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Research Article

Sunlet Decomposition of Certain Equipartite Graphs

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Let L_{2n} stand for the sunlet graph which is a graph that consists of a cycle and an edge terminating in a vertex of degree one attached to each vertex of cycle C_n. The necessary condition for the equipartite graph $K_n + I * \overline{K}_m$ to be decomposed into L_{2n} for $n \geq 2$ is that the order of L_{2n} must divide $n^2m^2/2$, the order of $K_n + I * \overline{K}_m$. In this work, we show that this condition is sufficient for the decomposition. The proofs are constructive using graph theory techniques.

1. Introduction

Let C_r, K_n, $\overline{K}_m$ denote cycle of length r, complete graph on n vertices, and complement of complete graph on m vertices. For n even, $K_n + I$ denotes the multigraph obtained by adding the edges of a 1-factor to K_n, thus duplicating $n/2$ edges. The total number of edges in $K_n + I$ is $n^2/2$. The *lexicographic product*, $G * H$, of graphs G and H, is the graph obtained by replacing every vertex of G by a copy of H and every edge of G by the complete bipartite graph $K_{|H|,|H|}$.

For a graph H, an *H-decomposition* of a graph G, $H \mid G$, is a set of subgraphs of G, each isomorphic to H, whose edge set partitions the edge set of G. Note that for any graph G and H and any positive integer m, if $H \mid G$ then $(H * \overline{K}_m) \mid (G * \overline{K}_m)$.

Let G be a graph of order n and H any graph. The *corona* (crown) of G with H, denoted by $G \odot H$, is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H. A special corona graph is $C_n \odot K_1$, that is, a cycle with pendant points which has $2n$ vertices. This is called *sunlet* graph and denoted by L_q, $q = 2n$.

Obvious necessary condition for the existence of a *k-cycle* decomposition of a simple connected graph G is that G has at least k vertices (or trivially, just one vertex), the degree of every vertex in G is even, and the total number of edges in G is a multiple of the cycle length k. These conditions have been shown to be sufficient in the case that G is the complete graph K_n, the complete graph minus a 1-factor $K_n - I$ [1, 2], and the complete graph plus a 1-factor $K_n + I$ [3].

The study of cycle decomposition of $K_n * \overline{K}_m$ was initiated by Hoffman et al. [4]. The necessary and sufficient conditions for the existence of a C_p-decomposition of $K_n * \overline{K}_m$, where $p \geq 5$ (p is prime) that (i) $m(n-1)$ is even and (ii) p divides $n(n-1)m^2$, were obtained by Manikandan and Paulraja [5, 6]. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of a C_{2p}-decomposition of $K_n * \overline{K}_m$ were given by Smith [7]. For a prime number $p \geq 3$, Smith [8] showed that C_{3p}-decomposition of $K_n * \overline{K}_m$ exists if the obvious necessary conditions are satisfied. In [9], Anitha and Lekshmi proved that the complete graph K_n and the complete bipartite graph $K_{n,n}$ for n even have decompositions into sunlet graph L_n. Similarly, in [10], it was shown that the complete equipartite graph $K_n * \overline{K}_m$ has a decomposition into sunlet graph of length $2p$, for a prime p.

We extend these results by considering the decomposition of $K_n + I * \overline{K}_m$ into sunlet graphs and prove the following result.

Let $m \geq 2$, $n > 2$, and $q \geq 6$ be even integers. The graph $K_n + I * \overline{K}_m$ can be decomposed into sunlet graph of length q if and only if q divides $n^2m^2/2$, the number of edges in $K_n + I * \overline{K}_m$.

2. Proof of the Result

To prove the result, we need the following.

Lemma 1 (see [10]). *For $r \geq 3$, L_{2r} decomposes $C_r * \overline{K}_2$.*

Lemma 2. *For any integer $r > 2$ and a positive even integer m, the graph $C_r * \overline{K}_m$ has a decomposition into sunlet graph L_q, for $q = rm$.*

Proof

Case 1 (r is even). First observe that $C_r * \overline{K}_2$ can be decomposed into 2 sunlet graphs with $2r$ vertices. Now, set $m = 2t$ and decompose $C_r * \overline{K}_t$ into cycles C_{rt}. To decompose $C_r * \overline{K}_t$ into t-cycles C_{rt}, denote vertices in ith part of $C_r * \overline{K}_t$ by $x_{i,j}$ for $j = 1, \ldots, t$, $i = 1, 2, \ldots, r$ and create t base cycles $x_{1,j}x_{2,j}x_{3,j} \cdots x_{r-1,j}x_{r,j}$. Next, combine these base cycles into one cycle C_{rt} by replacing each edge $x_{1,j}x_{2,j}$ with $x_{1,j}x_{2,j+1}$. To create the remaining cycles C_{rt}, we apply mappings ϕ_s for $s = 0, 1, \ldots, t-1$ defined on the vertices as follows.

