

# SUBLINEAR FUNCTIONALS ERGODICITY AND FINITE INVARIANT MEASURES

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**ABSTRACT.** By introducing a sublinear functional involving infinite matrices, we establish its connection with ergodicity and measure preserving transformation. Further, we characterize the existence of a finite invariant measure by means of a condition involving the above sublinear functional.

**KEY WORDS AND PHRASES.** Sublinear functional, Banach limit, conservative, regular, transitive, null invariant, measure preserving, ergodic.

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## 1. INTRODUCTION AND DEFINITIONS.

Let  $\ell_\infty$  be the set of all real bounded sequence  $\{x_n\}$ , normed by  $\|x\| = \sup_{n \geq 0} |x_n|$

Linear functional  $\phi$  on  $\ell_\infty$  are called Banach limit [1] satisfying the conditions,

$$i) \quad \phi(x_n) \geq 0, \text{ if } x_n \geq 0, n = 0, 1, 2 \dots$$

$$ii) \quad \phi(x_{n+1}) = \phi(x_n)$$

$$iii) \quad \lim_{n \rightarrow \infty} x_n \leq \phi(x_n) \leq \overline{\lim}_{n \rightarrow \infty} x_n$$

If there is a number for all Banach limits  $\phi$ , the sequence  $x = \{x_n\}$  is called almost convergent and we write;  $F - \lim_{n \rightarrow \infty} x_n = s$ . It is shown by Lorentz [2] that a sequence  $\{x_n\}$  is almost convergent with  $F$ -limit  $s$ , if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=i}^{i+n-1} x_k = s \quad (1.1)$$

uniformly in  $i$ .

Let,  $A = (a_{n,k}^{(i)})$  be a sequence of real or complex matrices for each  $i = 0, 1, 2, \dots$  such that  $a_{n,k}^{(i)} = 0$ , if any  $n, k, i$ , is a negative integer. The sequence  $\{x_n\}$  is called  $A$  summable to  $s$  if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k = s \quad (1.2)$$

uniformly in  $i$  and in this case we write:

$$A - \lim x_n = s, \text{ or } x_n \rightarrow s(A).$$

In the case  $a_{n,k}^{(i)} = 1/n+1$  ( $i \leq k \leq i+n$ ) and 0 otherwise, (A) reduces to the method (F). If  $A = A = a_{n,k}^{(i)}$ , then we obtain the usual summability method (A). It is significant to note that there does not exist any regular method (A) equivalent to method (F) (See Lorentz [2]. Theorem 11 and 12). In the case  $a_{n,k}^{(i)} = \frac{1}{n+1} \sum_{r=i}^{n+i} a_{r,k}$ , then (A) reduces to the almost summability method introduced by King [3].

The method (A) is called conservative, if  $x \rightarrow s \Rightarrow x \rightarrow s^1(A)$ , regular, if  $s = s^1$ . The following characterization of regular matrices is due to Stieglitz [4]. The method (A) is called regular if and only if the following conditions hold:

$$\sum_{k=0}^{\infty} |a_{n,k}^{(i)}| < \infty, \text{ for all } n \text{ and } i \geq 0, \quad (1.3)$$

and there exists an integer  $m$  such that

$$\sup_{i \geq 0, n \geq m} \sum_{k=0}^{\infty} |a_{n,k}^{(i)}| < \infty \quad (1.4)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} = 1, \text{ uniformly in } i, \quad (1.5)$$

$$\lim_{n \rightarrow \infty} a_{n,k}^{(i)} = 0, \text{ for fixed } k, \text{ uniformly in } i. \quad (1.6)$$

We write

$$||A|| = \sup_{i \geq 0, n \geq 0} \sum_{k=0}^{\infty} |a_{n,k}^{(i)}|$$

The matrix  $A$  is called translative, if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |d_{n,k}^{(i)}| = 0 \quad (1.7)$$

uniformly in  $i$ , where

$$d_{n,k}^{(i)} = (a_{n,k-1}^{(i)} - a_{n,k}^{(i)}) \quad (1.8)$$

The matrix  $A$  is called positive, if

$$a_{n,k}^{(i)} \geq 0, \quad \forall n, k, i \quad (1.9)$$

For real  $\lambda$  we write,

$$\lambda^+ = \max(\lambda, 0), \quad \lambda^- = \max(-\lambda, 0).$$

The matrix  $A$  is called almost positive, if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{-(i)} = 0, \text{ uniformly in } i. \quad (1.10)$$

