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Research Article

The Structure of Reduced Sudoku Grids and the Sudoku Symmetry Group

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A Sudoku grid is a constrained Latin square. In this paper a reduced Sudoku grid is described, the properties of which differ, through necessity, from that of a reduced Latin square. The Sudoku symmetry group is presented and applied to determine a mathematical relationship between the number of reduced Sudoku grids and the total number of Sudoku grids for any size. This relationship simplifies the enumeration of Sudoku grids and an example of the use of this method is given.

1. Introduction

A Sudoku grid, $S^{x,y}$, is a $n \times n$ array subdivided into n minigrids of size $x \times y$ (where $n = xy$). $S^{x,y}$ consists of y *bands*, each composed of x horizontally consecutive minigrids, and x *stacks*, each composed of y vertically consecutive minigrids. Each $x \times y$ minigrid possesses x subrows, or *tiers* and y subcolumns, or *pillars*. The values $1, \ldots, n$ are contained within the array in such a way that each value occurs exactly once in every *row*, *column*, and *minigrid*. The discussion that follows requires analysis of the enumeration of Sudoku grids and therefore the following notation is required: $S^{x,y}(n)$ is the number of ways of arranging the values in $S^{x,y}$.

A Sudoku grid is a constrained Latin square. The calculation of the number of Latin squares, of some dimension, is greatly simplified by only counting those Latin squares which are in *reduced* form. A Latin square, or rectangle, is reduced if the values in the first row and column are in the natural order [1]. There are far fewer reduced Latin squares than Latin squares, and there exists a mathematical relationship allowing a direct calculation of the number of the latter from the number of the former. If the first row and first column of a Sudoku grid are in the natural order there is no longer a direct mathematical relationship between the number of these and the number of Sudoku grids. Therefore reduced Sudoku grids, defined in this paper, have different properties to reduced Latin squares.

In [2–5] the number of reduced Latin squares (for sizes 8 to 11) is calculated. In a recent article, Stones [6] surveys some well-known and some more recent formulae for Latin rectangles, their usefulness, and means to obtain approximate numbers. If $l(k,n)$ is the number of reduced Latin rectangles of size $k \times n$ then the total number of Latin rectangles, $L(k,n)$, [1] is as follows:

$$L(k,n) = n!(n-1)!\frac{l(k,n)}{(n-k)!}. \tag{1.1}$$

A similar relationship is developed here between the total number of Sudoku grids and the number of reduced Sudoku grids. Such relationships have previously been given for "NRC-Sudoku" [7] and "2-Quasi-Magic Sudoku" [8] where the focus is the symmetry groups for these structures; symmetry groups have been defined for $S^{3,2}$ [9], $S^{3,3}$ [10], and the symmetry group of $S^{n,n}$ is given in this paper.

2. The Sudoku Symmetry Group

The Sudoku symmetry group, $\mathfrak{S}$, (containing symmetry operations applicable to $S^{x,y}$) consists of all homomorphisms of the structure of a Sudoku grid. An element of the symmetry group when applied to a Sudoku grid preserves the underlying structure of that grid (including the relationship between the values) while permuting the values contained within the grid. An element of the symmetry group α operating on a Sudoku grid s is written α(s). For the symmetry group, identity i is defined as $i(s) = s$. For any element α in the symmetry group, if $\alpha(s) = s$ then the Sudoku grid s is said to be "fixed" by the symmetry operation α. For two Sudoku grids s_1 and s_2, and two elements of the symmetry group α and β, if $\alpha(s_1) = \beta(s_2)$ then s_1 and s_2 are said to be isomorphic.

Definition 2.1. The symmetry group, $\mathfrak{S}$, for a Sudoku grid is formed by the permutations in Table 1.

By Definition 2.1 the Sudoku symmetry group $\mathfrak{S}$ consists of all permutation operations listed in Table 1 and any combination thereof. Each permutation operation is given in general and the order of each gives the total number of such operations. The cardinality of the Sudoku symmetry group is given by the product of the orders of each permutation operation:

$$|\mathfrak{S}| = n!x!^{y+1}y!^{x+1}. \tag{2.1}$$

3. Reductions

The properties of a reduced Sudoku grid are given in Definition 3.1 and a direct mathematical relationship is given between the number of ways of arranging the values in a Sudoku grid and the number of ways of arranging the values in a reduced Sudoku grid, in Theorem 3.2. The reduced Sudoku properties are used in Section 4 to enable the simplification of the calculation of the number of ways of arranging the values in a Sudoku grid.

Table 1: Permutation operations generating $\mathfrak{S}$.

