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Research Article

Combinatorial Proofs of Some Identities for Nonregular Continued Fractions

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A combinatorial interpretation of nonregular continued fractions is studied. Using a modification of a tiling technique due to Benjamin and Quinn, combinatorial proofs of some identities for nonregular continued fractions are obtained.

1. Introduction

In the recently popular book *Proofs that Really Count* [1], many identities involving linear recurrences were proved by using beautiful tiling interpretations. Many researches provided tiling proofs for a variety of identities. See, for example, [2–4]. By making use of the combinatorial interpretation of continued fractions presented by Benjamin and Quinn in [1], Benjamin and Zeilberger [5] introduced a combinatorial proof of the statement related to prime numbers.

In this research, a combinatorial interpretation of nonregular continued fractions is investigated. The major part of this work is devoted to establishing combinatorial proofs of some identities for nonregular continued fractions.

A continued fraction of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}}, \tag{1.1}$$

where for $i > 0$, a_i, b_0, and b_i are positive integers, is called a nonregular continued fraction. It is more convenient to use the notation $[b_0; a_1, b_1; a_2, b_2; \ldots; a_n, b_n; \ldots]$ for the above continued fraction. If for every i, $a_i = 1$ we denote $[b_0; b_1, b_2, \ldots] := [b_0; a_1, b_1; a_2, b_2; \ldots]$ and $[b_0; b_1, b_2, \ldots]$ is said to be regular.

Corresponding to each continued fraction $[b_0; a_1, b_1; a_2, b_2; \ldots]$, two sequences $\{p_n\}$ and $\{q_n\}$ are defined inductively by

$$\begin{gathered} p_0 = b_0, \qquad p_1 = b_1 b_0 + a_1, \qquad p_n = b_n p_{n-1} + a_n p_{n-2} \quad (n \geq 2), \\ q_0 = 1, \qquad q_1 = b_1, \qquad q_n = b_n q_{n-1} + a_n q_{n-2} \quad (n \geq 2), \end{gathered} \tag{1.2}$$

p_n/q_n is called the nth convergent. An importance property of these numerators and denominators of continued fractions is

$$\frac{p_n}{q_n} = [b_0; a_1, b_1; \ldots; a_n, b_n] \quad (n \geq 0). \tag{1.3}$$

2. Combinatorial Interpretation of Nonregular Continued Fractions

As mentioned by Benjamin and Quinn in [1], a simple combinatorial interpretation of nonregular continued fractions can be realized.

Let $P(b_0; a_1, b_1; \ldots; a_n, b_n)$ be the number of ways to tile an $(n+1)$-board with dominoes and single tiles. All $n+1$ cells of the $(n+1)$-board are labeled with $0, 1, 2, \ldots, n$ from left to right, respectively. Figure 1 illustrates a 3-board, a single tile and a domino. Tiling must satisfy the following three conditions.

(1) All $n+1$ cells of the $(n+1)$-board must be covered.

(2) For $0 \leq i \leq n$, the ith cell can be covered by a stack of as many as b_i single tiles.

(3) For $1 \leq i \leq n$, two consecutive cells $i-1$ and i can be covered by a stack of as many as a_i dominoes.

It is obvious that $P(b_0) = b_0$ and by focusing on the last cell covering, it follows that $P(b_0; a_1, b_1) = b_1 b_0 + a_1$ and

$$\begin{aligned} P(b_0; a_1, b_1; \ldots; a_n, b_n) = {} & b_n P(b_0; a_1, b_1; \ldots; a_{n-1}, b_{n-1}) \\ & + a_n P(b_0; a_1, b_1; \ldots; a_{n-2}, b_{n-2}) \quad (n \geq 2). \end{aligned} \tag{2.1}$$

Since the sequence $\{P(b_0; a_1, b_1; \ldots; a_n, b_n)\}$ satisfy the same initial conditions and recurrence relation as the sequence $\{p_n\}$ defined by (1.2),

$$P(b_0; a_1, b_1; \ldots; a_n, b_n) = p_n, \quad \forall n \geq 0. \tag{2.2}$$

Next, we define $Q(b_0; a_1, b_1; \ldots; a_n, b_n) := P(b_1; a_2, b_2; \ldots; a_n, b_n)$.

