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Research Article

On the Existence of Infinite Size Costas Arrays Configurations of Nonattacking Queens on the Chessboard

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A procedure for the construction of Costas arrays of infinite size representing configurations on non-attacking queens on the chessboard is presented.

1. Introduction

Costas arrays [1] are square arrays of dots/1s and blanks/0s, such that there exists exactly one dot per row and column, and so that (a) no four dots form a parallelogram and (b) no three dots lying on a straight line are equidistant. They were first developed by Costas in 1965 in the context of SONAR detection ([2], and later in 1984 [3] as a journal publication), while Golomb, based on previously available empirical constructions, stated and proved in 1984 [4, 5] two algebraic construction methods based on finite fields, known as the Golomb and Welch methods, still the only ones available today, which work for infinitely many, but not all, sizes. It was then that Costas arrays acquired their present name and became an object of mathematical study.

In [6], in connection with the problem of determining the distance vectors between dots present in Costas arrays, the existence of Costas arrays representing configurations of nonattacking kings and queens on the chessboard, named Nonattacking Kings Costas Arrays (NAKCAs) and Nonattacking Queens Costas Arrays (NAQCAs), respectively, was considered. The conclusion was that NAKCAs exist and can even be systematically constructed for infinitely many sizes, provided a certain number-theoretic/algebraic conjecture holds true, while no known Costas array (i.e., either of size $n \leq 27$ [7] or algebraically constructed of any size, or one of the four Costas arrays presented in [8]) is a NAQCA. Thus, if finite size NAQCAs exist, they will necessarily have to be sporadic Costas arrays.

This work, after reviewing the basics on Costas arrays, proceeds to present a procedural constructive proof of the existence of NAQCAs of infinite size. The existence of NAQCAs of finite size remains an open problem.

2. Basics

Let $[n] := \{1, \ldots, n\}$, $n \in \mathbb{N}^*$. A Costas array/permutation of finite size is defined as follows.

Definition 2.1. Let $f : [n] \to [n]$, $n \in \mathbb{N}$, be a bijection; then f is a Costas permutation of size n if and only if the collection of vectors $\{(i-j, f(i)-f(j)) : 1 \leq j < i \leq n\}$, called the distance vectors, are all distinct. Equivalently, f is a Costas permutation if and only if

$$\forall i, j, k \in [n] : i+k, j+k \in [n], \quad f(i+k) - f(i) = f(j+k) - f(j) \Longrightarrow i = j. \tag{2.1}$$

The corresponding *Costas array* A_f is the square $n \times n$ array where the elements at $(f(i), i)$, $i \in [n]$, are equal to 1 (dots), while the remaining elements are equal to 0 (blanks):

$$A_f = [a_{ij}] = \begin{cases} 1 & \text{if } i = f(j), \\ 0 & \text{otherwise}, \end{cases} \quad j \in [n]. \tag{2.2}$$

For Costas arrays/permutations of infinite size, the definition is slightly modified.

Definition 2.2. Let $f : \mathbb{S}_1 \to \mathbb{S}_2$, where $\mathbb{S}_1, \mathbb{S}_2 \in \{\mathbb{N}, \mathbb{Z}\}$ is a bijection; then f is a Costas permutation if and only if the collection of vectors $\{(i-j, f(i)-f(j)) : j < i\}$, called the distance vectors, are all distinct. Equivalently, f is a Costas permutation if and only if

$$\forall i, j \in \mathbb{S}_1,\ k \in \mathbb{N}^*, \quad f(i+k) - f(i) = f(j+k) - f(j) \Longrightarrow i = j. \tag{2.3}$$

The corresponding *Costas array* A_f is the infinite array where the elements at $(f(i), i)$, $i \in \mathbb{S}_1$, are equal to 1 (dots), while the remaining elements are equal to 0 (blanks):

$$A_f = [a_{ij}] = \begin{cases} 1 & \text{if } i = f(j), \\ 0 & \text{otherwise}, \end{cases} \quad j \in \mathbb{S}_1. \tag{2.4}$$

There exist two algebraic construction techniques for Costas arrays, based on the theory of finite fields, known as the Golomb and Welch methods [1, 4, 5], which produce Costas arrays for infinitely many, though not all, sizes. Costas arrays not obtained by either of these two methods are known as *sporadic Costas arrays*.