Subcase 1.1 (i odd). Consider

$$\phi_s(x_{i,j}) = x_{i,j}. \tag{1}$$

This is the desired decomposition into cycles C_{rt}.

Subcase 1.2 (i even). Consider

$$\phi_s(x_{i,j}) = x_{i,j+s}. \tag{2}$$

This is the desired decomposition into cycles C_{rt}.

Now take each cycle C_{rt}, and make it back into $C_{rt} * \overline{K}_2$. Each $C_{rt} * \overline{K}_2$ decomposes into 2 sunlet graphs L_{2rt} (by Lemma 1), and we have $C_r * \overline{K}_m$ decomposing into sunlet graphs with length rm for r even. Note that

$$C_r * \overline{K}_{2t} = (C_r * \overline{K}_t) * \overline{K}_2. \tag{3}$$

Case 2 (r is odd)

Subcase 2.1 ($m \equiv 2 \pmod 4$). Set $m = 2t$. First create t cycles $C_{(r-1)t}$ in $C_{r-1} * \overline{K}_t$ as in Case 1. Then, take complete tripartite graph $K_{t,t,t}$ with partite sets $X_i = \{x_{i,j}\}$ for $i = 1, r-1, r$ and $j = 1, \ldots, t$ and decompose it into triangles using well-known construction via Latin square, that is, construct $t \times t$ Latin square and consider each element in the form (a, b, c) where a denotes the row, b denotes the column, and c denotes the entry with $1 \leq a, b, c \leq t$. Each cycle is of the form $x_{(1,a)}$, $x_{(r-1,b)}$, $x_{(r,c)}$. Then, for every triangle $x_{1,a}x_{r-1,b}x_{r,c}$, replace the edge $x_{1,a}x_{r-1,b}$ in each $C_{(r-1)t}$, by the edges $x_{r-1,b}x_{r,c}$ and $x_{r,c}x_{1,a}$ to obtain cycles C_{rt}. Therefore, $C_{rt} \mid C_r * \overline{K}_t$. Now take each cycle C_{rt}, make it into $C_{rt} * \overline{K}_2$, and by Lemma 1, $C_{rt} * \overline{K}_2$ has a decomposition into sunlet graphs $L_{2rt} = L_q$.

Subcase 2.2 ($m \equiv 0 \pmod 4$). Set $m = 2t$. The graph $C_r * \overline{K}_t$ decomposes into Hamilton cycle C_{rt} by [11]. Next, make each cycle C_{rt} into $C_{rt} * \overline{K}_2$. Each graph $C_{rt} * \overline{K}_2$ decomposes into sunlet graph L_{2rt} by Lemma 1. □

Theorem 3. *Let r, m be positive integers satisfying $r, m \equiv 0 \pmod 4$, then L_r decomposes $C_r * \overline{K}_m$.*

Proof. Let the partite sets (layers) of the r-partite graph $C_r * \overline{K}_m$ be $U_1, U_2, \ldots, U_r$. Set $m = 2t$. Obtain a new graph from $C_r * \overline{K}_m$ as follows.

Identify the subsets of vertices $\{x_{i,j}\}$, for $1 \leq i \leq r$ and $1 \leq j \leq m/2$ into new vertices x_i^1, and identify the subset of vertices $\{x_{i,j}\}$ for $1 \leq i \leq r$ and $m/2 + 1 \leq j \leq m$ into new vertices x_i^2 and two of these vertices x_i^k, where $k = 1, 2$, are adjacent if and only if the corresponding subsets of vertices in $C_r * \overline{K}_m$ induce $K_{t,t}$. The resulting graph is isomorphic to $C_r * \overline{K}_2$. Next, decompose $C_r * \overline{K}_2$ into cycles $C_{r/2}$ as follows: $x_{k,1}x_{k+1,1}, \ldots, x_{d,1}x_{d-1,2}, \ldots, x_{k+1,2}, x_{k,1}$

$$k = 1, \frac{r}{4} + 1, \frac{r}{2} + 1, \frac{3r}{4} + 1, \ldots, r - \frac{r}{4} + 1, \qquad d = \frac{r}{4} + k, \tag{4}$$

where k, d are calculated modulo r.