Similar to the case of $\{P(b_0; a_1, b_1; \ldots; a_n, b_n)\}$, we obtain

$$Q(b_0; a_1, b_1; \ldots; a_n, b_n) = q_n, \quad \forall n \geq 0. \tag{2.3}$$

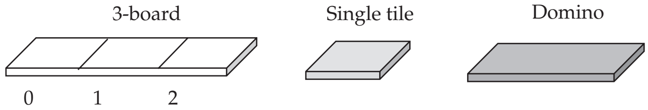

Figure 1: An empty 3-board, a single tile, and a domino.

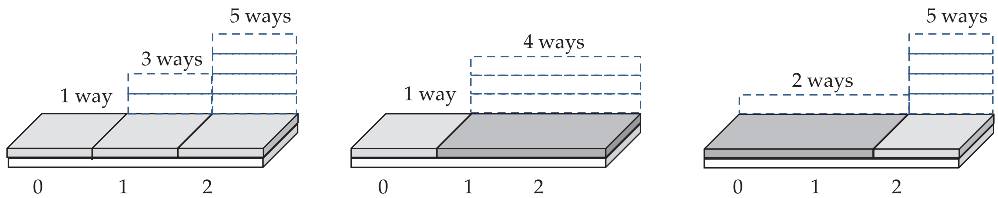

Figure 2: The ways to tile a 3-board with the height conditions $1; 2, 3; 4, 5$.

Equations (1.3), (2.2), and (2.3) yield

$$\frac{P(b_0; a_1, b_1; \ldots; a_n, b_n)}{Q(b_0; a_1, b_1; \ldots; a_n, b_n)} = [b_0; a_1, b_1; \ldots; a_n, b_n] \quad (n \ \geq 0). \tag{2.4}$$

For example, described by the Figures 2 and 3, we get

$$\begin{aligned} P(1; 2, 3; 4, 5) &= (1 \times 3 \times 5) + (1 \times 4) + (2 \times 5) = 29, \\ Q(1; 2, 3; 4, 5) &= P(3; 4, 5) = (3 \times 5) + 4 = 19. \end{aligned} \tag{2.5}$$

Thus, $[1; 2, 3; 4, 5] = 29/19$.

3. Combinatorial Proofs of Some Identities

In this section, the combinatorial interpretation presented in the previous is adopted to reach combinatorial proofs of some identities for nonregular continued fractions.

The reversal identity for the generalization of regular continued fractions recently investigated by Anselm and Weintraub in [6] can easily be verified as Theorem 3.1.

Theorem 3.1. *Let N be an arbitrary positive integer and p_n/q_n the nth convergent of $[b_0; N, b_1; \ldots; N, b_n]$. Then for all $n \geq 1$, one has*

$$[b_n; N, b_{n-1}; \ldots; N, b_0] = \frac{p_n}{p_{n-1}}. \tag{3.1}$$

Proof. This reversal identity follows immediately from (2.2)–(2.4) and the fact that the ways to tile an $(n+1)$-board with the height conditions $b_n; N, b_{n-1}; \ldots; N, b_0$ equals the ways to tile with the height conditions $b_0; N, b_1; \ldots; N, b_n$, which leads

$$P(b_n; N, b_{n-1}; \ldots; N, b_0) = P(b_0; N, b_1; \ldots; N, b_n) = \ p_n, \tag{3.2}$$

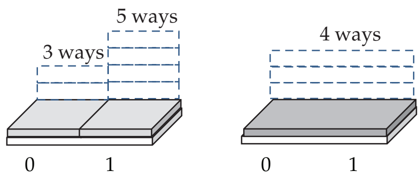

Figure 3: The ways to tile a 2-board with the height conditions $3; 4, 5$.

