Internat. J. Math. & Math. Sci.
VOL. 12 NO. 3 (1989) 503-506

CHANGES IN SIGNATURE INDUCED BY THE LYAPUNOV MAPPING $\mathcal{L}_A : X \to AX + XA^*$

TYLER HAYNES

Mathematics Department
Saginaw Valley State University
University Center, Michigan 48710

(Received October 7, 1987)

ABSTRACT. The Lyapunov mapping on n x n matrices over C is defined by $\mathcal{L}_A(X) = AX + XA^*$; a matrix is stable iff all its characteristic values have negative real parts; and the inertia of a matrix X is the ordered triple $In(X) = (\pi, \nu, \delta)$ where π is the number of eigenvalues of X whose real parts are positive, ν the number whose real parts are negative, and δ the number whose real parts are 0. It is proven that for any normal, stable matrix A and any hermitian matrix H, if $In(H) = (\pi, \nu, \delta)$ then $In(\mathcal{L}_A(H)) = (\nu, \pi, \delta)$. Further, if stable matrix A has only simple elementary divisors, then the image under $\mathcal{L}_A$ of a positive-definite hermitian matrix is negative-definite hermitian, and the image of a negative-definite hermitian matrix is positive-definite hermitian.

KEY WORDS AND PHRASES. Lyapunov, stable matrix, matrix inertia, positive-definite matrix
1980 AMS SUBJECT CLASSIFICATION CODES. 15A18, 15A42

For many years stable matrices have interested applied mathematicians because, for a system of linear homogeneous differential equations whose coefficients are constant, a stable matrix of coefficients is a necessary and sufficient condition that the solution be asymptotically stable. Recently, algebraists too have become interested in stable matrices.

Definition: A square matrix is stable $\Leftrightarrow$ all its characteristic values have negative real parts.

(In this article, the entries of all matrices are complex numbers unless stated otherwise.)

A classical test for stability of matrices is Lyapunov's theorem, whose statement is facilitated by some notation:

$\mathcal{S}$ = set of all nxn stable matrices

$\mathcal{H}$ = set of all nxn hermitian matrices

$i\mathcal{H}$ = set of all nxn skew-hermitian matrices

Π = set of all nxn positive-definite hermitian matrices

$\mathcal{N}$ = set of all nxn negative-definite hermitian matrices

$\mathcal{L}_A(X)$ = $AX+XA^*$, where A and X are nxn matrices and A^* is the conjugate transpose of A.

(It is trivial to verify that $\mathcal{L}_A(\bullet)$, the Lyapunov mapping, is a linear transformation on the linear space M_n of nxn matrices.)

Lyapunov's theorem is usually expressed as statement a) of

Theorem 1: The following three statements are equivalent:

a) $A \in \mathcal{S} \Leftrightarrow$ there exists $G \in \Pi$ such that $\mathcal{L}_A(G) = -I$;

b) $A \in \mathcal{S} \Leftrightarrow$ for every $G_1 \in \mathcal{N}$, there exists $G \in \Pi$ such that $\mathcal{L}_A(G) = G_1 \Leftrightarrow$ there exists $G_1 \in \mathcal{N}$ and there exists $G \in \Pi$ such that $\mathcal{L}_A(G) = G_1$ [Taussky, 1964; p. 6, thms 2-3];

c) Let $C = aI+S$ (a real and < 0, $S \in i\mathcal{H}$) and $D = \mathrm{diag}(d_1,\ldots,d_n)$ with d_i real ($i=1,\ldots,n$). Then $CD \in \mathcal{S} \Leftrightarrow d_i > 0$ for all i. [Taussky, 1961, J. Math Anal. & App.].

The equivalences are proven (essentially) in Taussky's articles. An analytic proof a) is in Bellman, pp. 242-245, and a topological proof in Ostrowski & Schneider.

Theorem 1 suggests that the operator $\mathcal{L}_A(\bullet)$ might give rise to other tests for stability; such usefulness is limited, however, by the following

Theorem 2: The range of $\mathcal{L}_A(H)$ as a function of $H \in \Pi$ and $A \in \mathcal{S}$ is that subset of $\mathcal{H}$ with $\nu \neq 0$ (where ν denotes the number of characteristic vectors with negative real parts). [Stein, p. 352, thm 2].

Some useful theorems result if further restrictions are imposed on A besides stability. These theorems are obtained via a topological route and require additional concepts.