To construct the remaining cycles, apply mapping ϕ defined on the vertices.

Subcase 1.1 (i odd in each cycle). Consider

$$\phi(x_{i,j}) = x_{i,j+1}. \tag{5}$$

This is the desired decomposition of $C_r * \overline{K}_2$ into cycles $C_{r/2}$.

Subcase 1.2 (i even in each cycle). Consider

$$\phi(x_{i,j}) = x_{i,j}. \tag{6}$$

This is the desired decomposition of $C_r * \overline{K}_2$ into cycles $C_{r/2}$.

By lifting back these cycles $C_{r/2}$ of $C_r * \overline{K}_2$ to $C_r * \overline{K}_{2t}$, we get edge-disjoint subgraphs isomorphic to $C_{r/2} * \overline{K}_t$. Obtain a new graph again from $C_{r/2} * \overline{K}_t$ as follows.

For each j, $1 \leq j \leq t/2$, identify the subsets of vertices $\{x_{i,2j-1}, x_{i,2j}\}$, where $1 \leq i \leq r/2$ into new vertices x_i^j, and two of these vertices x_i^j are adjacent if and only if the corresponding subsets of vertices in $C_{r/2} * \overline{K}_t$ induce $K_{2,2}$. The resulting graph is isomorphic to $C_{r/2} * \overline{K}_{t/2}$. Then, decompose $C_{r/2} * \overline{K}_{t/2}$ into cycles $C_{r/2}$. Each $C_{r/2} * \overline{K}_{t/2}$ decomposes into cycles $C_{r/2}$ by [12]. By lifting back these cycles $C_{r/2}$ of $C_{r/2} * \overline{K}_{t/2}$ to $C_{r/2} * \overline{K}_t$, we get edge-disjoint subgraph isomorphic to $C_{r/2} * \overline{K}_2$. Finally, each $C_{r/2} * \overline{K}_2$ decomposes into two sunlet graphs L_r (by Lemma 1), and we have $C_r * \overline{K}_m$ decomposing into sunlet graphs L_r as required. □

Theorem 4 (see [12]). *The cycle C_m decomposes $C_k * \overline{K}_m$ for every even $m > 3$.*

Theorem 5 (see [12]). *If m and $k \geq 3$ are odd integers, then C_m decomposes $C_k * \overline{K}_m$.*

Theorem 6. *The sunlet graph L_m decomposes $C_r * \overline{K}_m$ if and only if either one of the following conditions is satisfied.*

(1) *r is a positive odd integer, and m is a positive even integer.*

(2) *r, m are positive even integers with $m \equiv 0 \pmod 4$.*

Proof. (1) Set $m = 2t$, where t is a positive integer. Let the partite sets (layers) of the r-partite graph $C_r * \overline{K}_m$ be $U_1, U_2, \ldots, U_r$. For each j, where $1 \leq j \leq t$, identify the subsets of vertices $\{x_{i,2j-1}, x_{i,2j}\}$, for $1 \leq i \leq r$ into new vertices x_i^j, and two of these vertices x_i^j are adjacent if and only if the corresponding subsets of vertices in $C_r * \overline{K}_m$ induce $K_{2,2}$. The resulting graph is isomorphic to $C_r * \overline{K}_t$. Then, decompose $C_r * \overline{K}_t$ into cycles C_t, where t is a positive integer.

Now, $C_t \mid C_r * \overline{K}_t$ by Theorems 4 and 5.

By lifting back these t-cycles of $C_r * \overline{K}_t$ to $C_r * \overline{K}_{2t}$, we get edge-disjoint subgraphs isomorphic to $C_t * \overline{K}_2$. Each copy of $C_t * \overline{K}_2$ decomposes into sunlet graphs of length $2t$ (by Lemma 1), and we have $C_r * \overline{K}_m$ decomposing into sunlet graphs of length m as required.

(2) Set $m = 2t$, where t is an even integer since $m \equiv 0 \pmod 4$.