Permutation operation	Order
Permutation of the values (for example replacing all the values 1 in the entire grid with the value 2 and vice versa)	$n!$
Permutation of bands	$y!$
Permutation of stacks	$x!$
Permutation of the columns within the stacks	$y!^x$
Permutation of the rows within the bands	$x!^y$

Definition 3.1. A *reduced Sudoku grid*, $s^{x,y}$, is a Sudoku grid, $S^{x,y}$, having the following properties:

(i) the values in $S^{x,y}{}_{1,1}$ are in canonical form, $[S^{x,y}{}_{1,1}]_{i,j} = (i-1)y + j$;

(ii) for each minigrid $S^{x,y}{}_{1,b}$, for $b = 2,\ldots,x$, the values in $[S^{x,y}{}_{1,b}]_{1,j}$ for $j = 1,\ldots,y$ are increasing;

(iii) for each minigrid $S^{x,y}{}_{a,1}$, for $a = 2,\ldots,y$, the values in $[S^{x,y}{}_{a,1}]_{i,1}$ for $i = 1,\ldots,x$ are increasing;

(iv) $[S^{x,y}{}_{1,b}]_{1,1} < [S^{x,y}{}_{1,b+1}]_{1,1}$ for $b = 2,\ldots,x-1$;

(v) $[S^{x,y}{}_{a,1}]_{1,1} < [S^{x,y}{}_{a+1,1}]_{1,1}$ for $a = 2,\ldots,y-1$.

An example reduced Sudoku grid is given in Figure 1(a), and an isomorphic Sudoku grid that can be formed from it (by permuting the values $(1,9,2,5,8)(3,7,6)(4)$ and permuting the rows and columns) is given in Figure 1(b).

Theorem 3.2. *If $s^{x,y}(n)$ is the number of reduced Sudoku grids of size $n \times n$ with minigrids of size $x \times y$ (where $n = xy$) then $S^{x,y}(n)$ is given by the following:*

$$S^{x,y}(n) = (n-1)!x!^y y!^x s^{x,y}(n). \tag{3.1}$$

Proof. Let a set X contain all reduced Sudoku grids. There exists an $x, y \in \mathfrak{S}$ such that $x(S) = y(T)$ for $S, T \in X$. Hence a count of the number of possible isomorphisms by applying the symmetry operations in $\mathfrak{S}$ will be larger than the number of distinct isomorphisms. The permutations in Table 2 contain all symmetry operations of $\mathfrak{S}$, and form the group $\mathbb{R}$, and is such that if $p(S) = r(T)$ for $S, T \in X$ then $x = y$ and $S = T$ for $p, r \in \mathbb{R}$.

The cardinality of $\mathbb{R}$ is given by the product of the order of the permutation operations in Table 2, and thus gives the number of isomorphic Sudoku grids which may be formed from a given reduced Sudoku grid. Thus the total number of Sudoku grids may be calculated from the total number of reduced Sudoku grids using the following:

$$S^{x,y}(n) = n!y!^{(x-1)}x!^{(y-1)}(x-1)!(y-1)!s^{x,y}(n). \tag{3.2}$$

Equation (3.2) can be simplified, since $n = xy$, to give (3.1). $\square$

1	2	3	4	5	9	6	7	8
4	5	6	7	1	8	3	9	2
7	8	9	3	6	2	1	5	4
2	9	4	5	7	6	8	1	3
6	7	1	8	4	3	5	2	9
8	3	5	9	2	1	4	6	7
3	6	7	1	9	4	2	8	5
5	4	2	6	8	7	9	3	1
9	1	8	2	3	5	7	4	6

(a) A reduced Sudoku grid

9	5	7	6	1	3	2	8	4
4	8	3	2	5	7	1	9	6
6	1	2	8	4	9	5	3	7
1	7	8	3	6	4	9	5	2
5	2	4	9	7	1	3	6	8
3	6	9	5	2	8	7	4	1
8	4	5	7	9	2	6	1	3
2	9	1	4	3	6	8	7	5
7	3	6	1	8	5	4	2	9

(b) A Sudoku grid isomorphic to (a)

Figure 1: An example Sudoku grid in reduced form and a sudoku grid isomorphic to it.

Table 2: Permutation operations used to form isomorphic Sudoku grids.

Permutation operation	Order
Permutation of values $1, \ldots, n$ (equivalent to rearranging the values in $S^{x,y}{}_{1,1}$)	$n!$
Permutation of rows in bands $2, \ldots, x$	$(y!)^{(x-1)}$
Permutation of columns in stacks $2, \ldots, y$	$(x!)^{(y-1)}$
Permutation of bands $2, \ldots, x$	$(x-1)!$
Permutation of stacks $2, \ldots, y$	$(y-1)!$

4. Example of the Enumeration Technique

Let a 4×4 Sudoku grid be in reduced form such that: the values in $S^{2,2}{}_{1,1}$ are in canonical form, the values in $[S^{2,2}{}_{1,2}]_{1,j}$ for $j = 1, 2$ are increasing; the values in $[S^{2,2}{}_{2,1}]_{i,1}$ for $i = 1, 2$ are increasing (note that the remaining two properties of reduced Sudoku grids are not needed for $S^{2,2}$ since there are only two bands and two stacks). A partially filled reduced Sudoku grid is given in Figure 2 (containing only those values which are prearranged by the grid being in reduced form).