Let  $(X, F, m)$  be a finite measure space and let,  $T; X \rightarrow X$  be a measurable transformation. (This is assumed throughout). The measure  $m$  is called null invariant, if  $m(A) = 0 \iff m(T^{-1}A) = 0$ ,  $A \in F$ . It is conservative, if  $A \cap T^{-n}A = \emptyset \implies m(A) = 0$ , for all  $n$ , and  $A \in F$ . A measure  $\mu$  is called equivalent to measure  $m$ , if  $m(A) \iff \mu(A) = 0$ , for  $A \in F$ . The transformation  $T$  is called measure preserving or invariant, if  $m(A) = m(T^{-1}A)$ ,  $A \in F$ . It is called ergodic if,  $T^{-1}A = A \implies m(A) = 0$  or  $m(X/A) = 0$ . The set  $A \in F$  is called invariant, if  $A = T^{-1}A$ . It is called wandering, if  $A, T^{-1}A, T^{-2}A \dots A \in F$ , are mutually disjoint. It is called weakly wandering, if there is an increasing sequence of positive integers  $\{r_k, k = 1, 2, 3 \dots\}$  such that  $A, T^{-r_1}A, T^{-r_2}A \dots$  are mutually disjoint. A measure  $q$  is called  $m$ -continuous if, for  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $m(A) < \delta \implies q(A) < \epsilon$ . A sequence of measures  $\{q_n\}$  is called uniformly  $m$  continuous if for each  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $m(A) < \delta \implies q_n(A) < \epsilon$  for all  $n$ .

Write:

$$t(x) = \overline{\lim}_{n \rightarrow \infty} \sup_i \frac{1}{n} \sum_{k=i}^{i+n-1} x_k, \quad x_k \in l_\infty. \quad (1.11)$$

Let  $\{l_\infty, t\}$  denote the set of linear functionals  $\phi$ , such that  $\phi(x) \leq t(x)$ . It is known (see Sucheston [5] Das and Misra [6]) that  $\{l_\infty, t\}$  is the set of all Banach limits on  $l_\infty$  and  $\phi \in \{l_\infty, t\}$  is unique if and only if  $\phi(x) = -\phi(-x)$  and this happens when

$$\frac{1}{n} \sum_{k=i}^{i+n-1} x_k \rightarrow \text{a limit}$$

as  $n \rightarrow \infty$ , uniformly in  $i$ . Lorentz [2] calls all such sequences as almost convergent sequences. Let  $A$  be real and such that  $||A|| < \infty$ . Then we define,  $R: l_\infty \rightarrow l_\infty$  by

$$R(x) = \overline{\lim}_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k. \quad (1.12)$$

Since, for all  $x \in l_\infty$

$$|R(x)| \leq ||x|| ||A|| < \infty,$$

$R$  is finite valued. It is easy to see that it is a sublinear functional on  $l_\infty$ . By Hahn-Banach theorem there exists a linear functional  $\phi$  on  $l_\infty$  such that

$$-R(-x) \leq \phi(x) \leq R(x), \quad x \in l_\infty \quad (1.13)$$

Let  $\{l_\infty, R\}$  be the set of all linear functional  $\phi$  satisfying (1.13). It is easily seen that  $\phi$  is unique if and only if

$$-R(-x) = R(x) \quad (1.14)$$

and this happens if and only if

$$\sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k \rightarrow \text{a limit}$$

as  $n \rightarrow \infty$ , uniformly in  $i$ .

We now state a lemma.

LEMMA 1. Let  $x \in l_\infty$ , then

$$(a) \quad \underline{\lim} x_n \leq R(x) \leq \overline{\lim} x_n$$

if and only if  $A$  is real, regular and almost positive.

$$(b) \quad -t(-x) \leq -R(-x) \leq R(x) \leq t(x)$$

if and only if  $A$  is regular, almost positive and translative.

(c) If  $x$  is almost convergent to  $s$ , then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k = s, \text{ uniformly in } i.$$

## 2. ERGODICITY.

In this section, we establish that the ergodicity and invariance can be established in terms of summability of a particular sequence and thus generalizes a result of (Sucheston [5], Theorem 3) involving almost convergence.