Obtain a new graph $C_r * \overline{K}_t$ from the graph $C_r * \overline{K}_m$ as in Case 1. By Theorem 4, $C_t \mid C_r * \overline{K}_t$. By lifting back these t-cycles of $C_r * \overline{K}_t$ to $C_r * \overline{K}_{2t}$, we get edge-disjoint subgraphs isomorphic to $C_t * \overline{K}_2$. Each copy of $C_t * \overline{K}_2$ decomposes into sunlet graph of length $2t$ (by Lemma 1). Therefore, $L_m \mid C_r * \overline{K}_m$ as required. □

Remark 7. In [10], it was shown that

$$L_{2r} * \overline{K}_l \text{ can be decomposed into } l^2 \text{ copies of } L_{2r}. \quad (7)$$

This, coupled with Lemma 1, gives the following.

Theorem 8 (see [10]). *The graph $C_r * \overline{K}_{2l}$ decomposes into sunlet graphs L_{2r} for any positive integer l.*

Lemma 9 (see [3]). *Let $n \geq 4$ be an even integer. Then, $K_n + I$ is C_n-decomposable.*

Lemma 10 (see [3]). *Let m and n be integers with m odd, $n \equiv 2 \pmod 4$, $3 \leq m \leq n < 2m$, and $n^2 \equiv 0 \pmod{2m}$. Then, $K_n + I$ is C_m-decomposable.*

Lemma 11 (see [3]). *Let m and n be integers with m odd, $n \equiv 0 \pmod 4$, $3 \leq m \leq n < 2m$, and $n^2 \equiv 0 \pmod{2m}$. Then, $K_n + I$ is C_m-decomposable.*

We can now prove the major result.

Theorem 12. *For any even integers $m \geq 2$, $n > 2$, and $q \geq 6$, the sunlet graph L_q decomposes $K_n + I * \overline{K}_m$ if and only if $n^2m^2/2 \equiv 0 \pmod q$.*

Proof. The necessity of the condition is obvious, and so we need only to prove its sufficiency. We split the problem into the following two cases.

Case 1 ($q \mid n$)

Subcase 1.1 ($n > q$). Cycle C_n decomposes $K_n + I$ by Lemma 9, and we have

$$C_n * \overline{K}_m \mid K_n + I * \overline{K}_m. \quad (8)$$

Each graph $C_n * \overline{K}_m$ decomposes into sunlet graph L_q, where $q = nm$ by Lemma 2, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph L_q, where $q > n$.

Subcase 1.2 ($q = n$). First, consider $n \equiv 0 \pmod 4$.

Cycle C_q decomposes $K_q + I$ by Lemma 9, and we have

$$C_q * \overline{K}_m \mid K_q + I * \overline{K}_m. \quad (9)$$

Now, sunlet graph $L_q \mid (C_q * \overline{K}_m)$ by Theorem 3, and hence sunlet graph L_q decomposes $K_n + I * \overline{K}_m$.

Also, consider $n \equiv 2 \pmod 4$.

Suppose $m = 2t$. Cycle $C_{q/2}$ decomposes $K_q + I$ by Lemma 10, and we have

$$C_{q/2} * \overline{K}_{2t} \mid K_q + I * \overline{K}_{2t}. \quad (10)$$

Now, sunlet graph L_q decomposes $C_{q/2} * \overline{K}_{2t}$ by Theorem 8, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph of length q.

Case 2 ($q \mid m$)

Subcase 2.1 ($m \equiv 0 \pmod 4$). Suppose $m = q$, and by Lemma 9, cycle C_n decomposes $K_n + I$, and we have

$$C_n * \overline{K}_q \mid K_n + I * \overline{K}_q. \quad (11)$$

Also, sunlet graph L_q decomposes each $C_n * \overline{K}_q$ by Theorem 6, and we have sunlet graph L_q decomposing $K_n + I * \overline{K}_m$.

Subcase 2.2 ($m \equiv 2 \pmod 4$). Let $m = q$ and $r \leq n$ an odd integer. Cycle C_r decomposes $K_n + I$, by Lemmas 9, 10, and 11, and we have

$$C_r * \overline{K}_q \mid K_n + I * \overline{K}_q. \quad (12)$$

Now, each $C_r * \overline{K}_q$ decomposes into sunlet graph L_q by Theorem 6, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph L_q as required.

