634	614123	17	016634	614231	18	016643	612271
19	016653	612313	20	016724	643121	21	016727	612440
22	016756	611280	23	016774	612363	24	016817	614320
25	017262	641243	26	017356	616123	27	017374	616242
28	017375	641243	29	017632	614620	30	017646	612640
31	017664	614520	32	017674	612462	33	017674	614272
34	018265	612772	35	018342	614620	36	018382	612770
37	018767	613123	38	018767	613230	39	061186	164231
40	061246	166323	41	061256	164341	42	061264	162273
43	061264	162433	44	061284	164231	45	061462	162452
46	061462	164322	47	061463	162262	48	061472	128681
49	061473	164251	50	061476	162431	51	061476	164320
52	061547	127621	53	061575	126232	54	061618	126232
55	061624	126332	56	061644	126341	57	061764	126242
58	061774	126343	59	063144	168281	60	063412	168481
61	063512	162642	62	063515	162441	63	063515	164321
64	063541	162660	65	063541	164420	66	063551	163241
67	063811	164261	68	063824	162660	69	063824	164420
70	063825	162772	71	063828	162273	72	063828	162433
73	063842	164261	74	063858	163320	75	063877	128160
76	063882	163423	77	063884	163421	78	064143	164251
79	064146	162431	80	064146	164320	81	064615	122632
82	064826	126262	83	064838	126451	84	064842	126451

Theorem 2.1. *For $(A;B;C;D) \in \mathrm{BS}(n+1,n)$, the sum of the four quad entries is $2 \pmod 4$ for the first quad of the pair $(A;B)$ and is $0 \pmod 4$ for all other quads of $(A;B)$ and also for all quads of the pair $(C;D)$.*

Thus there are 8 possibilities for the first quad of the pair $(A;B)$:

$$\begin{gathered} 1' = \begin{bmatrix} - & + \\ + & + \end{bmatrix}, \quad 2' = \begin{bmatrix} + & - \\ + & + \end{bmatrix}, \quad 3' = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \quad 4' = \begin{bmatrix} + & + \\ - & + \end{bmatrix}, \\ 5' = \begin{bmatrix} + & - \\ - & - \end{bmatrix}, \quad 6' = \begin{bmatrix} - & + \\ - & - \end{bmatrix}, \quad 7' = \begin{bmatrix} - & - \\ - & + \end{bmatrix}, \quad 8' = \begin{bmatrix} - & - \\ + & - \end{bmatrix}. \end{gathered} \tag{2.10}$$

These eight quads occur in the study of Golay sequences (see, e.g., [9]), and we refer to them as the *Golay quads*.

Table 6: Equivalence classes of BS(13, 12).

	AB	CD		AB	CD		AB	CD
1	0611863	164521	2	0611871	166143	3	0612360	164844
4	0612573	164215	5	0612760	128287	6	0612760	162443
7	0612870	162424	8	0614230	164864	9	0614533	164226
10	0614670	162327	11	0614671	128266	12	0614830	162624
13	0615230	126876	14	0615230	161883	15	0615643	161234
16	0615733	126452	17	0616433	126452	18	0616533	126143
19	0616733	126143	20	0617212	126847	21	0617220	126876
22	0617220	161883	23	0617623	126413	24	0617651	122818
25	0617811	126428	26	0617820	161624	27	0618833	126413
28	0631412	168644	29	0634170	168382	30	0635120	164558
31	0635152	164341	32	0635340	164621	33	0635440	164242
34	0635483	162246	35	0635483	162432	36	0635513	164132
37	0635550	163214	38	0635810	163422	39	0635810	164142
40	0635872	162134	41	0638112	163822	42	0638121	164423
43	0638212	164242	44	0638220	164278	45	0638241	164423
46	0638781	162167	47	0641282	162424	48	0641363	164522
49	0641370	164265	50	0641470	162732	51	0641471	128642
52	0641471	164217	53	0641481	164215	54	0643513	164242
55	0643822	164522	56	0643880	162424	57	0644833	163214
58	0646430	126143	59	0648112	126283	60	0648212	161624
61	0648213	161644	62	0648363	126461	63	0648382	126716
64	0648433	126452	65	0658112	116273	66	0658363	116342
67	0658463	112645	68	0661363	116245	69	0661453	112634
70	0663640	116342	71	0663853	116324	72	0663880	116245
73	0664360	116245	74	0685733	116245	75	0685860	112645
76	0685871	112766	77	0686130	116245	78	0686230	116636
79	0686240	116273	80	0686433	116245	81	0687623	112645
82	0688613	116245	83	0688671	112764	84	0811283	182788
85	0812661	182668	86	0812883	186247	87	0816621	181128
88	0817883	181128	89	0826620	186247	90	0826782	182266
91	0826783	182641	92	0826851	186444	93	0835382	182266
94	0836121	186621	95	0837383	182278	96	0837383	182771
97	0838521	186627	98	0838652	182777	99	0838673	182668
100	0838751	182777	101	0838871	186644	102	0862230	164278
103	0862270	164413	104	0862312	164226	105	0862322	166128
106	0862383	162748	107	0862441	164265	108	0862483	128674
109	0862650	163341	110	0862651	162138	111	0863220	164287
112	0863353	162642	113	0863353	164612	114	0863482	166144
115	0864121	164324	116	0864311	164226	117	0864343	164226
118	0864382	164413	119	0864463	162451	120	0864781	162167
121	0865261	126237	122	0865310	126452	123	0865343	161642
124	0865382	126238	125	0865382	161267	126	0866310	126413
127	0866443	126314	128	0867740	126174	129	0868761	126238
130	0868761	161267	131	0871130	162747	132	0871332	162624
133	0872231	162838	134	0872833	164432	135	0873111	166361
136	0873470	162842	137	0873470	166413	138	0873481	162862

Table 6: Continued.