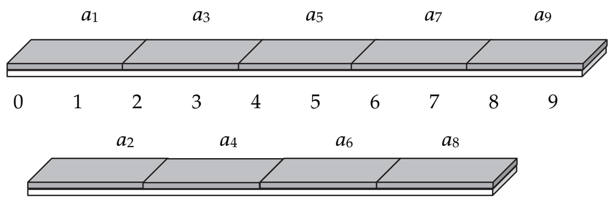

Figure 4: An illustration of the fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ for the case n is odd.

and the fact that the ways to tile an n-board with the height conditions $b_{n-1}; N, b_{n-2}; \ldots; N, b_0$ equals the ways to tile with the height conditions $b_0; N, b_1; \ldots; N, b_{n-1}$, which implies

$$Q(b_n; N, b_{n-1}; \ldots; N, b_0) = P(b_{n-1}; N, b_{n-2}; \ldots; N, b_0) = P(b_0; N, b_1; \ldots; N, b_{n-1}) = p_{n-1}. \tag{3.3}$$

□

Theorems 3.2 and 3.3 are proved by modifying the proofs of Identity 110 and Identity 111 in [1] for regular continued fractions to nonregular continued fractions.

Theorem 3.2. *The difference between consecutive convergents of* $[b_0; a_1, b_1; \ldots; a_n, b_n]$ *is*

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1} \prod_{i=1}^{n} a_i}{q_n q_{n-1}} \quad (n \geq 1). \tag{3.4}$$

Equivalently, after multiply both sides by $q_n q_{n-1}$, *we have*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \prod_{i=1}^{n} a_i \quad (n \geq 1). \tag{3.5}$$

Proof. Denote $\mathbb{P}_n \times \mathbb{Q}_{n-1}$:= the set of tiling of two boards, where on the top board has cells $0, 1, 2, \ldots, n$ with height conditions $b_0; a_1, b_1; \ldots; a_n, b_n$, and the bottom board has cells $1, 2, \ldots, n-1$ with height conditions $b_1; \ldots; a_{n-1}, b_{n-1}$ and $\mathbb{P}_{n-1} \times \mathbb{Q}_n$:= the set of tiling of two boards, where on the top board has cells $0, 1, 2, \ldots, n-1$ with height conditions $b_0; a_1, b_1; \ldots; a_{n-1}, b_{n-1}$, and the bottom board has cells $1, 2, \ldots, n$ with height conditions $b_1; \ldots; a_n, b_n$.

Any element (A, B) in $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ or $\mathbb{P}_{n-1} \times \mathbb{Q}_n$ is said to has a fault at cell $i \geq 1$, if A and B have tiles that end at i.

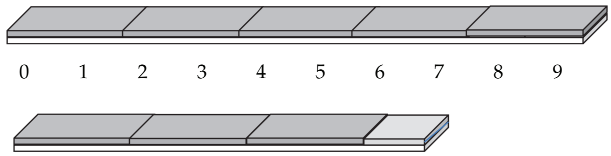

Figure 5: There are no fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-2}$ for the case n is odd.

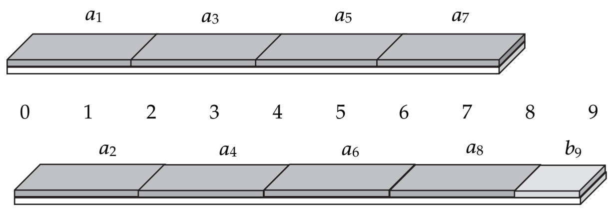

Figure 6: An illustration of the fault-free elements of $\mathbb{P}_{n-2} \times \mathbb{Q}_n$ for the case n is odd.

It can be seen from the definitions that

(1) $|\mathbb{P}_n \times \mathbb{Q}_{n-1}| = p_n q_{n-1}$,

(2) $|\mathbb{P}_{n-1} \times \mathbb{Q}_n| = p_{n-1} q_n$,

(3) when A or B contains a single tile, (A, B) must have a fault.

Next, we consider the numbers of fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ and $\mathbb{P}_{n-1} \times \mathbb{Q}_n$.

Case 1 (n is odd). There are $\prod_{i=1}^{n} a_i$ fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ as shown in Figure 4 and no fault-free element of $\mathbb{P}_{n-1} \times \mathbb{Q}_n$, since for all $(A, B) \in \mathbb{P}_{n-1} \times \mathbb{Q}_n$, A and B both cover an odd number of cells that mean A and B must contain a single tile.

Case 2 (n is even). Similar to the case of n is odd, there are no fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ and $\prod_{i=1}^{n} a_i$ fault-free elements of $\mathbb{P}_{n-1} \times \mathbb{Q}_n$.