We now examine the following conditions:

(I) For some  $\phi \in \{1_\infty, R\}$   $\phi[m(T^{-n}B \cap C)] = m(B) \cap m(C)$ ,  $n = 0, 1, 2, \dots$

(II)  $\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} m(T^{-n}B \cap C) = m(B) m(C)$

uniformly in  $i$ ,  $\forall B, C \in F$ .

(III)  $T$  is ergodic and measure preserving.

THEOREM 1. Let  $(X, F, m)$  be a finite measure space and let  $\|A\| < \infty$ . Then

(a) (II)  $\Rightarrow$  (I)

(b) (i) (I)  $\Rightarrow T$  is ergodic

(ii) If  $A$  is translative. Then

(I)  $\Rightarrow$  (III)

(c) If  $A$  is regular, almost positive, and translative, then

(I)  $\Leftrightarrow$  (II)  $\Leftrightarrow$  (III)

We need the following lemma for the proof of the theorem.

LEMMA 2. Let  $\|A\| < \infty$ ,  $\phi \in \{1_\infty, R\}$ ,  $s: 1_\infty \rightarrow 1_\infty$  be the shift operator i.e.

$$s(x_n) = x_{n+1}, \quad s^2(x_n) = x_{n+2}.$$

Then

$$(a) \quad |\phi(SX) - \phi(x)| \leq \|x\| \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} |d_{n,k}^{(i)}|$$

where  $d_{n,k}^{(i)}$  is defined by (1.8).

Let  $A$  be translative, then for  $x \in 1_\infty$ .

(b) (i)  $R(SX - x) = R(x - SX) = 0$

(ii)  $\phi(Sx) = \phi(x)$

(c)  $R(Sx) = R(x)$

Let, further

$$\lim_{n \rightarrow \infty} a_{n,k}^{(i)} = 0, \text{ fixed } k, \text{ uniformly in } i. \quad (2.1)$$

Then

$$(d) \quad R\left(\sum_{j=0}^p s^{r_j} x\right) = p \cdot R(x)$$

Where  $r_0 = 1, r_1, r_2, \dots, r_p$  is a sequence of fixed positive integers.

PROOF: Since

$$R(Sx - x) = \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} (Sx_k - x_k)$$

$$= \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} d_{n,k}^{(i)} x_k .$$

It follows that

$$|R(Sx-x)| \leq ||x|| \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} |d_{n,k}^{(i)}| . \quad (2.2)$$

Now as  $\phi$  is linear, we obtain

$$\phi(sx) - \phi(x) = \phi(sx-x) \leq R(sx-x) . \quad (2.3)$$

Changing the role of  $sx$  and  $x$  in (2.2) and (2.3) we obtain (a). When  $A$  is trans-  
lative (b) (i), (ii) follows from (2.2), and changing the role of  $sx$  and  $x$  in  
(2.2) (b) (ii) follows from (a). Since,  $R$  is sublinear,

$$R(Sx) = R(Sx-x+x) \leq R(Sx-x) + R(x) = R(x)$$

by b (i). Changing the role of  $Sx$  and  $x$ , we obtain  $R(x) \leq R(Sx)$ . So (c) follows.

Lastly  $R(S^{r_1}_x + S^{r_2}_x) = R(S^{r_1}_x - x + S^{r_2}_x - x + 2x) \leq R(S^{r_1}_{x-x}) + R(S^{r_2}_{x-x}) + 2R(x)$ .  
i.e.

$$R(S^{r_1}_x + S^{r_2}_x) - 2R(x) \leq R(S^{r_1}_{x-x}) + R(S^{r_2}_{x-x}) \quad (2.3)$$

But,

$$\begin{aligned} R(S^{r_1}_{x-x}) &= \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} (x_{k+r_1} - x_k) \\ &= \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} (a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}) x_k \\ &= \lim_{n \rightarrow \infty} \sup_i \left[ \sum_{k=r_1}^{\infty} (a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}) x_k - \sum_{k=0}^{r_1-1} a_{n,k}^{(i)} x_k \right] \\ &= \lim_{n \rightarrow \infty} \sup_i \left[ \sum_{k=r_1}^{\infty} (a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}) x_k \right] \quad \text{by (2.1)} \\ &= \lim_{n \rightarrow \infty} \sup_i \sum_{k=r_1}^{\infty} x_k \sum_{j=0}^{r_1-1} (a_{n,k-j-1}^{(i)} - a_{n,k-j}^{(i)}) \\ &\leq ||x|| \sum_{j=0}^{r_1-1} \lim_{n \rightarrow \infty} \sup_i \sum_{k=r_1}^{\infty} |d_{n,k-j}^{(i)}| \\ &= 0 \quad ( \because A \text{ is translatable} ) \end{aligned}$$

Similarly

$$R(S^{r_2}_{x-x}) \leq 0 .$$

Hence,

$$R(S^{r_1}_x + S^{r_2}_x) \leq 2R(x), \quad x \in l_{\infty} .$$

Again, since

$$\begin{aligned} 2R(x) &= R(2x - S^{r_1}_x - S^{r_2}_x + S^{r_1}_x + S^{r_2}_x) \\ &\leq R(x - S^{r_1}_x) + R(S - S^{r_2}_x) + R(S^{r_1}_x + S^{r_2}_x) . \end{aligned}$$

Proceeding as above, we have

$$2R(x) \leq R(S^{r_1}_x + S^{r_2}_x), \quad x \in l_{\infty} .$$

Hence,

$$R(S^{r_1}_x + S^{r_2}_x) = 2R(x), \quad x \in l_{\infty} . \quad (2.4)$$

(d) follows by repeated application of (2.4).

PROOF OF THEOREM 1.

(a) Let (II) hold. Then

$$-R [-m(T^{-n}B \cap C)] = R [m(T^{-n}B \cap C)]$$

Since

$$-R(x) \leq Q(x) \leq R(x), \quad x \in l_\infty \quad (2.5)$$

It follows that

$$\phi [m(T^{-n}B \cap C)] = m(B) \cdot m(C), \quad n = 0, 1, 2, \dots$$

This proves (II)  $\Rightarrow$  (I).

(b) Take,  $T^{-1}B = B$ ,  $C = x/B = B^{-1}$  in (I).

Hence it follows that

$$0 = \phi(0) = m(B) \cdot m(B^{-1})$$

either  $m(B) = 0$  or  $m(B^{-1}) = 0$ .

i.e.  $T$  is ergodic.

Writing,  $C = X$  in (I), we obtain

$$\phi [m(T^{-n}B)] = m(B) \cdot m(X) \quad (2.6)$$

Replacing  $B$  by  $T^{-1}B$  in (2.6), we obtain

$$\phi [m(T^{-n-1}B)] = m(T^{-1}B) \cdot m(X) \quad (2.7)$$

If further,  $A$  is transitive, by Lemma 2 (b)

$$\phi [m(T^{-n-1}B)] = \phi [m(T^{-1}B)]$$

Again, since  $0 < m(X) < \infty$ , it follows from (2.6) and (2.7) that

$$m(T^{-1}B) = m(B)$$

Hence, (I)  $\Rightarrow$  (III).

(c) In view of (a) and (b), it is enough to show that (III)  $\Rightarrow$  (II). Take any fixed  $B \in \mathcal{F}$  such that  $m(B) > 0$ . Define, for  $\phi \in \{l_\infty, R\}$  and  $C \in \mathcal{F}$

$$\begin{aligned} q_n(c) &= \frac{m(T^{-n}B \cap C)}{m(B)}, \quad n = 0, 1, 2, \dots \\ q(c) &= \phi(q_n(c)) \end{aligned} \quad (2.8)$$

We now show that  $q$  is an invariant measure and  $m = q$ .

Since,  $A$  is almost positive

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{-(i)} x_k = 0, \quad \text{uniformly in } i, \quad (2.9)$$

for  $x_k \in l_\infty$ .

Write

$$R^+(x) = \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{+(i)} x_k.$$

So, by (2.9)

$$R(x) = R^+(x)$$

Since,  $x \geq 0 \Rightarrow R^+(x) \geq 0$

$$x \geq 0 \Rightarrow R(x) \geq 0. \quad (2.10)$$

Again, since  $m$  is a measure,  $q_n(c) \geq 0$ . So it follows from (2.10) that  $R(q_n(c)) \geq 0$ .

Since  $R$  is sublinear, we have

$$-R[-q_n(c)] \geq 0.$$

Now, it follows from (2.5) that  $q(c) \geq 0$ ,  $c \in F$ . Let  $B_i \in F$  be a countable sequence of disjoint sets. Then

$$\begin{aligned} q\left(\bigcup_{i=1}^{\infty} B_i\right) &= \phi\left[q_n\left(\bigcup_{i=1}^{\infty} B_i\right)\right] \\ &= \phi\left[\sum_{i=1}^{\infty} q_n(B_i)\right] \quad (\cdot, \cdot \text{ is a measure}) \\ &= \sum_{i=1}^{\infty} \phi[q_n(B_i)] \quad (\phi \text{ is continuous linear functional}) \end{aligned}$$

So,  $q$  is countably additive and hence it is a measure.

Next,

$$\begin{aligned} q(T^{-1}C) &= \phi\left[\frac{m(T^{-n}B \cap T^{-1}C)}{m(B)}\right] \\ &= \phi\left[\frac{m(T^{-n+1}B \cap C)}{m(B)}\right] \quad (\cdot, \cdot \text{ is a measure preserving}) \\ &= \phi[q_{n-1}(C)] \end{aligned}$$

Since  $\phi$  is shift invariant by Lemma 2 (b),

$$\begin{aligned} &= \phi[q_n(C)] \\ &= q(C), \quad C \in F. \end{aligned}$$

This proves that  $q$  is an invariant measure. Since  $T$  is ergodic, the invariant sets are of measure 0 or 1. Since  $m$  and  $q$  are invariant measures, an invariant measure is determined by the value it takes on invariant sets (See Sucheston [8], Theorem 2), it follows that  $q = m$ .

Now, we have

$$q(C) = m(C) = \phi\left[\frac{m(T^{-n}B \cap C)}{m(B)}\right] \quad \text{i.e.} \quad \phi[m(T^{-n}B \cap C)] = m(B) \cdot m(C).$$

Hence,  $\phi$  is unique on  $\{T^{-n}B \cap C\}$ ,  $n = 0, 1, 2, \dots$ . But,  $\phi \in \{l_{\infty}, R\}$  has unique value  $\lambda$  if and only if

$$R(x) = -R(-x) = \lambda.$$

Hence, it follows that

$$R[m(T^{-n}B \cap C)] = -R[-m(T^{-n}B \cap C)] = m(B) \cdot m(C).$$

i.e. (II) holds and hence proves (c) completely.

### 3. EQUIVALENT MEASURES.

Many necessary and sufficient conditions have been determined for the existence of equivalent invariant measures (see Sucheston [7], [8], Mrs. Dowker [9], Calderon [10], and Hajian and Kakutani [11]). In the pointwise ergodic theorem of Birkhoff [12], it was necessary to take invariant measure, but Halmos [13] has shown that even if a measure is null invariant and conservative, an equivalent measure need not exist. Sucheston [7], [8] has used Banach limit technique to prove the existence of invariant measures. We now generalize some of the theorems of Sucheston [5] involving almost convergence and some results of Mrs. Dowker on  $(C, 1)$  convergence and establish the existence of invariant measure by using linear functional  $\phi \in \{l_{\infty}, R\}$ .

We now prove

**THEOREM 2.** Let  $A$  be a real matrix such that  $\|A\| < \infty$  and let  $A$  be almost positive and translative. Let  $(x, F, m)$  be a finite measure space and  $T$  be a measurable

transformation. Then, the following condition are equivalent.

(I) There exists an equivalent finite invariant measure.

(II) For some  $\phi \in \{1_\infty, R\}$  and all  $B \in F$

$$m(B) > 0 \Rightarrow \phi [m(T^{-n}B)] > 0$$

(III)  $m(B) > 0 \Rightarrow R [m(T^{-n}B)] > 0$

PROOF. (I)  $\Rightarrow$  (II). Suppose that  $p$  is an invariant measure which is equivalent to  $m$ . Suppose that (II) fails to hold. Then there exists a  $B \in F$  such that  $m(B) > 0$  and

$$\phi [m(T^{-n}B)] = 0.$$

But, since  $-R(-x) \leq \phi(x) \leq R(x)$ ,  $x \in 1_\infty$  and by Lemma 1

$$\lim_{n \rightarrow \infty} x_n \leq -R(-x) \leq R(x) \leq \lim_{n \rightarrow \infty} x_n.$$