Definition: The inertia of an nxn matrix X is the ordered triple of integers $(\pi(X), \nu(X), \delta(X)) = \mathrm{In}(X)$ where $\pi(X)$ is the number of characteristic values of X whose real parts are positive, $\nu(X)$ the number whose real parts are negative, and $\delta(X)$ the number whose real parts are 0. If nxn matrices M and N possess the same inertia, this will be denoted by $M \overset{i}{\sim} N$.

Let M and N be nxn hermitian matrices. M and N are congruent (denoted $M \overset{c}{\sim} N$) $\Leftrightarrow \exists$ P non-singular such that $M = P^*NP$.

Recall that all norms in the set of all nxn matrices M_n induce the same topology. In M_n so topologized, matrices M and N are connected $\Leftrightarrow$ there exists a connected set containing both M and N. The relationship of being connected is an equivalence relation, which will be denoted by $\overset{u}{\sim}$. M and N are arc-wise connected $\Leftrightarrow$ there exists a continuous function f from the real interval [0,1] into M_n such that $f(0) = M$ and $f(1) = N$. This, too, is an equivalence relation in M_n and will be denoted by $\overset{a}{\sim}$.

The preceding concepts are brought together by the following theorem:

Theorem 3: In the set N_n of all non-singular nxn matrices with the relative topology induced by any norm, $A \overset{u}{\sim} B$ and $A \overset{a}{\sim} B$ ($\forall A, B \in N_n$). [Schneider; pp. 818-819, lemmata 1 & 2]. Let $\mathcal{H}^n_r$ denote the set of all nxn hermitian matrices of rank r. In $\mathcal{H}^n_r$ with the relative topology induced by any norm the four equivalence relations $\overset{u}{\sim}$, $\overset{a}{\sim}$, $\overset{i}{\sim}$, $\overset{c}{\sim}$ coincide. [Schneider; p. 820].

The relationship between algebraic features of hermitian matrices and topological features expressed by theorem 3 makes it possible to discover the variation in signature induced by the Lyapunov mapping $\mathcal{L}_A(\bullet)$ whenever $A \in \mathcal{S}$ is normal and $H \in \mathcal{H}$.

Theorem 4: If $A \in \mathcal{S}$ is normal, then for any $H \in \mathcal{H}$ with $\mathrm{In}(H) = (\pi,\nu,\delta)$, $\mathrm{In}(\mathcal{L}_A(H)) = (\nu,\pi,\delta)$.

Proof: Let $A \in \mathcal{S}$ be normal, $\{a_i\}_1^n$ be its characteristic values, $H \in \mathcal{H}$, $\mathrm{In}(H) = (\pi,\nu,\delta)$, and $\mathcal{L}_A(H) = AH+HA^* = C$.

Since A is normal, it is unitarily similar to a diagonal matrix: $VAV^* = \mathrm{diag}(a_1,\ldots,a_n)$, V unitary. Also a basis for n-dimensional space can be formed from the characteristic vectors of A, $\{\alpha_i\}_1^n$.

For any i, $\alpha_i C = \alpha_i(AH+HA^*) = \alpha_i a_i H + \alpha_i HA^* = \alpha_i H(a_i I + A^*)$. The number of independent $\alpha_i C$ is the rank of C; it is also the rank of $H(a_i I + A^*)$ = rank of H (since $a_i I + A^*$ is non-singular, for the characteristic values of $-A^*$ are $\{-\bar{a}_i\}_1^n$ and $\{a_i\}_1^n \cap \{-a_i\}_1^n = \emptyset$ because real part of $\bar{a}_i$ = real part of $a_i < 0$ $(i=1,\ldots,n)$.) Therefore, rank (H) = rank ($\mathcal{L}_A(H)$).

Because $\mathcal{L}_A$ is a linear transformation of M_n onto itself, it is continuous. If $\mathcal{L}_A$ is restricted to $\mathcal{H} \subseteq M_n$ it is continuous and onto $\mathcal{H}$. Therefore, $\mathcal{L}_A$ maps topologically connected components of $\mathcal{H}_r^n$ onto components of $\mathcal{H}_r^n$ since rank is preserved by $\mathcal{L}_A$. But by theorem 3 topologically connected components coincide with inertial components. Therefore, $\mathcal{L}_A$ maps In(H) on In(C).