Finally, let (S, T) be an element in $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ with (S, T) has a fault. If we swap the tails of S and T after the rightmost fault, then we get the element (S', T') in $\mathbb{P}_{n-1} \times \mathbb{Q}_n$ that has the same rightmost fault as (S, T). Hence, by using this swapping, a one-to-one correspondence between the set of the elements that has a fault in $\mathbb{P}_n \times \mathbb{Q}_{n-1}$ and the set of the elements that has a fault in $\mathbb{P}_{n-1} \times \mathbb{Q}_n$ can be constructed.

Therefore, $p_n q_{n-1} - p_{n-1} q_n = |\mathbb{P}_n \times \mathbb{Q}_{n-1}| - |\mathbb{P}_{n-1} \times \mathbb{Q}_n| = (-1)^{n-1} \prod_{i=1}^{n} a_i$. □

Theorem 3.3. *For $n \geq 2$, let p_n / q_n be the nth convergent of $[b_0; a_1, b_1; \ldots; a_n, b_n]$. Then*

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n b_n \prod_{i=1}^{n-1} a_i. \tag{3.6}$$

Proof. Denote $\mathbb{P}_n \times \mathbb{Q}_{n-2}$:= the set of tiling of two boards, where on the top board has cells $0, 1, 2, \ldots, n$ with height conditions $b_0; a_1, b_1; \ldots; a_n, b_n$, and the bottom board has cells $1, 2, \ldots, n-2$ with height conditions $b_1; \ldots; a_{n-2}, b_{n-2}$ and $\mathbb{P}_{n-2} \times \mathbb{Q}_n$:= the set of tiling of two boards, where on the top board has cells $0, 1, 2, \ldots, n-2$ with height conditions $b_0; a_1, b_1; \ldots; a_{n-2}, b_{n-2}$, and the bottom board has cells $1, 2, \ldots, n$ with height conditions $b_1; \ldots; a_n, b_n$.

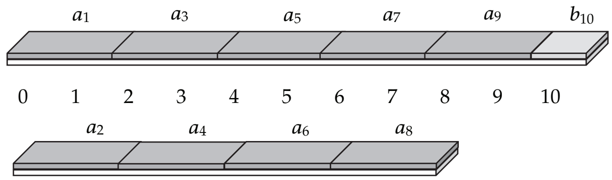

Figure 7: An illustration of the fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-2}$ for the case n is even.

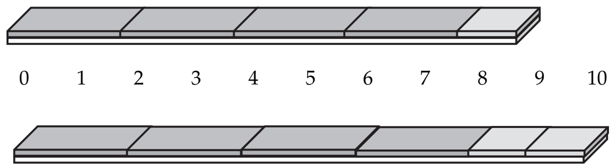

Figure 8: There are no fault-free elements of $\mathbb{P}_{n-2} \times \mathbb{Q}_n$ for the case n is even.

Hence $|\mathbb{P}_n \times \mathbb{Q}_{n-2}| = p_n q_{n-2}$ and $|\mathbb{P}_{n-2} \times \mathbb{Q}_n| = p_{n-2} q_n$.

Similar to the proof of Theorem 3.2, we can construct a one-to-one correspondence between the set of the elements that has a fault in $\mathbb{P}_n \times \mathbb{Q}_{n-2}$ and the set of the elements that has a fault in $\mathbb{P}_{n-2} \times \mathbb{Q}_n$ by swapping the tails after the rightmost fault.

Thus, it suffices to consider the numbers of fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-2}$ and $\mathbb{P}_{n-2} \times \mathbb{Q}_n$.

Case 1 (n is odd). There are no fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-2}$ and $b_n \prod_{i=1}^{n-1} a_i$ fault-free elements of $\mathbb{P}_{n-2} \times \mathbb{Q}_n$, described by the examples illustrated in Figures 5 and 6.

Case 2 (n is even). There are $b_n \prod_{i=1}^{n-1} a_i$ fault-free elements of $\mathbb{P}_n \times \mathbb{Q}_{n-2}$ and no fault-free element of $\mathbb{P}_{n-2} \times \mathbb{Q}_n$, see Figures 7 and 8.

Therefore, $p_n q_{n-2} - p_{n-2} q_n = |\mathbb{P}_n \times \mathbb{Q}_{n-2}| - |\mathbb{P}_{n-2} \times \mathbb{Q}_n| = (-1)^n b_n \prod_{i=1}^{n-1} a_i$. □

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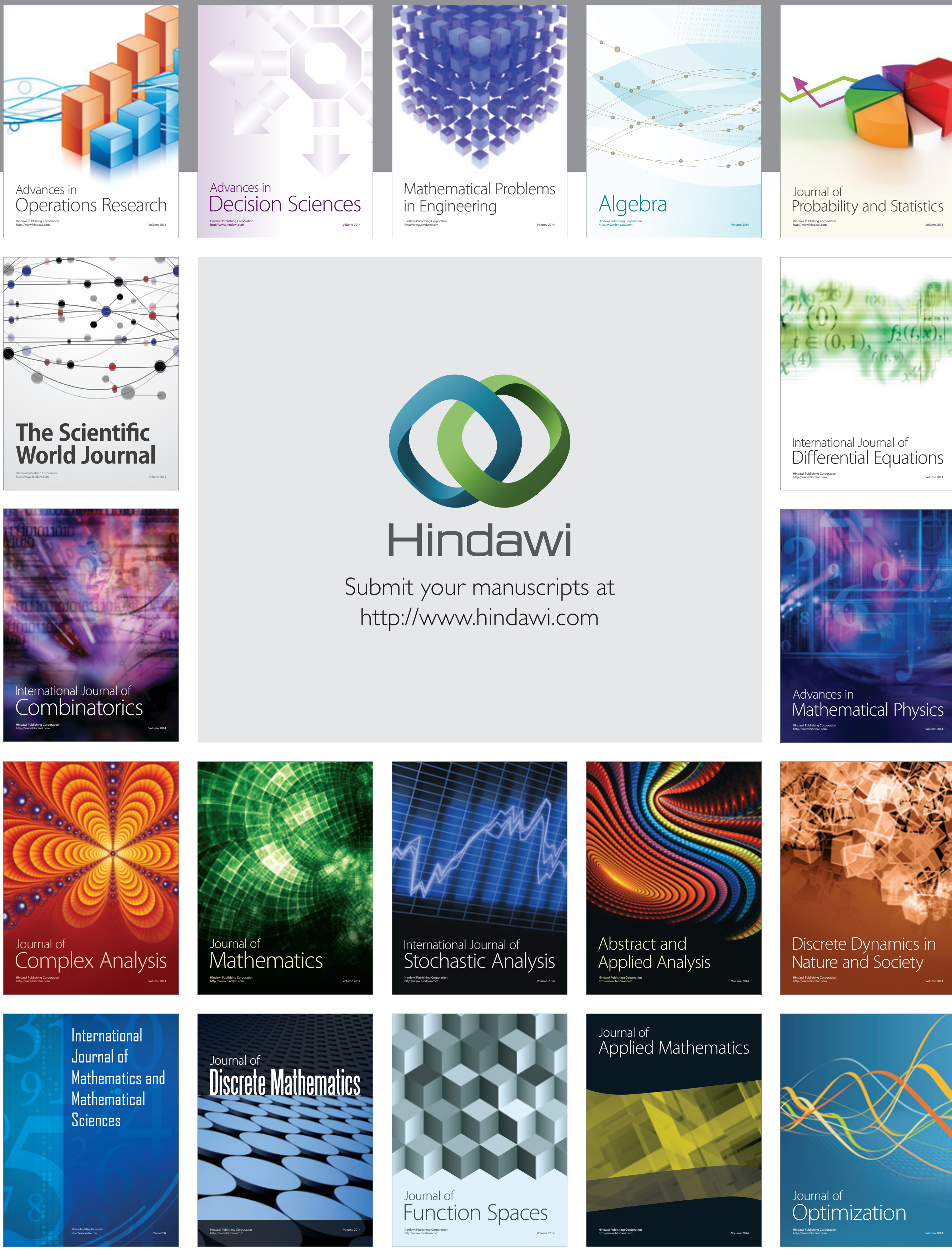

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