it follows that for all  $B \in F$

$$0 = \phi [m(T^{-n}B)] \geq \lim_{n \rightarrow \infty} m(T^{-n}B).$$

But, since  $\lim_{n \rightarrow \infty} m(T^{-n}B) \geq 0$ , it follows that

$$\lim_{n \rightarrow \infty} m(T^{-n}B) = 0.$$

Hence, there exists a sub sequence  $\{x_k\}$  such that

$$\lim_{k \rightarrow \infty} m(T^{-nk}B) = 0.$$

Since  $p$  is equivalent to  $m$ , we obtain

$$p(B) > 0 \text{ and } \lim_{k \rightarrow \infty} p(T^{-nk}B) = 0.$$

Since  $p$  is invariant, we have

$$p(T^{-nk}B) = p(B).$$

Hence  $p(B) = 0$ . This is a contradiction and this proves the fact that (I)  $\Rightarrow$  (II).

(II)  $\Rightarrow$  (III). Let II hold. Since,  $\phi [m(T^{-n}B)] \leq R [m(T^{-n}B)]$  it follows that

$$\phi [m(T^{-n}B)] > 0 \Rightarrow R [m(T^{-n}B)] > 0.$$

(III)  $\Rightarrow$  (I). Suppose (III) holds and (I) fails. Since Condition (I) is equivalent to non-existence of weakly wandering set (See Sucheston [7], Theorem 6) it follows that there exists positive integers  $r_0 = 1, r_1, r_2 \dots$  and a set  $B \in F$  with  $m(B) > 0$  such that

$$B, T^{-r_1} B, T^{-r_2} B, \dots, T^{-r_k} B \dots$$

are mutually disjoint. Since,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n a_{n,k}^{(i)} = 1, \text{ uniformly in } i, \text{ it follows that}$$

$$R[m(T^{-k}X)] = R[m(X)] = m(X) \quad R(1) = m(X).$$

Again

$$m(X) = R [m(T^{-n}X)] \geq R [m(\bigcup_{j=0}^s T^{-r_j} B)]$$



$$\begin{aligned} m(X) &= \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} m \left[ \bigcup_{j=0}^S T^{-rj-k} B \right] \\ &= \lim_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} \sum_{j=0}^S s^{rj} x_k . \end{aligned}$$

Where  $X_k = m(T^{-k} B)$ ,  $S$  is a shift operator. Then by Lemma 2(d),

$$m(X) \geq S \cdot R(X) \quad (3.1)$$

Then it follows from (3.1) that

$$m(X) \geq s \cdot R \quad m(T^{-k} B) . \quad (3.2)$$

Since  $m(T^{-k} S) > 0$  by hypothesis and  $s$  is an arbitrary positive integer. This contradicts (3.2). This proves (III)  $\Rightarrow$  (I).

In the next theorem we give yet another characterization of existence of invariant measure in terms of the sublinear functional  $R(x)$ .

**THEOREM 3.** Let  $A$  satisfy the condition of Theorem 2. Let  $(x, F, m)$  be the finite measure space. Then there exists an invariant measure equivalent to measure  $m$  on  $X$ , if and only if,

- (i)  $m$  is null-preserving.
- (ii)  $T$  is conservative.
- (iii)  $\sum_{n=0}^{\infty} a_{n,k}^{(i)} m(T^{-k} B)$  converges uniformly in  $i$ , for every  $B \in F$ .

Again, whenever it has equivalent invariant measure, then the map  $q : F \rightarrow R$  defined by  $q(B) = R[m(T^{-n} B)]$  is an invariant measure equivalent to  $m$  and agrees with  $m$  on invariant sets.

**PROOF: NECESSITY .**

Let us assume that  $m$  admits an invariant equivalent measure  $\mu$ . Then  $\mu$  is  $m$  continuous (See Halmos [9] p. 125).

Write for  $\phi \in \{1_{\infty}, R\}$

$$q(B) = \phi[m(T^{-n} B)]$$

We want to show

- (a)  $q$  is a measure
- (b)  $q$  is a  $m$  continuous
- (c)  $q$  is invariant.

As in the proof of Theorem 1, we can show that

$$q(B) \geq 0, \text{ for all } B \in F .$$

It is easy to show that

$$B, C \in F, \quad B \subset C \Rightarrow q(B) \leq q(C) .$$

Since  $\phi$  is linear, it also follows that  $q$  is finitely additive. Since  $\mu$  is  $m$ -continuous, for given  $\varepsilon > 0$ ,  $\delta > 0$ . Such that

$$m(T^{-n} B) < \varepsilon \text{ when } \mu(B) = \mu(T^{-n} B) < \delta, \text{ and } m(T^{-n} B) < \varepsilon \Rightarrow q(B) < \varepsilon .$$

So  $q$  is  $m$ -continuous. The countably additivity of  $m$  and  $m$ -continuity of  $q$  (See Halmos [9] p. 39).

Next,

$$\begin{aligned} q(T^{-1} B) - q(B) &= \phi[m(T^{-n-1} B)] - \phi[m(T^{-n} B)] \\ &= \phi[m(T^{-n-1} B) - m(T^{-n} B)] \quad (\phi \text{ is linear}). \end{aligned}$$

$$\begin{aligned} &\leq R [m(T^{-n-1}B) - m(T^{-n}B)] \\ &= \overline{\lim}_n \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} [m(T^{-k-1}B) - m(T^{-k}B)] . \end{aligned}$$

Since  $a_{n-1}^{(i)} = 0, \forall n$  and  $i$

$$\begin{aligned} &= \overline{\lim}_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} [a_{n,k-1}^{(i)} - a_{n,k}^{(i)}] m(T^{-k}B) \\ &\leq m(X) \overline{\lim}_{n \rightarrow \infty} \sup_i \sum_{k=0}^{\infty} |d_{n,k}^{(i)}| . \end{aligned}$$

Since  $A$  is translatable,

$$\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ uniformly in } i.$$

Hence,

$$q(T^{-1}B) \leq q(B).$$

Changing the role of  $T^{-1}B$  and  $B$ , we obtain

$$q(B) \leq q(T^{-1}B) .$$

Hence,

$$q(T^{-1}B) = q(B)$$

i.e.  $q$  is invariant under  $T$ .

Now if  $T^{-1}B = B$ . Then,

$$\begin{aligned} q(B) &= \phi[m(T^{-1}B)] = \phi[m(B)] , \\ &= m(B) \cdot \phi(1) = m(B) . \end{aligned}$$

So  $q = m$  on invariant sets. Hence (Sucheston [8], Theorem 2)  $q = m$  on  $F$ . Thus

$q(B) = \phi[m(T^{-n}B)]$  is unique. But  $\phi \in \{1_{\infty}, R\}$  is unique if and only if  $R(x) = -R(-x) = q(B)$  and this happens if and only if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} m(T^{-k}B) = q(B)$$

uniformly in  $i$ .

Now since,  $q(T^{-1}B) = q(B)$ ,  $B \in F$  and  $q = m$  on  $F$ , we have  $m(T^{-1}B) = m(B)$ ,  $B \in F$  so  $m(B) = 0 \Rightarrow m(T^{-1}B) = 0$

i.e.  $m$  is null-preserving.

Again (See Sucheston [7], Theorem 6) existence of invariant measure is equivalent to non-existence of weakly wandering sets and non-existence of weakly wandering sets is the same as conservativeness of  $T$ .

SUFFICIENCY:

Let (i), (ii) and (iii) hold. Define

$$q(B) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} m(T^{-k}B)$$

Then it can be proved as before that  $q$  is an invariant measure. So only we have to prove  $q$  is equivalent to  $m$ . Since  $T$  is null preserving,

$$m(B) = 0 \Rightarrow m(T^{-1}B) = 0.$$

Then

$$q(B) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} m(T^{-k}B) = 0$$

uniformly in  $i$ .

Conversely, let  $q(B) = 0$ .

Write;

$$\begin{aligned} A^* &= \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^{-i}B. \text{ Then} \\ q(A^*) &= q\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^{-i}B\right) \\ &\leq q\left(\bigcup_{i=1}^{\infty} T^{-i}B\right) \\ &\leq \sum_{i=1}^{\infty} q(T^{-i}B) \quad (q \text{ is a measure}) \\ &= \sum_{i=n}^{\infty} q(B) \quad (q \text{ is invariant}) \\ &= 0 \end{aligned}$$

Since,  $q$  and  $m$  agrees on invariant sets, we have  $m(A^*) = 0$ . Since,  $T$  is conservative by recurrence theorem  $m(B/A^*) = 0 \Rightarrow m(B) = 0$ .

Hence  $q$  is equivalent to  $m$ .

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