	AB	CD		AB	CD		AB	CD
139	0873581	164413	140	0873583	162445	141	0873670	162248
142	0873681	162268	143	0873711	164226	144	0873750	164413
145	0876221	127763	146	0876510	126342	147	0876510	126451
148	0876570	122864	149	0876570	161247	150	0876581	126238
151	0876581	161267	152	0876612	126341	153	0876663	126314
154	0876750	126238	155	0876750	161267	156	0877382	126876
157	0877382	161883	158	0877871	126748	159	0878631	127778
160	0878663	127647	161	0878861	161886	162	0878870	161884
163	0881211	168382	164	0882762	168242	165	0883571	168242
166	0883671	168422	167	0886261	162874	168	0886280	162867
169	0886471	162842	170	0886471	166413	171	0886560	164215
172	0886613	164521	173	0886671	162741	174	886760	164215
175	0887780	162748						

There are also 8 possibilities for each of the remaining quads of $(A;B)$ and all quads of $(C;D)$:

$$\begin{gathered} 1=\begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad 2=\begin{bmatrix} + & + \\ - & - \end{bmatrix}, \quad 3=\begin{bmatrix} - & + \\ - & + \end{bmatrix}, \quad 4=\begin{bmatrix} + & - \\ - & + \end{bmatrix}, \\ 5=\begin{bmatrix} - & + \\ + & - \end{bmatrix}, \quad 6=\begin{bmatrix} + & - \\ + & - \end{bmatrix}, \quad 7=\begin{bmatrix} - & - \\ + & + \end{bmatrix}, \quad 8=\begin{bmatrix} - & - \\ - & - \end{bmatrix}. \end{gathered} \tag{2.11}$$

We will refer to these eight quads as the BS-*quads*. We say that a BS-quad is *symmetric* if its two columns are the same, and otherwise we say that it is *skew*. The quads $1,2,7$, and 8 are symmetric and $3,4,5$, and 6 are skew. We say that two quads have the *same symmetry type* if they are both symmetric or both skew.

There are 4 possibilities for the central column:

$$0=\begin{bmatrix} + \\ + \end{bmatrix}, \quad 1=\begin{bmatrix} + \\ - \end{bmatrix}, \quad 2=\begin{bmatrix} - \\ + \end{bmatrix}, \quad 3=\begin{bmatrix} - \\ - \end{bmatrix}. \tag{2.12}$$

We encode the pair $(A;B)$ by the symbol sequence

$$p_1p_2\cdots p_mp_{m+1}, \tag{2.13}$$

where p_i is the label of the ith quad except in the case where n is even and $i=m+1$ in which case p_{m+1} is the label of the central column.

Similarly, we encode the pair $(C;D)$ by the symbol sequence

$$q_1q_2\cdots q_m \quad \text{respectively} \quad q_1q_2\cdots q_mq_{m+1} \tag{2.14}$$

Table 7: Equivalence classes of BS(14, 13).

	AB	CD		AB	CD		AB	CD
1	0116455	6616380	2	0116536	6645183	3	0116546	6645153
4	0116734	6641822	5	0117653	6618273	6	0117653	6618422
7	0117663	6618450	8	0118176	6618441	9	0118323	6618222
10	0118324	6618241	11	0118327	6612712	12	0118327	6614122
13	0118345	6614212	14	0123628	6645153	15	0123644	6641450
16	0123672	6614413	17	0123827	6611811	18	0131554	6618272
19	0131657	6618273	20	0131657	6618422	21	0131676	6612763
22	0131735	6641851	23	0131743	6644513	24	0131745	6645560
25	0131842	6618222	26	0132157	6641822	27	0132284	6614580
28	0132354	6614153	29	0132427	6641810	30	0132463	6618843
31	0133154	6612743	32	0133414	6618451	33	0133425	6611813
34	0134174	6641273	35	0134174	6641853	36	0134234	6641513
37	0134246	6645113	38	0134273	6612412	39	0134274	6612280
40	0134314	6641851	41	0134416	6641822	42	0136167	6636440
43	0136764	6634111	44	0137536	6631223	45	0138324	6631411
46	0161327	6441863	47	0161533	6418643	48	0161633	6414853
49	0161655	6456330	50	0161742	6418643	51	0161755	6412743
52	0161762	6418430	53	0162173	6162842	54	0162283	6415680
55	0162328	6418650	56	0162556	6451340	57	0162582	6451343
58	0162617	6451282	59	0162766	6413151	60	0163157	6168433
61	0163174	6168441	62	0163255	6164143	63	0163287	6162131
64	0163325	6168241	65	0163352	6162760	66	0163372	6414161
67	0163375	6412612	68	0163417	6414653	69	0163462	6412681
70	0163562	6451883	71	0163822	6455630	72	0163844	6451220
73	0164133	6456613	74	0164143	6168242	75	0164156	6456550
76	0164234	6164450	77	0164314	6162862	78	0164317	6162871
79	0164327	6412623	80	0164367	6164311	81	0164471	6418541
82	0164481	6411811	83	0164624	6451133	84	0165173	6142832
85	0165174	6142582	86	0165243	6141643	87	0165274	6122463
88	0165327	6142412	89	0165347	6141622	90	0165367	6142312
91	0165413	6144831	92	0165428	6142152	93	0165523	6143513
94	0165716	6143581	95	0165816	6114242	96	0165826	6114531
97	0165826	6123142	98	0166137	6128182	99	0166153	6127433
100	0166173	6128371	101	0166317	6127481	102	0166324	6124841
103	0166413	6127832	104	0166423	6141830	105	0166543	6112471
106	0167125	6142682	107	0167134	6148463	108	0167156	6434350
109	0167162	6127643	110	0167162	6142763	111	0167238	6128222
112	0167286	6122162	113	0167345	6144161	114	0167356	6141461
115	0167356	6183881	116	0167365	6144311	117	0167385	6116270
118	0167416	6144581	119	0167426	6142280	120	0167455	6431311
121	0167457	6435150	122	0167465	6141272	123	0167466	6124512
124	0167583	6112740	125	0167584	6123240	126	0167817	6112763
127	0168171	6118282	128	0168171	6143841	129	0168276	6114521
130	0168286	6114531	131	0168286	6123142	132	0168425	6143411
133	0168465	6112422	134	0168476	6123420	135	0168486	6114530
136	0171655	6416273	137	0171656	6416851	138	0171831	6416481

Table 7: Continued.