3. NAQCAs of Infinite Size

NAQCAs of infinite size exist in abundance. A procedural construction is described below, where, starting with an empty array, dots are added to its rows and columns one at a time, so that, eventually, an NAQCA is formed after countably infinitely many steps. The infinite size of the array guarantees that, at each step, any column or row that does not contain a dot yet has finitely many forbidden positions, so there are infinitely many available positions for placing the dot therein.

Theorem 3.1. *NAQCAs of infinite size exist.*

Proof. The proof is procedural and begins with an array $A = [a_{ij}]$, $i \in \mathbb{S}_1$, $j \in \mathbb{S}_2$, $\mathbb{S}_1, \mathbb{S}_2 \in \{\mathbb{N}, \mathbb{Z}\}$, whose elements are valueless, namely neither 0 nor 1. The steps of the procedure are the following.

(1) *Initialization*. Set $a_{00} \leftarrow 1$; set $a_{ij} \leftarrow 0$ for all pairs (i, j) such that either $i \neq 0$, $j = 0$, or $j \neq 0$, $i = 0$, or $i = j \neq 0$, or $i = -j \neq 0$.

(2) *Positive Row*. Find the smallest $i_0 > 0$ for which for all j, $a_{i_0 j} \neq 1$; choose randomly a j for which $a_{i_0 j}$ is valueless, say j_0, and set $a_{i_0 j_0} \leftarrow 1$; set $a_{i-i_0, j-j_0} \leftarrow 0$ for all (i, j) such that either $i \neq i_0$, $j = j_0$, or $j \neq j_0$, $i = i_0$, or $i - i_0 = j - j_0 \neq 0$, or $i - i_0 = -(j - j_0) \neq 0$. Set $a_{ij} \leftarrow 0$ for all (i, j) such that the distance vector $\operatorname{sgn}(i - i_0)(i - i_0, j - j_0)$ already appears in the construction so far between elements equal to 1.

(3) *Positive Column*. Find the smallest $j_0 > 0$ for which for all i, $a_{ij_0} \neq 1$; choose randomly an i for which a_{ij_0} is valueless, say i_0, and set $a_{i_0 j_0} \leftarrow 1$; set $a_{i-i_0, j-j_0} \leftarrow 0$ for all (i, j) such that either $i \neq i_0$, $j = j_0$, or $j \neq j_0$, $i = i_0$, or $i - i_0 = j - j_0 \neq 0$, or $i - i_0 = -(j - j_0) \neq 0$. Set $a_{ij} \leftarrow 0$ for all (i, j) such that the distance vector $\operatorname{sgn}(i - i_0)(i - i_0, j - j_0)$ already appears in the construction so far between elements equal to 1.

(4) *Negative Row, Applicable If and Only If* $\mathbb{S}_1 = \mathbb{Z}$. Find the largest $i_0 < 0$ for which for all j, $a_{i_0 j} \neq 1$; choose randomly a j for which $a_{i_0 j}$ is valueless, say j_0, and set $a_{i_0 j_0} \leftarrow 1$; set $a_{i-i_0, j-j_0} \leftarrow 0$ for all (i, j) such that either $i \neq i_0$, $j = j_0$, or $j \neq j_0$, $i = i_0$, or $i - i_0 = j - j_0 \neq 0$, or $i - i_0 = -(j - j_0) \neq 0$. Set $a_{ij} \leftarrow 0$ for all (i, j) such that the distance vector $\operatorname{sgn}(i - i_0)(i - i_0, j - j_0)$ already appears in the construction so far between elements equal to 1.