Subcase 2.3 ($m > q$). Set $m = wq$, where w is any positive integer, then by Subcases 2.1 and 2.2, we have

$$L_q * \overline{K}_w \mid \left(K_n + I * \overline{K}_q\right) * \overline{K}_w. \quad (13)$$

Each graph $L_q * \overline{K}_w$ decomposes into sunlet graph L_q by Remark 7, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph L_q. □

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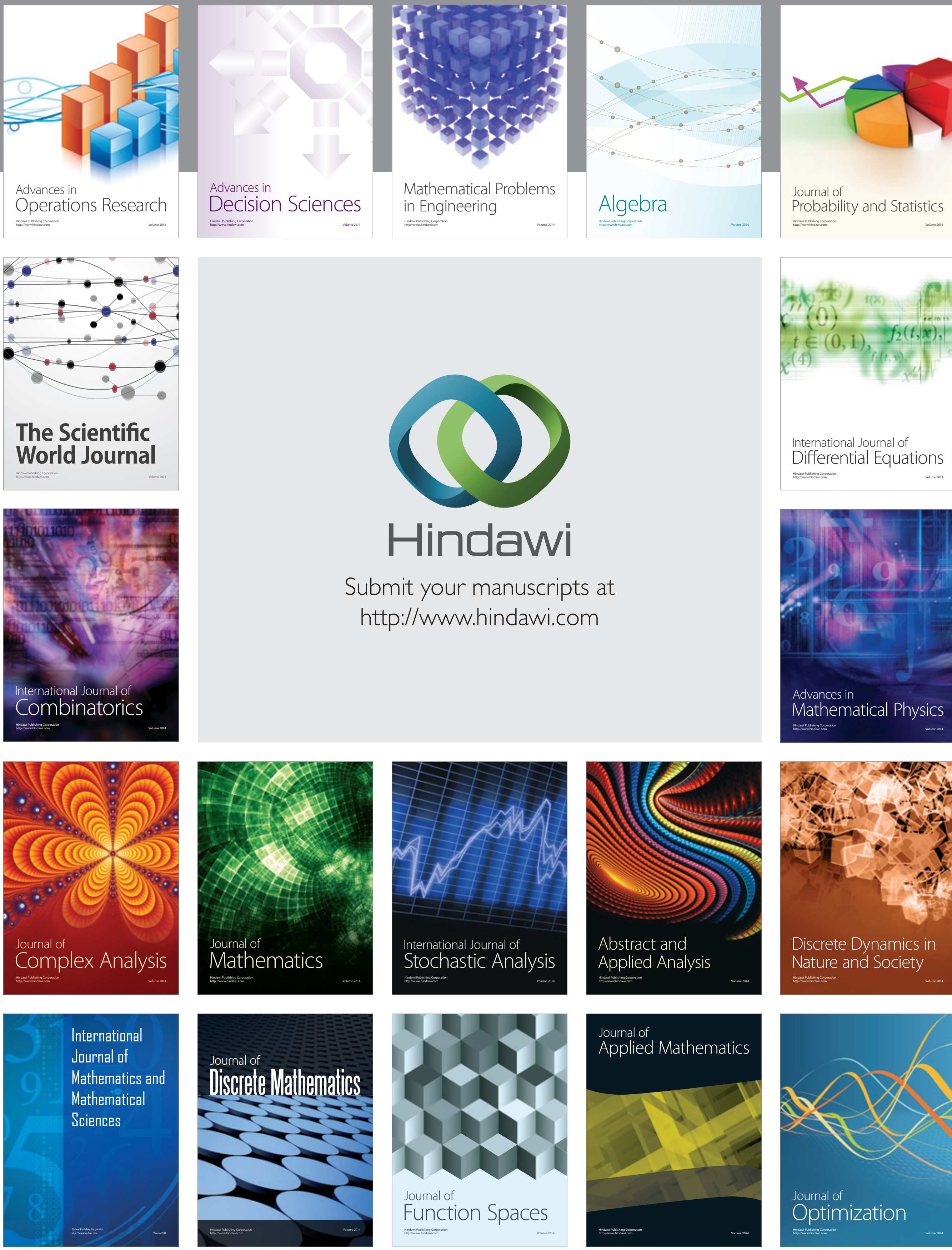

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