Consider the minigrid $S^{2,2}{}_{1,2}$; the values 1 and 2 must be contained in the second tier and there are two ways of arranging these values. Consider the minigrid $S^{2,2}{}_{2,1}$; the values 1 and 3 must be contained in the second pillar and there are two ways of arranging these values. Four $S^{2,2}$ grids are formed and are given in Figure 3. In Figures 3(a), 3(b), and 3(c) there is one way of arranging the values in $S^{2,2}{}_{2,2}$ and in Figure 3(d) there are no valid ways of arranging the values in $S^{2,2}{}_{2,2}$; therefore $s^{2,2}(4) = 3$.

The enumeration of the number of $S^{2,2}$ grids has been greatly simplified by only calculating the number of reduced Sudoku grids. Using (3.1) in Theorem 3.2 the total number of 4×4 grids can be calculated such that $s^{2,2}(4) = 288$.

5. Comparison of the Number of Known and Reduced Sudoku Grids

Exact values for $S^{x,y}(n)$ are known for $x \leq 3$ and $y \leq 4$. All results were calculated computationally and appeared around late 2005 on the Sudoku Players Forum [11], except for $S^{3,3}(9)$ which appeared in [12] in January 2006. All results have been confirmed computationally by other contributors to the forum. $S^{2,2}(4)$ is attributed to many different authors, $S^{2,3}(6)$, $S^{2,5}(10)$, and $S^{2,6}(12)$ are attributed to Pettersen [11]. The number $S^{2,4}(8)$ was calculated by Russel [11], $S^{3,3}(9)$ by Felgenhauer and Jarvis [12], and $S^{3,4}(12)$ by Pettersen

Table 3: Comparison of $S^{x,y}(n)$ and $s^{x,y}(n)$.

x, y	$S^{x,y}(n)$	$s^{x,y}(n)$
2, 2	288	3
2, 3	28,200,960	1,632
2, 4	29,136,487,207,403,520	3,763,703,808
3, 3	6,670,903,752,021,072,936,960	56,738,340,804,608
5, 2	1,903,816,047,972,624,930,994,913,280,000	273,250,346,759,004,487,680

1	2	3	4
3	4		
2			
4			

Figure 2: A partially filled reduced $S^{2,2}$.

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

(a)

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

(b)

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

(c)

1	2	3	4
3	4	2	1
2	3		
4	1		

(d)

Figure 3: Four arrangements of the values in the partially filled reduced $S^{2,2}$.

and Silver [11]. Currently no other results are available in the academic or nonpeer-reviewed literature. Known numbers of Sudoku grids, $S^{x,y}{}_{1,x}(n)$, for some values of x and y are extended using Theorem 3.2 to produce a comparison of $s^{x,y}(n)$ and $S^{x,y}(n)$; these are shown in Table 3.

The link between Latin squares and reduced Latin squares has been employed to great advantage, especially within the field of enumerative combinatorics. Here we have introduced a reduced Sudoku grid which likewise simplifies the enumeration of Sudoku grids. It can be seen from Table 3 that there are far fewer reduced Sudoku grids than their regular counterparts and in Section 3 the mathematical link between them is demonstrated.

References

[1] C. F. Laywine and G. L. Mullen, *Discrete Mathematics Using Latin Squares*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, NY, USA, 1998.

[2] M. B. Wells, "The number of Latin squares of order eight," *Journal of Combinatorial Theory*, vol. 3, pp. 98–99, 1967.

[3] S. E. Bammel and J. Rothstein, "The number of 9 × 9 Latin squares," *Discrete Mathematics*, vol. 11, pp. 93–95, 1975.

[4] B. D. McKay and E. Rogoyski, "Latin squares of order 10," *Electronic Journal of Combinatorics*, vol. 2, no. 3, pp. 1–4, 1995.

[5] B. D. McKay and I. M. Wanless, "On the number of Latin squares," *Annals of Combinatorics*, vol. 9, no. 3, pp. 335–344, 2005.

[6] D. S. Stones, "The many formulae for the number of Latin rectangles," *Electronic Journal of Combinatorics*, vol. 17, no. 1, Article 1, 46 pages, 2010.

[7] B. Michel, "Mathematics of NRC-Sudoku," Tech. Rep., Universiteit Utrecht, Utrecht, The Netherlands, 2007.

[8] S. K. Jones, S. Perkins, and P. A. Roach, “Properties, isomorphisms and enumeration of 2-Quasi-Magic Sudoku grids,” *Discrete Mathematics*, vol. 311, no. 1, pp. 1098–1110, 2011.

[9] E. Russell and F. Jarvis, There are 49 essentially different Sudoku 2×3 grids... and the 2×3 Sudoku symmetry group, http://www.afjarvis.staff.shef.ac.uk/sudoku/sud23gp.html.

[10] E. Russell and F. Jarvis, “Mathematics of Sudoku II,” *Mathematical Spectrum*, vol. 39, no. 2, pp. 54–58, 2007.

[11] Forum Contributors, 4×3 Sudoku counting (pg. 2), 2005, http://www.sudoku.com/boards/viewtopic.php?t=2511&start=17.

[12] B. Felgenhauer and F. Jarvis, “Mathematics of Sudoku I,” *Mathematical Spectrum*, vol. 39, pp. 15–22, 2006.