(5) *Negative Column, Applicable If and Only If* $\mathbb{S}_2 = \mathbb{Z}$. Find the largest $j_0 < 0$ for which for all i, $a_{ij_0} \neq 1$; choose randomly an i for which a_{ij_0} is valueless, say i_0, and set $a_{i_0 j_0} \leftarrow 1$; set $a_{i-i_0, j-j_0} \leftarrow 0$ for all (i, j) such that either $i \neq i_0$, $j = j_0$, or $j \neq j_0$, $i = i_0$, or $i - i_0 = j - j_0 \neq 0$, or $i - i_0 = -(j - j_0) \neq 0$. Set $a_{ij} \leftarrow 0$ for all (i, j) such that the distance vector $\operatorname{sgn}(i - i_0)(i - i_0, j - j_0)$ already appears in the construction so far between elements equal to 1.

For $\mathbb{S}_1 = \mathbb{S}_2 = \mathbb{N}$, steps are performed in the order 1232323..., for $\mathbb{S}_1 = \mathbb{Z}$, $\mathbb{S}_2 = \mathbb{N}$ in the order 1234234..., for $\mathbb{S}_1 = \mathbb{N}$, $\mathbb{S}_2 = \mathbb{Z}$ in the order 1235235..., and, finally, for $\mathbb{S}_1 = \mathbb{S}_2 = \mathbb{Z}$ in the order 123452345....

Observe that, once a value is assigned, the procedure does not alter it any further. To check the procedure for correctness, it remains to be shown that

(i) it produces a permutation;

(ii) it retains the nonattacking queens property;

(iii) it retains the Costas property.

This is done inductively. To begin with, initialization certainly retains both the nonattacking queens and the Costas property. Assume now that, up to a certain step, each row and column has at most one element equal to 1 and that both the nonattacking queens and Costas properties are preserved.

(i) As each step assigns exactly one value of 1, at that point finitely many rows and columns will contain a value of 1, so that any of steps (2)–(5) is allowed to follow at this point (the procedure "does not get stuck"), and a new value of 1 is indeed assigned.

(ii) The new value of 1 cannot possibly be assigned to a row or a column that already contains a value of 1, as 1s get explicitly assigned by the procedure to rows/columns that contain no value of 1. So, it is always true that each row and column has at most one element equal to 1.

(iii) Any given row or column can remain without an element equal to one, namely, full of either unassigned elements or elements equal to 0, for a finite number of steps only. This is because, at each step, finitely many elements of a row or column are 0s, and, since for row i/column j it holds true that $i-1$ rows/$j-1$ columns lie between themselves and row 0/column 0, row i/column j is bound to acquire an element equal to 1 after at most i/j runs of the procedure step cycle 23(4)(5), and at that point, all remaining valueless elements of this row/column will become equal to 0.

(iv) The Costas and nonattacking queens properties are preserved, by the very definitions of the steps of the procedure. Note that the Costas property constraint assigns finitely many values of 0 at the end of each step while the nonattacking queens constraint assigns values of 0 on four "rays" (the row, the column, and the two diagonals) emanating from the point where a 1 was assigned, so that every row or column of the array not containing this specific point gets at most three values of 0 assigned by this constraint on every step.

This concludes the proof. □

Note that the description of the procedure in the proof above is unnecessarily restrictive. In reality, the crucial feature is that each row and column gets exactly one value of 1, so that a bijection is obtained; the two properties (Costas and nonattacking queens) are satisfied by default thanks to the definition of the steps. Therefore, the first dot can be placed anywhere, while, subsequently, steps can be applied in an arbitrary order, as long as any two consecutive applications of any particular step lie finitely many steps apart.

Note also that the term "procedure" has been used above to characterize the proposed construction technique, instead of the more common perhaps term "algorithm." This was a deliberate decision, in order to respect the fact that an "algorithm" is usually defined as a *finite* sequence of actions (or instructions, or steps...) leading to the solution of a problem. The procedure described above involves a countably infinite number of steps and lacks an explicit stopping condition; hence it should not be characterized as an "algorithm."

4. Conclusion

A procedural and constructive proof of the existence of Costas arrays of infinite size representing configurations on nonattacking queens on the chessboard was presented. The existence of such arrays in finite sizes remains an open problem: the techniques used in this work rely crucially on the infinity of the size and do not readily generalize to finite sizes.

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