	AB	*CD*		*AB*	*CD*		*AB*	*CD*
139	0172647	6411422	140	0172656	6415223	141	0173362	6416122
142	0173413	6441863	143	0173474	6445160	144	0173521	6162781
145	0173523	6414151	146	0173554	6161242	147	0173612	6164880
148	0173612	6414863	149	0173658	6412511	150	0173744	6162760
151	0173843	6412141	152	0173843	6412710	153	0176164	6182672
154	0176421	6146273	155	0176421	6184643	156	0176424	6146511
157	0176515	6434350	158	0176526	6431511	159	0176541	6124682
160	0176542	6142280	161	0176554	6141272	162	0176557	6145313
163	0176583	6145310	164	0176636	6142172	165	0176641	6142780
166	0176744	6431413	167	0176756	6124831	168	0176834	6127172
169	0176834	6141241	170	0176843	6116413	171	0176847	6121632
172	0176865	6121730	173	0176873	6121742	174	0177663	6431413
175	0177683	6128280	176	0177686	6145161	177	0178262	6116271
178	0178283	6141231	179	0178325	6142411	180	0178346	6142420
181	0178356	6124281	182	0178365	6141440	183	0178365	6431310
184	0178377	6183422	185	0178384	6121641	186	0178683	6112422
187	0178686	6123123	188	0178737	6118273	189	0182357	6126411
190	0182652	6124842	191	0182655	6431311	192	0182657	6141222
193	0182662	6127481	194	0182663	6122863	195	0182664	6124263
196	0182665	6124712	197	0182668	6141522	198	0182761	6127773
199	0182766	6141223	200	0182883	6121763	201	0183241	6184683
202	0183262	6126473	203	0183277	6182231	204	0183521	6124681
205	0183533	6141650	206	0183553	6121641	207	0183557	6127130
208	0183624	6142283	209	0183644	6142281	210	0183657	6122481
211	0183734	6128421	212	0183734	6142240	213	0183744	6142411
214	0183753	6124411	215	0183757	6141232	216	0183767	6124233
217	0183773	6141272	218	0183774	6121642	219	0183776	6124142
220	0186553	6132311	221	0186652	6131422	222	0186774	6132312
223	0186827	6131223	224	0186847	6131222	225	0187626	6132281
226	0187667	6131422	227	0611364	1662871	228	0611453	1648653
229	0611455	1648422	230	0611546	1626832	231	0611554	1627841
232	0611645	1643851	233	0611654	1286740	234	0611686	1286320
235	0611763	1624852	236	0611764	1642733	237	0611853	1626233
238	0611853	1642523	239	0611874	1638220	240	0612286	1663480
241	0612356	1648241	242	0612457	1662171	243	0612517	1642753
244	0612528	1638242	245	0612536	1287762	246	0612547	1624820
247	0612548	1644511	248	0612556	1286422	249	0612585	1623820
250	0612586	1286420	251	0612645	1286441	252	0612646	1624730
253	0612744	1642280	254	0612752	1624282	255	0612752	1624670
256	0612753	1644131	257	0612753	1661181	258	612758	1624250
259	0612764	1286620	260	0612844	1661271	261	0612853	1626142
262	0613515	1663843	263	0613554	1628243	264	0613647	1628323
265	0613753	1628263	266	0613753	1664122	267	0614236	1664450
268	0614271	1663481	269	0614274	1662142	270	0614475	1628213
271	0614527	1622843	272	0614571	1286372	273	0614571	1286671
274	0614577	1643513	275	0614638	1286321	276	0614642	1286341
277	0614675	1623412	278	0614681	1623233	279	0614773	1643450

Table 7: Continued.

	AB	*CD*		*AB*	*CD*		*AB*	*CD*
280	0614774	1643270	281	0614784	1624250	282	0614873	1643211
283	0614884	1645130	284	0615118	1268632	285	0615146	1268462
286	0615147	1268452	287	0615471	1276443	288	0615487	1268111
289	0615627	1612342	290	0615741	1263482	291	0615743	1616242
292	0615783	1228631	293	0615784	1271660	294	0615785	1614350
295	0615874	1612341	296	0615883	1262313	297	0616234	1264760
298	0616237	1264280	299	0616281	1228632	300	0616358	1267113
301	0616472	1614551	302	0616481	1262343	303	0616481	1614232
304	0616534	1261732	305	0616841	1611822	306	0617413	1618861
307	0617426	1268423	308	0617526	1616431	309	0617556	1271642
310	0617556	1612741	311	0617572	1614651	312	0617586	1271660
313	0617625	1262433	314	0617626	1262343	315	0617626	1614232
316	0617655	1613422	317	0617682	1613640	318	0617786	1271642
319	0617786	1612741	320	0617824	1264620	321	0617824	1616231
322	0617844	1616450	323	0617845	1266212	324	0617853	1612461
325	0617853	1614222	326	0617874	1271662	327	0617884	1262450
328	0618516	1262443	329	0618517	1262433	330	0618527	1612441
331	0618557	1613450	332	0618616	1613441	333	0618625	1613441
334	0618714	1612662	335	0618748	1228631	336	0618824	1264142
337	0618825	1263172	338	0618847	1612361	339	0618874	1613241
340	0618883	1613451	341	0631147	1686422	342	0631352	1686222
343	0631458	1682421	344	0631557	1681411	345	0631778	1681411
346	0634135	1686220	347	0634187	1682321	348	0635124	1644340
349	0635132	1644270	350	0635137	1638220	351	0635138	1626240
352	0635152	1624740	353	0635311	1645860	354	0635381	1626710
355	0635441	1646161	356	0635857	1621341	357	0636178	1634410
358	0636178	1641321	359	0636882	1631443	360	0638171	1644131
361	0638171	1661181	362	0638177	1286420	363	0638188	1286620
364	0638221	1645681	365	0638237	1642620	366	0638248	1642470
367	0638281	1644141	368	0638285	1643440	369	0638372	1624821
370	0638384	1661242	371	0638427	1638221	372	0638428	1638421
373	0638457	1624420	374	0638487	1626132	375	0638522	1633822
376	0638744	1632281	377	0638747	1641272	378	0641147	1638422
379	0641163	1286372	380	0641163	1286671	381	0641264	1643430
382	0641278	1624260	383	0641361	1624842	384	0641361	1643833
385	0641451	1286372	386	0641451	1286671	387	0641457	1622453
388	0641517	1632482	389	0641712	1632783	390	0641754	1621662
391	0641867	1621370	392	0643113	1648422	393	0643412	1662713
394	0643536	1286881	395	0643545	1661421	396	0643611	1624573
397	0643814	1638221	398	0643836	1661222	399	0643842	1661271
400	0643843	1646150	401	0644124	1624570	402	0644134	1627611
403	0644145	1286240	404	0644381	1626122	405	0644813	1621263
406	0644813	1634121	407	0644823	1634141	408	0645172	1616441
409	0645387	1263821	410	0645651	1226343	411	0645857	1261243
412	0645871	1261623	413	0646123	1261373	414	0646138	1261622
415	0646387	1261233	416	0648121	1264760	417	0648175	1262282
418	0648176	1614430	419	0648225	1263482	420	0648272	1612662

Table 7: Continued.

	AB	CD		AB	CD		AB	CD
421	0648275	1614460	422	0648276	1262282	423	0648376	1264620
424	0648376	1616231	425	0648412	1264523	426	0648426	1263323
427	0648472	1264141	428	0648481	1264161	429	0648586	1226480
430	0648587	1226380	431	0648635	1226481	432	0648863	1226453
433	0655416	1126471	434	0655416	1162182	435	0655417	1126371
436	0658125	1164341	437	0658135	1166450	438	0658146	1162461
439	0658147	1162451	440	0658173	1127621	441	0658275	1126760
442	0658364	1162461	443	0658463	1127762	444	0658487	1127631
445	0661274	1127641	446	0663582	1164522	447	0663614	1163441
448	0663875	1163241	449	0663885	1163422	450	0682412	1186822
451	0685236	1128263	452	0685424	1128623	453	0685536	1162740
454	0685637	1126370	455	0685724	1162323	456	0685753	1126341
457	0685827	1162742	458	0685846	1162762	459	0685863	1126432
460	0686142	1163422	461	0686154	1122671	462	0686213	1162471
463	0686215	1163322	464	0686253	1126452	465	0686273	1126760
466	0686357	1163222	467	0686374	1164650	468	0686413	1164560
469	0686424	1162731	470	0686451	1126341	471	0687515	1162451
472	0687525	1162631	473	0687561	1126361	474	0687645	1126760
475	0688763	1164561						

when n is even respectively odd. Here q_i is the label of the ith quad for $i \leq m$, and q_{m+1} is the label of the central column (when n is odd).

3. The Equivalence Relation

We start by defining five types of elementary transformations of base sequences $\mathrm{BS}(n+1,n)$. These elementary transformations include the standard ones, as described in [3, 6]. But we also introduce one additional elementary transformation, see item (T4), which made its first appearance in [10] in the context of near-normal sequences. The quad notation was instrumental in the discovery of this new elementary operation.

The *elementary transformations* of $(A;B;C;D) \in \mathrm{BS}(n+1,n)$ are the following.

(T1) Negate one of the sequences $A;B;C;D$.

(T2) Reverse one of the sequences $A;B;C;D$.

(T3) Interchange the sequences $A;B$ or $C;D$.

(T4) Replace the pair $(C;D)$ with the pair $(\widetilde{C};\widetilde{D})$ which is defined as follows: if (2.14) is the encoding of $(C;D)$, then the encoding of $(\widetilde{C};\widetilde{D})$ is $\tau(q_1)\tau(q_2)\cdots\tau(q_m)q_{m+1}$ or $\tau(q_1)\tau(q_2)\cdots\tau(q_m)$ depending on whether n is even or odd, where τ is the transposition (45). In other words, the encoding of $(\widetilde{C};\widetilde{D})$ is obtained from that of $(C;D)$ by replacing each quad symbol 4 with the symbol 5, and vice versa. (We verify below that $N_{\widetilde{C}} + N_{\widetilde{D}} = N_C + N_D$.)

(T5) Alternate all four sequences $A;B;C;D$.

In order to justify (T4) one has to verify that $N_{\widetilde{C}} + \mathrm{N}_{\widetilde{D}} = N_C + N_D$. For that purpose let us fix two quads q_k and q_{k+i} and consider their contribution δ_i to $N_C(i) + N_D(i)$. We claim that δ_i is equal to the contribution $\widetilde{\delta}_i$ of $\tau(q_k)$ and $\tau(q_{k+i})$ to $N_{\widetilde{C}}(i) + N_{\widetilde{D}}(i)$. If neither q_k nor q_{k+i} belongs to $\{4,5\}$, then $\tau(q_k) = q_k$ and $\tau(q_{k+i}) = q_{k+i}$ and so $\delta_i = \widetilde{\delta}_i$. If $q \in \{4,5\}$, then $\tau(q)$ is the negation of q. Hence if $\{q_k, q_{k+i}\} \subseteq \{4,5\}$, then again $\delta_i = \widetilde{\delta}_i$. Otherwise, say $q_k \in \{4,5\}$ while $q_{k+i} \notin \{4,5\}$, and it is easy to verify that $\delta_i = 0 = \widetilde{\delta}_i$. The pairs q_k and $q_{n+1-k-i}$ also make a contribution to $N_C(i) + N_D(i)$, but they can be treated in the same manner. Finally, if n is odd then the pair $(C;D)$ also has a central column with label q_{m+1}. In that case, if $k = m+1-i$ and $q_k \in \{4,5\}$, the contribution of q_k and q_{m+1} to $N_C(i) + N_D(i)$ is 0. This completes the verification.

We say that two members of $\mathrm{BS}(n+1,n)$ are *equivalent* if one can be transformed to the other by applying a finite sequence of elementary transformations. One can enumerate the equivalence classes by finding suitable representatives of the classes. For that purpose we introduce the canonical form.

Definition 3.1. Let $S = (A;B;C;D) \in \mathrm{BS}(n+1,n)$ and let (2.13) respectively (2.14) be the encoding of the pair $(A;B)$ respectively $(C;D)$. One says that S is in the *canonical form* if the following eleven conditions hold.

(i) $p_1 = 3'$, $p_2 \in \{6,8\}$ for n even and $p_2 \in \{1,6\}$ for n odd.

(ii) The first symmetric quad (if any) of $(A;B)$ is 1 or 8.

(iii) If n is even and $p_i \in \{3,4,5,6\}$ for $2 \le i \le m$, then $p_{m+1} \in \{0,3\}$.

(iv) The first skew quad (if any) of $(A;B)$ is 3 or 6.

(v) $q_1 = 1$ for n even and $q_1 \in \{1,6\}$ for n odd.

(vi) The first symmetric quad (if any) of $(C;D)$ is 1.

(vii) The first skew quad (if any) of $(C;D)$ is 6.

(viii) If i is the least index such that $q_i \in \{2,7\}$, then $q_i = 2$.

(ix) If i is the least index such that $q_i \in \{4,5\}$, then $q_i = 4$.

(x) If n is odd and $q_i \in \{1,3,6,8\}$, for all $i \le m$, then $q_{m+1} \neq 2$.

(xi) If n is odd and $q_i \in \{3,4,5,6\}$, for all $i \le m$, then $q_{m+1} = 0$.

We can now prove that each equivalence class has a member which is in the canonical form. The uniqueness of this member will be proved in the next section.

Proposition 3.2. *Each equivalence class $\mathcal{E} \subseteq \mathrm{BS}(n+1,n)$ has at least one member having the canonical form.*

Proof. Let $S = (A;B;C;D) \in \mathcal{E}$ be arbitrary and let (2.13) respectively (2.14) be the encoding of $(A;B)$ respectively $(C;D)$. By applying the first three types of elementary transformations and by Theorem 2.1 we can assume that $p_1 = 3'$ and $c_1 = d_1 = +1$. By Theorem 2.1, $q_1 \in \{1,6\}$. If n is even and $q_1 = 6$ we apply the elementary transformation (T5). Thus we may assume that $p_1 = 3'$, and that condition (v) for the canonical form is satisfied. To satisfy conditions (ii) and (iii), replace B with $-B'$ (if necessary). To satisfy condition (iv), replace A with A' (if necessary).

We now modify S in order to satisfy the second part of condition (i). Note that p_2 is a BS-quad by Theorem 2.1. If $p_2 = 6$ there is nothing to do.

Assume that the quad p_2 is symmetric. By (ii), $p_2 \in \{1,8\}$. From (2.8) for $i = n-1$, we deduce that $p_2 = 8$ if $q_1 = 1$ and $p_2 = 1$ if $q_1 = 6$. Note that if n is even, then $q_1 = 1$ by (v). If

n is odd and $q_1 = 1$, we switch A and B and apply the elementary transformation (T5). After this change we still have $p_1 = 3'$, $q_1 = 1$, conditions (ii), (iii), and (iv) remain satisfied, and moreover $p_2 = 6$.

Now assume that p_2 is skew. In view of (iv), we may assume that $p_2 = 3$. Then the argument above based on (2.8) shows that $q_1 = 6$, and so n must be odd. After applying the elementary transformation (T5), we obtain that $p_2 = 1$. Hence condition (i) is fully satisfied.

To satisfy (vi), in view of (v) we may assume that n is odd and $q_1 = 6$. If the first symmetric quad in $(C; D)$ is 2 respectively 7, we reverse and negate C respectively D. If it is 8, we reverse and negate both C and D. Now the first symmetric quad will be 1.

To satisfy (vii), (if necessary) reverse C, D, or both. To satisfy (viii), (if necessary) interchange C and D. Note that in this process we do not violate the previously established properties. To satisfy (ix), (if necessary) apply the elementary transformation (T4). To satisfy (x), switch C and D (if necessary). To satisfy (xi), (if necessary) replace C with $-C'$, D with $-D'$, or both.

Hence S is now in the canonical form. □

4. The Symmetry Group of $\mathrm{BS}(n+1, n)$

We will construct a group G_{BS} of order 2^{12} which acts on $\mathrm{BS}(n+1, n)$. Our (redundant) generating set for G_{BS} will consist of 12 involutions. Each of these generators is an elementary transformation, and we use this information to construct G_{BS}, that is, to impose the defining relations. We denote by $S = (A; B; C; D)$ an aritrary member of $\mathrm{BS}(n+1, n)$.

To construct G_{BS}, we start with an elementary abelian group E of order 2^8 with generators ν_i, ρ_i, $i \in \{1, 2, 3, 4\}$. It acts on $\mathrm{BS}(n+1, n)$ as follows:

$$\begin{aligned} \nu_1 S &= (-A; B; C; D), & \rho_1 S &= (A'; B; C; D), \\ \nu_2 S &= (A; -B; C; D), & \rho_2 S &= (A; B'; C; D), \\ \nu_3 S &= (A; B; -C; D), & \rho_3 S &= (A; B; C'; D), \\ \nu_4 S &= (A; B; C; -D), & \rho_4 S &= (A; B; C; D'), \end{aligned} \tag{4.1}$$

that is, ν_i negates the ith sequence of S and ρ_i reverses it.

Next we introduce two commuting involutory generators σ_1 and σ_2. We declare that σ_1 commutes with ν_3, ν_4, ρ_3, and ρ_4, and σ_2 commutes with ν_1, ν_2, ρ_1, and ρ_2 and that

$$\sigma_1\nu_1 = \nu_2\sigma_1, \qquad \sigma_1\rho_1 = \rho_2\sigma_1, \qquad \sigma_2\nu_3 = \nu_4\sigma_2, \qquad \sigma_2\rho_3 = \rho_4\sigma_2. \tag{4.2}$$

The group $H = \langle E, \sigma_1, \sigma_2 \rangle$ is the direct product of two isomorphic groups of order 32:

$$H_1 = \langle \nu_1, \rho_1, \sigma_1 \rangle, \qquad H_2 = \langle \nu_3, \rho_3, \sigma_2 \rangle. \tag{4.3}$$

The action of E on $\mathrm{BS}(n+1, n)$ extends to H by defining

$$\sigma_1 S = (B; A; C; D), \qquad \sigma_2 S = (A; B; D; C). \tag{4.4}$$

We add a new generator θ which commutes elementwise with H_1, commutes with $\nu_3\rho_3, \nu_4\rho_4$, and σ_2, and satisfies $\theta\rho_3 = \rho_4\theta$. Let us denote this enlarged group by $\widetilde{H}$. It has the direct product decomposition

$$\widetilde{H} = \langle H, \theta \rangle = H_1 \times \widetilde{H}_2, \tag{4.5}$$

where the second factor is itself direct product of two copies of the dihedral group D_8 of order 8:

$$\widetilde{H}_2 = \langle \rho_3, \rho_4, \theta \rangle \times \langle \nu_3\rho_3, \nu_4\rho_4, \theta\sigma_2 \rangle. \tag{4.6}$$

The action of H on $\mathrm{BS}(n+1, n)$ extends to $\widetilde{H}$ by letting θ act as the elementary transformation (T4).

Finally, we define G_{BS} as the semidirect product of $\widetilde{H}$ and the group of order 2 with generator α. By definition, α commutes with each ν_i and satisfies

$$\begin{aligned} \alpha\rho_i\alpha &= \rho_i\nu_i^n, \quad i = 1, 2; \\ \alpha\rho_j\alpha &= \rho_j\nu_j^{n-1}, \quad j = 3, 4; \\ \alpha\theta\alpha &= \theta\sigma_2^{n-1}. \end{aligned} \tag{4.7}$$

The action of $\widetilde{H}$ on $\mathrm{BS}(n+1, n)$ extends to G_{BS} by letting α act as the elementary transformation (T5), that is, we have

$$\alpha S = (A^*; B^*; C^*; D^*). \tag{4.8}$$

We point out that the definition of the subgroup $\widetilde{H}$ is independent of n, and its action on $\mathrm{BS}(n+1, n)$ has a quadwise character. By this we mean that the value of a particular quad, say p_i, of $S \in \mathrm{BS}(n+1, n)$ and $h \in \widetilde{H}$ determine uniquely the quad p_i of hS. In other words $\widetilde{H}$ acts on the Golay quads, the BS-quads, and the set of central columns such that the encoding of hS is given by the symbol sequences

$$h(p_1)h(p_2)\cdots h(p_{m+1}), \qquad h(q_1)h(q_2)\cdots. \tag{4.9}$$

On the other hand the full group G_{BS} has neither of these two properties.

An important feature of the action of $\widetilde{H}$ on the BS-quads is that it preserves the symmetry type of the quads.

The following proposition follows immediately from the construction of G_{BS} and the description of its action on $\mathrm{BS}(n+1, n)$.

Proposition 4.1. *The orbits of G_{BS} in $\mathrm{BS}(n+1, n)$ are the same as the equivalence classes.*

The main tool that we use to enumerate the equivalence classes of $\mathrm{BS}(n+1, n)$ is the following theorem.

Theorem 4.2. *For each equivalence class $\mathcal{E} \subseteq \mathrm{BS}(n+1,n)$ there is a unique $S = (A;B;C;D) \in \mathcal{E}$ having the canonical form.*

Proof. In view of Proposition 3.2, we just have to prove the uniqueness assertion. Let

$$S^{(k)} = \left(A^{(k)}; B^{(k)}; C^{(k)}; D^{(k)}\right) \in \mathcal{E}, \quad (k = 1,2) \tag{4.10}$$

be in the canonical form. We have to prove that in fact $S^{(1)} = S^{(2)}$.

By Proposition 4.1, we have $gS^{(1)} = S^{(2)}$ for some $g \in G_{\mathrm{BS}}$. We can write g as $g = \alpha^s h$ where $s \in \{0,1\}$ and $h \in \widetilde{H}$. Let $p_1^{(k)} p_2^{(k)} \cdots p_{m+1}^{(k)}$ be the encoding of the pair $(A^{(k)}; B^{(k)})$ and $q_1^{(k)} q_2^{(k)} \cdots$ the encoding of the pair $(C^{(k)}; D^{(k)})$. The symbols (i-xi) will refer to the corresponding conditions of Definition 3.1. Observe that $p_1^{(1)} = p_1^{(2)} = 3'$ by (i).

We prove first preliminary claims (a-c).

(a) $q_1^{(1)} = q_1^{(2)}$.

For n even see (v). Let n be odd. When we apply the generator α to any $S \in \mathrm{BS}(n+1,n)$, we do not change the first quad of $(C;D)$. It follows that the quads $q_1^{(1)}$ and $q_1^{(2)}$ have the same symmetry type. The claim now follows from (v).

(b) $g \in \widetilde{H}$, that is, $s = 0$.

Assume first that n is even. By (v), $q_1^{(1)} = q_1^{(2)} = 1$. For any $S \in \mathcal{E}$ the first quad of $(C;D)$ in S and the one in αS have different symmetry types. As quad $h(1)$ is symmetric, the equality $\alpha^s h S^{(1)} = S^{(2)}$ forces s to be 0. Assume now that n is odd. Then for any $S \in \mathcal{E}$ the second quad of $(A;B)$ in S and the one in αS have different symmetry types. Recall that $q_1^{(1)} = q_1^{(2)} \in \{1,6\}$ and that (see (i)) $p_2^{(1)}$ and $p_2^{(2)}$ belong to $\{1,6\}$. From (2.8), with $i = n-1$, we deduce that $p_2^{(k)} \neq q_1^{(k)}$ for $k = 1,2$. We conclude that $p_2^{(1)} = p_2^{(2)}$. The claim now follows from the fact that h preserves while α alters the symmetry type of the quad p_2.

As an immediate consequence of (b), we point out that a quad $p_i^{(1)}$ is symmetric if and only if $p_i^{(2)}$ is, and the same is true for the quads $q_i^{(1)}$ and $q_i^{(2)}$.

(c) $p_2^{(1)} = p_2^{(2)}$.

This was already proved above in the case when n is odd. In general, the claim follows from (b) and the equality $h(p_2^{(1)}) = p_2^{(2)}$. Observe that each of the sets $\{6,8\}$ and $\{1,6\}$ consists of one symmetric and one skew quad and that h preserves the symmetry type of quads.

Recall that $\widetilde{H} = H_1 \times \widetilde{H}_2$. Since $s = 0$, we have $g = h = h_1 h_2$ with $h_1 \in H_1$ and $h_2 \in \widetilde{H}_2$. Consequently, $h_1(p_i^{(1)}) = p_i^{(2)}$ and $h_2(q_i^{(1)}) = q_i^{(2)}$ for all i's.

We will now prove that $A^{(1)} = A^{(2)}$ and $B^{(1)} = B^{(2)}$. Since $p_1^{(1)} = p_1^{(2)} = 3'$, the equality $h_1(p_1^{(1)}) = p_1^{(2)}$ implies that $h_1(3') = 3'$. Thus $h_1 = \rho_1^e (\nu_2 \rho_2)^f$ for some $e, f \in \{0,1\}$.

Assume first that $p_2^{(1)}$ is symmetric. By (ii), $p_2^{(1)} \in \{1,8\}$. Then $h_1(p_2^{(1)}) = p_2^{(2)} = p_2^{(1)}$ implies that $f = 0$. Hence, $h_1 = \rho_1^e$ and so $B^{(1)} = B^{(2)}$. If all quads $p_i^{(1)}$, $i \neq 1$, are symmetric, then also $A^{(2)} = h_1 A^{(1)} = A^{(1)}$. Otherwise let i be the least index for which the quad $p_i^{(1)}$ is skew. Since $B^{(1)} = B^{(2)}$ and $p_i^{(1)}$ is 3 or 6 (see (iv)), we infer that $e = 0$. Hence $h_1 = 1$ and so $A^{(1)} = A^{(2)}$.

Now assume that $p_2^{(1)}$ is skew. By (ii), $p_2^{(1)} = 6$. Then $h_1(p_2^{(1)}) = p_2^{(2)} = p_2^{(1)}$ implies that $e = 0$. Thus $h_1 = (\nu_2\rho_2)^f$ and so $A^{(1)} = A^{(2)}$. If all quads $p_i^{(1)}$, $i \neq 1$, are skew, then by invoking condition (iii) we deduce that $f = 0$ and so $B^{(1)} = B^{(2)}$. Otherwise let i be the least index for which the quad $p_i^{(1)}$ is symmetric. Since $A^{(1)} = A^{(2)}$ and $p_i^{(1)}$ is 1 or 8 (see (ii)), we infer that $f = 0$. Hence $h_1 = 1$ and so $B^{(1)} = B^{(2)}$.

It remains to prove that $C^{(1)} = C^{(2)}$ and $D^{(1)} = D^{(2)}$. We set $Q = \{q_i^{(1)} : 1 \leq i \leq m\}$. By (v) and claim (a) we have $q_1^{(1)} = q_1^{(2)} \in \{1, 6\}$.

We first consider the case $q_1^{(1)} = q_1^{(2)} = 6$ which occurs only for n odd. Then $h_2(6) = 6$ and so $h_2 \in \langle \nu_3\rho_3, \nu_4\rho_4, \theta, \sigma_2 \rangle$. It follows that $h_2(3) = 3$.

If some $q \in Q$ is symmetric, let i be the least index such that $q_i^{(1)}$ is symmetric. Then (vi) implies that $q_i^{(1)} = q_i^{(2)} = 1$. Thus h_2 must fix the quad 1. As the stabilizer of the quad 1 in $\langle \nu_3\rho_3, \nu_4\rho_4, \theta, \sigma_2 \rangle$ is $\langle \theta, \sigma_2 \rangle$, we infer that h_2 must also fix the quad 8. Similarly, if $2 \in Q$, then (viii) implies that h_2 fixes 2 and 7. If $4 \in Q$, then (ix) implies that h_2 fixes 4 and 5. These facts imply that h_2 fixes all quads in Q, that is, $q_i^{(1)} = q_i^{(2)}$ for all $i \leq m$. It remains to show that, for odd n, $q_{m+1}^{(1)} = q_{m+1}^{(2)}$. If $Q \subseteq \{3, 4, 5, 6\}$, this follows from (xi). Otherwise Q contains a symmetric quad and so $h_2 \in \langle \theta, \sigma_2 \rangle$. If $Q \not\subseteq \{1, 3, 6, 8\}$ then Q contains one of the quads 2, 4, 5, or 7. Since h_2 fixes all quads in Q, we infer that $h_2 \in \langle \theta \rangle$, and so $q_{m+1}^{(1)} = q_{m+1}^{(2)}$. If $Q \subseteq \{1, 3, 6, 8\}$, the equality $q_{m+1}^{(1)} = q_{m+1}^{(2)}$ follows from (x).

Finally, we consider the case $q_1^{(1)} = q_1^{(2)} = 1$. Since $h_2(q_1^{(1)}) = q_1^{(2)}$, $h_2 \in \langle \rho_3, \rho_4, \theta, \sigma_2 \rangle$. Hence h_2 fixes the quads 1 and 8.

If some $q \in Q$ is skew, then (vii) implies that h_2 fixes the quads 3 and 6. If $2 \in Q$, then (viii) implies that h_2 fixes the quads 2 and 7. If $4 \in Q$, then (ix) implies that h_2 fixes the quads 4 and 5. These facts imply that h_2 fixes all quads in Q. If n is odd, then we invoke conditions (x) and (xi) to conclude that h_2 also fixes the central column of $(C^{(1)}; D^{(1)})$. Hence $C^{(1)} = C^{(2)}$ and $D^{(1)} = D^{(2)}$ also in this case. □

5. Representatives of the Equivalence Classes

We have computed a set of representatives for the equivalence classes of base sequences $BS(n+1, n)$ for all $n \leq 30$. Due to their excessive size, we tabulate these sets only for $n \leq 13$. Each representative is given in the canonical form which is made compact by using our standard encoding. The encoding is explained in detail in Section 2.

As an example, the base sequences

$$\begin{aligned} A &= +,+,+,+,-,-,+,-,+, \\ B &= +,+,+,-,+,+,+,-,-, \\ C &= +,+,-,-,+,-,-,+, \\ D &= +,+,+,+,-,+,-,+, \end{aligned} \tag{5.1}$$

are encoded as $3'6142$, 1675. In the tables we write 0 instead of $3'$. This convention was used in our previous papers on this and related topics.

This compact notation is used primarily in order to save space, but also to avoid introducing errors during decoding. For each n, the representatives are listed in the lexicographic order of the symbol sequences (2.13) and (2.14).

In Table 2 we list the codes for the representatives of the equivalence classes of $\mathrm{BS}(n+1,n)$ for $n \leq 8$. This table also records the values $N_X(k)$ of the nonperiodic autocorrelation functions for $X \in \{A,B,C,D\}$ and $k \geq 0$. For instance let us consider the first item in the list of base sequences $\mathrm{BS}(8,7)$ given in Table 2. The base sequences are encoded in the first column as 0165, 6123. The first part 0165 encodes the pair $(A;B)$, and the second part 6123 encodes the pair $(C;D)$. The function N_A, at the points $0,1,\ldots,7$, takes the values $8,-1,2,-1,0,1,2$, and 1 listed in the second column. Just below these values one finds the values of N_B at the same points. In the third column we list likewise the values of N_C and N_D at the points $i=0,1,\ldots,6$.

Tables 3, 4, 5, 6, and 7 contain only the list of codes for the representatives of the equivalence classes of $\mathrm{BS}(n+1,n)$ for $9 \leq n \leq 13$.

Let us say that the base sequences $S=(A;B;C;D) \in \mathrm{BS}(n+1,n)$ are *normal* respectively *near-normal* if $b_i = a_i$ respectively $b_i = (-1)^{i-1} a_i$ for all $i \leq n$. We denote by $NS(n)$ respectively $NN(n)$ the set of all normal respectively near-normal sequences in $\mathrm{BS}(n+1,n)$. Let us say also that an equivalence class $\mathcal{E} \subseteq \mathrm{BS}(n+1,n)$ is *normal* respectively *near-normal* if $\mathcal{E} \cap NS(n)$ respectively $\mathcal{E} \cap NN(n)$ is not void. Our canonical form has been designed so that if $\mathcal{E}$ is normal, then its canonical representative S belongs to $NS(n)$. The analogous statement for near-normal classes is false. It is not hard to recognize which representatives S in our tables are normal sequences. Let (2.13) be the encoding of the pair $(A;B)$. Then $S \in NS(n)$ if and only if all the quads p_i, $i \neq 1$, belong to $\{1,3,6,8\}$, and, in the case when $n=2m$ is even, the central column symbol p_{m+1} is 0 or 3.

It is an interesting question to find the necessary and sufficient conditions for two binary sequences to have the same norm. The group of order four generated by the negation and reversal operations acts on binary sequences. We say that two binary sequences are *equivalent* if they belong to the same orbit. Note that the equivalent binary sequences have the same norm. However the converse is false. Here is a counter-example which occurs in Table 2 for the case $n=8$. The base sequences 15 and 16 differ only in their first sequences, which we denote here by U and V, respectively:

$$\begin{aligned} U &= + + - - - + - - +, \\ V &= + + - + + - - - +. \end{aligned} \tag{5.2}$$

Their associated polynomials are

$$\begin{aligned} U(x) &= 1 + x - x^2 - x^3 - x^4 + x^5 - x^6 - x^7 + x^8, \\ V(x) &= 1 + x - x^2 + x^3 + x^4 - x^5 - x^6 - x^7 + x^8. \end{aligned} \tag{5.3}$$

It is obvious that U and V are not equivalent in the above sense. On the other hand, from the factorizations $U(x) = p(x)q(x)$ and $V(x) = p(x)r(x)$, where $p(x) = 1+x-x^2$, $q(x) = 1-x^3-x^6$, and $r(x) = 1+x^3-x^6 = -x^6 q(x^{-1})$, we deduce immediately that $N_U(x) = N_V(x)$.

This counter-example can be easily generalized. Let us define *binary polynomials* as polynomials associated to binary sequences. If $f(x)$ is a polynomial of degree d with $f(0) \neq 0$, we define its *dual* polynomial f^* by $f^*(x) = x^d f(x^{-1})$. Then for any positive integer k we have $f^*(x^k) = f(x^k)^*$, that is, $f^*(x^k) = g^*(x)$ where g^* is the dual of the polynomial $f(x^k)$. In general we can start with any number of binary sequences, but here we take only three of them: A; B; C of lengths m, n, k, respectively. From the associated binary polynomials $A(x), B(x)$, and $C(x)$ we can form several binary polynomials of degree $mnk - 1$. The basic one is $A(x)B(x^m)C(x^{mn})$. The others are obtained from this one by replacing one or more of the three factors by their duals. It is immediate that the binary sequences corresponding to these binary polynomials all have the same norm. In general many of these sequences will not be equivalent. However, note that if we replace all three factors with their duals, we will obtain a binary sequence equivalent to the basic one.

Acknowledgments

The author has pleasure to thank the three anonymous referees for their valuable comments. The author is grateful to NSERC for the continuing support of his research. This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET: http://www.sharcnet.ca) and Compute/Calcul Canada.

References

[1] H. Kharaghani and C. Koukouvinos, "Complementary, base and Turyn sequences," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz, Eds., pp. 317–321, CRC Press, Boca Raton, Fla, USA, 2nd edition, 2007.
[2] S. Kounias and K. Sotirakoglou, "Construction of orthogonal sequences," in *Proceedings of the 14th Greek Statistics Conference*, pp. 229–236, Skiathos, Greece, 2001.
[3] C. Koukouvinos, S. Kounias, and K. Sotirakoglou, "On base and Turyn sequences," *Mathematics of Computation*, vol. 55, no. 192, pp. 825–837, 1990.
[4] C. H. Yang, "On composition of four-symbol δ-codes and Hadamard matrices," *Proceedings of the American Mathematical Society*, vol. 107, no. 3, pp. 763–776, 1989.
[5] J. Seberry and M. Yamada, "Hadamard matrices, sequences, and block designs," in *Contemporary Design Theory*, Wiley-Intersci. Ser. Discrete Math. Optim., pp. 431–560, Wiley, New York, NY, USA, 1992.
[6] D. Ž. Đoković, "Aperiodic complementary quadruples of binary sequences," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 27, pp. 3–31, 1998, Correction: ibid 30, p. 254, 1999.
[7] D. Ž. Đoković, "On the base sequence conjecture," *Discrete Mathematics*, vol. 310, pp. 1956–1964, 2010.
[8] C. Koukouvinos, S. Kounias, J. Seberry, C. H. Yang, and J. Yang, "Multiplication of sequences with zero autocorrelation," *The Australasian Journal of Combinatorics*, vol. 10, pp. 5–15, 1994.
[9] D. Ž. Đoković, "Equivalence classes and representatives of Golay sequences," *Discrete Mathematics*, vol. 189, no. 1–3, pp. 79–93, 1998.
[10] D. Ž. Đoković, "Classification of near-normal sequences," *Discrete Mathematics, Algorithms and Applications*, vol. 1, no. 3, pp. 389–399, 2009.