$H \in \mathcal{H}$ and since VHV^* is congruent to H, In(VHV^*) = In(H). Hence, In($\mathcal{L}_A(VHV^*)$) = In($\mathcal{L}_A(H)$) = In(C).

Let $D = \mathcal{L}_A(VHV^*) = A(VHV^*) + (VHV^*)A^*$. Then $V^*DV = (V^*AV)H + H(V^*A^*V)$. Because $D \in \mathcal{H}$, In($\mathcal{L}_{VAV^*}(H)$) = In(V^*DV) = In(D) = In(C).

H is congruent to $K = I_\pi \oplus -I_\nu \oplus 0_\delta$, so In(K) = In(H), whence In($\mathcal{L}_{VAV^*}(K)$) = In($\mathcal{L}_{VAV^*}(H)$) = In(C). $\mathcal{L}_{VAV^*}(K)$ is of the form

$$\mathrm{diag}(a_1,\ldots,a_n)(I_\pi \oplus -I_\nu \oplus 0_\delta) + (I_\pi \oplus -I_\nu \oplus 0_\delta)\,\mathrm{diag}(\bar{a}_1,\ldots,\bar{a}_n)$$
$$= 2\,\mathrm{diag}\,(R(a_1),\ldots, R(a_\pi), -R(a_{\pi+1}),\ldots, -R(a_{\pi+\nu}), 0,\ldots,0),$$

where R(a) denotes the real part of complex number a, I_m the mxm identity matrix, and 0_m the mxm zero matrix. Since $R(a_i) < 0$ $(i=1,\ldots,n)$, In($\mathcal{L}_{VAV^*}(K)$) = (ν,π,δ). Therefore, In(C) = (ν,π,δ). QED

The preceding theorem was based on the unitary similarity of A to a diagonal matrix; this property was used first to show the invariance of rank and then to display the inertia when both A and H were expressed in canonical form. The next theorem generalizes the last in that A need be similar (not unitarily similar) to a diagonal matrix, but it is more restrictive of the inertia of H.

<u>Theorem 5</u>: If $A \in \mathcal{S}$ has only simple elementary divisors, then $\mathcal{L}_A(\Pi) = \mathcal{N}$ and $\mathcal{L}_A(\mathcal{N}) = \Pi$.

<u>Proof</u>: Since A has only simple elementary divisors, it is similar to a diagonal matrix. As in the proof of the preceding theorem, rank (H) = rank ($\mathcal{L}_A(H)$). Likewise, $\mathcal{L}_A$ maps In(H) on In($\mathcal{L}_A(H)$). By Lyapunov's theorem (1a), $\exists H \in \Pi : \mathcal{L}_A(H) = -I \in \mathcal{N}$. Therefore, $\mathcal{L}_A(\Pi) \subseteq \mathcal{N}$. But by the alternative version (1b) of Lyapunov's theorem, $\mathcal{N} \subseteq \mathcal{L}_A(\Pi)$.

The second equation follows from $-\mathcal{L}_A(H) = \mathcal{L}_A(-H) = I$. QED

REFERENCES

1. Bellman, Richard. <u>Introduction to Matrix Analysis</u>. New York: McGraw-Hill Book Co., Inc., 1960.

2. Ostrowski, Alexander and Schneider, Hans. "Some Theorems on the Inertia of General Matrices." <u>Journal of Mathematical Analysis and Applications</u>, Vol. 4, No. 1 (1962 February), 72-84.

3. Schneider, Hans. "Topological Aspects of Sylvester's Theorem on the Inertia of Hermitian Matrices." <u>The American Mathematical Monthly</u> Vol. 73, No. 8 (1966 October), 817-821.

4. Stein, P. "On the Ranges of Two Functions of Positive Definite

Matrices." Journal of Algebra, Vol. 2, No. 3 (1965 September), 350-353.

5. Taussky, Olga. "A Remark on a Theorem of Lyapunov." Journal of Mathematical Analysis and Applications, Vol. 2, No. 1 (1961 February), 105-107.

6. Taussky, Olga. "A Generalization of a Theorem of Lyapunov." Journal of the Society for Industrial and Applied Mathematics, Vol. 9, No. 4 (1961 December), 640-643.

7. Taussky, Olga. "Matrices C with $C^n \rightarrow 0$." Journal of Algebra, Vol. 1, No. 1 (1964 April), 5-10.

Mathematical Problems in Engineering

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru