

RIEMANN-STIELTJES OPERATORS BETWEEN BERGMAN-TYPE SPACES AND α -BLOCH SPACES

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Received 8 December 2005; Revised 11 May 2006; Accepted 28 May 2006

We study the following integral operators: $J_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi$; $I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi$, where g is an analytic function on the open unit disk in the complex plane. The boundedness and compactness of J_g, I_g between the Bergman-type spaces and the α -Bloch spaces are investigated.

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1. Introduction

Let D be the open unit disk in the complex plane. Denote by $H(D)$ the class of all analytic functions on D . An analytic function f in D is said to belong to the α -Bloch space $\mathcal{B}^\alpha$, or Bloch-type space, if

$$\|f\|_\alpha = \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty. \quad (1.1)$$

The expression $\|f\|_\alpha$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. It makes $\mathcal{B}^\alpha$ into a Banach space.

A positive continuous function ϕ on $[0, 1)$ is normal, if there exists $0 < s < t$ such that (see [7])

$$\frac{\phi(r)}{(1-r)^s} \downarrow 0, \quad \frac{\phi(r)}{(1-r)^t} \uparrow \infty, \quad \text{as } r \rightarrow 1^-. \quad (1.2)$$

For $0 < p < \infty$ and a normal function ϕ , let $H(p, p, \phi)$ denote the space of all analytic functions f on D such that

$$\|f\|_{H(p, p, \phi)} = \int_D |f(z)|^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) < \infty. \quad (1.3)$$

Here dA denotes the normalized Lebesgue area measure on the unit disk D such that $A(D) = 1$. We call $H(p, p, \phi)$ the Bergman-type space. If $1 \leq p < \infty$, $H(p, p, \phi)$ is a Banach

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space equipped with the norm $\|f\|_{H(p,p,\phi)}$. When $0 < p < 1$, $H(p,p,\phi)$ is a Fréchet space. In particular, if $\phi(r) = (1-r)^{1/p}$, then $H(p,p,\phi)$ is the Bergman space A^p .

For an analytic function $f(z)$ on the unit disk D with the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the Cesàro operator acting on f is

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n. \quad (1.4)$$

The integral form of $\mathcal{C}$ is

$$\mathcal{C}(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta} d\zeta = \frac{1}{z} \int_0^z f(\zeta) \left(\ln \frac{1}{1-\zeta} \right)' d\zeta, \quad (1.5)$$

taking simply as a path the segment joining 0 and z , we have that

$$\mathcal{C}(f)(z) = \int_0^1 f(tz) \left(\ln \frac{1}{1-\zeta} \right)' \Big|_{\zeta=tz} dt. \quad (1.6)$$

The following operator:

$$\mathcal{Z}\mathcal{C}(f)(z) = \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta \quad (1.7)$$

is closely related to the previous operator and on many spaces the boundedness of these two operators is equivalent. It is well known that Cesàro operator acts as a bounded linear operator on various analytic function spaces (see, e.g., [6, 9, 13, 15, 16, 18, 20], and the references therein).

Suppose that $g: \mathbb{D} \rightarrow \mathbb{C}^1$ is an analytic map, $f \in H(\mathbb{D})$. A class of integral operator introduced by Pommerenke is defined by (see [11])

$$J_g f(z) = \int_0^z f dg = \int_0^1 f(tz) z g'(tz) dt = \int_0^z f(\xi) g'(\xi) d\xi, \quad z \in \mathbb{D}. \quad (1.8)$$

The operator J_g can be viewed as a generalization of the Cesàro operator which was called the Riemann-Stieltjes operator (see [21]).

In [11], Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. Aleman and Siskakis showed that J_g is bounded (compact) on the Hardy space H^p , $1 \leq p < \infty$, if and only if $g \in BMOA$ ($g \in VMOA$), and that J_g is bounded (compact) on the Bergman space A^p if and only if $g \in \mathcal{B}$ ($g \in \mathcal{B}_0$), see [2, 3]. Recently, J_g acting on various function spaces, including the Bloch space, the weighted Bergman space, the BMOA, and VMOA spaces have been studied (see [1–3, 17, 22], and the related references therein).

Another integral operator has recently been defined as the following (see [22]):

$$I_g f(z) = \int_0^z f'(\xi) g(\xi) d\xi. \quad (1.9)$$

In this paper, we study the boundedness and compactness of the operators J_g , I_g between the Bergman-type space and the α -Bloch space.

Constants are denoted by C in this paper, they are positive and may differ from one occurrence to the other. $a \leq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \leq b$ and $b \leq a$ hold, then one says that $a \asymp b$.

2. $J_g, I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$

In this section, we consider the boundedness and compactness of $J_g, I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$. First, let us state some useful lemmas.

LEMMA 2.1. Assume that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. If $f \in H(p, p, \phi)$, then

$$|f(z)| \leq C \frac{\|f\|_{H(p, p, \phi)}}{\phi(|z|)(1 - |z|^2)^{1/p}}. \quad (2.1)$$

Proof. Let $\beta(z, w)$ denote the Bergman metric between two points z and w in D . It is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}. \quad (2.2)$$

For $a \in D$ and $r > 0$, the set $D(a, r) = \{z \in D : \beta(a, z) < r\}$ is a Bergman metric disk with center a and radius r . It is well known that (see [25])

$$\frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} \asymp \frac{1}{(1 - |z|^2)^2} \asymp \frac{1}{(1 - |a|^2)^2} \asymp \frac{1}{|D(a, r)|}, \quad (2.3)$$

when $z \in D(a, r)$. For $0 < r < 1$ and $z \in D$, by the subharmonicity of $|f(z)|^p$ and the normality of ϕ , we get

$$\begin{aligned} |f(z)|^p &\leq \frac{C}{(1 - |z|^2)^2} \int_{D(z, r)} |f(a)|^p dA(a) \\ &\leq \frac{C}{(1 - |z|^2)\phi^p(|z|)} \int_{D(z, r)} (1 - |a|)^{-1} \phi^p(|a|) |f(a)|^p dA(a) \\ &\leq \frac{C}{(1 - |z|^2)\phi^p(|z|)} \int_D (1 - |a|)^{-1} \phi^p(|a|) |f(a)|^p dA(a) \leq \frac{C\|f\|_{H(p, p, \phi)}^p}{(1 - |z|^2)\phi^p(|z|)}, \end{aligned} \quad (2.4)$$

from which we get the desired result. $\square$

The following lemma can be found in [7, Theorem 2].

LEMMA 2.2. Assume that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. Then for $f \in H(D)$,

$$\|f\|_{H(p, p, \phi)}^p \asymp |f(0)|^p + \int_D |f'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z). \quad (2.5)$$

From the proof of Lemmas 2.1 and 2.2, we get the following result.

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LEMMA 2.3. Assume that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. If $f \in H(p, p, \phi)$ and $z \in D$, then

$$|f'(z)| \leq C \frac{\|f\|_{H(p, p, \phi)}}{\phi(|z|)(1 - |z|^2)^{1/p+1}}, \quad (z \in D). \quad (2.6)$$

The following lemma can be found in [14].

LEMMA 2.4. For $\beta > -1$ and $m > 1 + \beta$,

$$\int_0^1 (1 - \rho r)^{-m} (1 - r)^\beta dr \leq C(1 - \rho)^{1+\beta-m}, \quad 0 < \rho < 1. \quad (2.7)$$

The next lemma can be proved in a standard way (see [5]).

LEMMA 2.5. The operator J_g (or I_g): $H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact if and only if for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $H(p, p, \phi)$ which converges to zero uniformly on compact subsets of D , $J_g f_k$ (or $I_g f_k$) $\rightarrow 0$ in $\mathcal{B}^\alpha$ as $k \rightarrow \infty$.

THEOREM 2.6. Assume that $0 < p < \infty$, $\alpha > 0$, and ϕ is normal on $[0, 1)$. Then the operator $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z \in D} \frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| < \infty. \quad (2.8)$$

Moreover, the following relationship:

$$\|J_g\|_{H(p, p, \phi) \rightarrow \mathcal{B}^\alpha} \asymp \sup_{z \in D} \frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| \quad (2.9)$$

holds.

Proof. By (1.8), it is easy to see that $(J_g f)'(z) = f(z)g'(z)$, $(J_g f)(0) = 0$. Let $f(z) \in H(p, p, \phi)$. We have

$$(1 - |z|^2)^\alpha |(J_g f)'(z)| \leq C \|f\|_{H(p, p, \phi)} \frac{(1 - |z|^2)^\alpha}{\phi(|z|)(1 - |z|^2)^{1/p}} |g'(z)|. \quad (2.10)$$

Taking supremum over the unit disk in this inequality, we obtain

$$\|J_g f\|_{\mathcal{B}^\alpha} \leq C \|f\|_{H(p, p, \phi)} \sup_{z \in D} \frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)|. \quad (2.11)$$

Therefore, (2.8) implies that $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded.

Conversely, suppose J_g is a bounded operator from $H(p, p, \phi)$ to $\mathcal{B}^\alpha$. For $w \in D$, let

$$f_w(z) = \frac{(1 - |w|^2)^{t+1}}{\phi(|w|)(1 - \bar{w}z)^{1/p+t+1}}. \quad (2.12)$$

It is easy to see that

$$f_w(w) = \frac{1}{\phi(|w|)(1-|w|^2)^{1/p}}, \quad |f'_w(w)| = \left(\frac{1}{p} + t + 1\right) \frac{|\bar{w}|}{\phi(|w|)(1-|w|^2)^{1/p+1}}. \quad (2.13)$$

By [12], we get

$$M_p(f_w, r) \leq C \frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-r|w|)^{t+1}}. \quad (2.14)$$

Since ϕ is normal, by Lemma 2.4,

$$\begin{aligned} \|f_w\|_{p,p,\phi}^p &= \int_0^1 r(1-r)^{-1} \phi^p(r) M_p^p(f_w, r) dr \leq \int_0^1 (1-r)^{-1} \phi^p(r) \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}} dr \\ &\leq \int_0^{|w|} (1-r)^{-1} \frac{\phi^p(r)(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}} dr + \int_{|w|}^1 (1-r)^{-1} \frac{\phi^p(r)(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}} dr \\ &\leq (1-|w|^2)^p \int_0^{|w|} \frac{(1-r)^{pt-1} dr}{(1-r|w|)^{p(t+1)}} + (1-|w|^2)^{p(t+1)-ps} \int_{|w|}^1 \frac{(1-r)^{ps-1} dr}{(1-r|w|)^{p(t+1)}} \\ &\leq (1-|w|^2)^p \int_0^1 \frac{(1-r)^{pt-1} dr}{(1-r|w|)^{p(t+1)}} + (1-|w|^2)^{p(t+1)-ps} \int_0^1 \frac{(1-r)^{ps-1} dr}{(1-r|w|)^{p(t+1)}} \leq C. \end{aligned} \quad (2.15)$$

Therefore, $f_w \in H(p, p, \phi)$ (or see [23]). Moreover, there is a positive constant C such that $\|f_w\|_{H(p,p,\phi)} \leq C$. Hence

$$(1-|z|^2)^\alpha |f_w(z)g'(z)| \leq \|J_g f_w\|_{\mathcal{B}^\alpha} \leq \|J_g\|_{H(p,p,\phi) \rightarrow \mathcal{B}^\alpha} \|f_w\|_{H(p,p,\phi)} \quad (2.16)$$

for every $z, w \in D$.

From this and (2.13), we have

$$\frac{(1-|w|^2)^\alpha}{\phi(|w|)(1-|w|^2)^{1/p}} |g'(w)| \leq (1-|w|^2)^\alpha |f_w(w)g'(w)| \leq C \|J_g\|_{H(p,p,\phi) \rightarrow \mathcal{B}^\alpha}, \quad (2.17)$$

from which (2.8) follows. Combining (2.11) with (2.17), we get (2.9). $\square$

THEOREM 2.7. Assume that $0 < p < \infty$, $\alpha > 0$, and ϕ is normal on $[0, 1)$. Then $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z \in D} \frac{(1-|z|^2)^{\alpha-1/p-1}}{\phi(|z|)} |g(z)| < \infty. \quad (2.18)$$

Moreover, the following relationship:

$$\|I_g\|_{H(p,p,\phi) \rightarrow \mathcal{B}^\alpha} \asymp \sup_{z \in D} \frac{(1 - |z|^2)^{\alpha-1/p-1}}{\phi(|z|)} |g(z)| \quad (2.19)$$

holds.

Proof. Similar to the case of I_g , we have $(I_g f)' = f'(z)g(z)$, $(I_g f)(0) = 0$. Assume (2.18) holds. Let $f(z) \in H(p, p, \phi)$. Then

$$(1 - |z|^2)^\alpha |(I_g f)'(z)| \leq C \|f\|_{H(p,p,\phi)} \frac{(1 - |z|^2)^\alpha}{\phi(|z|)(1 - |z|^2)^{1/p+1}} |g(z)|. \quad (2.20)$$

It follows that $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded.

Conversely, if $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded. For $w \in D$, let $f_w(z)$ be defined by (2.12). From (2.3) and (2.13),

$$\begin{aligned} & \left(\frac{1}{p} + t + 1 \right)^2 \frac{|\bar{w}|^2}{\phi^2(|w|)(1 - |w|^2)^{2(1/p+1)}} |g(w)|^2 \\ &= |f'_w(w)g(w)|^2 \leq \frac{C}{(1 - |w|^2)^2} \int_{D(w,r)} |f'_w(z)|^2 |g(z)|^2 dA(z) \\ &\leq \frac{C}{(1 - |w|^2)^2} \int_{D(w,r)} |f'_w(z)|^2 |g(z)|^2 (1 - |z|^2)^{2\alpha} \frac{1}{(1 - |z|^2)^{2\alpha}} dA(z) \\ &\leq C \int_{D(w,r)} \frac{dA(z)}{(1 - |z|^2)^{2\alpha+2}} \sup_{z \in D(w,r)} (1 - |z|^2)^{2\alpha} |f'_w(z)|^2 |g(z)|^2 \\ &\leq \frac{C}{(1 - |w|^2)^{2\alpha}} \|I_g f_w\|_{\mathcal{B}^\alpha}^2, \end{aligned} \quad (2.21)$$

that is,

$$\frac{|\bar{w}|(1 - |w|^2)^\alpha}{\phi(|w|)(1 - |w|^2)^{1/p+1}} |g(w)| \leq C \|I_g f_w\|_{\mathcal{B}^\alpha} \leq C \|I_g\|_{H(p,p,\phi) \rightarrow \mathcal{B}^\alpha}. \quad (2.22)$$

Taking supremum in the last inequality over the set $1/2 \leq |w| < 1$ and noticing that by the maximum modulus principle there is a positive constant C independent of $g \in H(D)$ such that

$$\sup_{|w| \leq 1/2} \frac{(1 - |w|^2)^\alpha}{\phi(|w|)(1 - |w|^2)^{1/p+1}} |g(w)| \leq C \sup_{1/2 \leq |w| < 1} \frac{|\bar{w}|(1 - |w|^2)^\alpha}{\phi(|w|)(1 - |w|^2)^{1/p+1}} |g(w)|, \quad (2.23)$$

the result follows. From (2.20) and (2.22), we obtain (2.19). $\square$

THEOREM 2.8. Assume that $0 < p < \infty$, $\alpha > 0$, and ϕ is normal on $[0, 1)$. Then the operator $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| = 0. \quad (2.24)$$

Proof. First, we assume that (2.24) holds. In order to prove that J_g is compact, by Lemma 2.5, it suffices to show that if $\{f_n\}$ is a bounded sequence in $H(p, p, \phi)$ that converges to 0 uniformly on compact subsets of D , then $\|J_g f_n\|_{\mathcal{B}^\alpha} \rightarrow 0$. Let $\{f_n\}$ be a sequence in $H(p, p, \phi)$ with $\|f_n\|_{H(p, p, \phi)} \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of D . By the assumption, for any $\epsilon > 0$, there is a constant δ , $0 < \delta < 1$, such that $\delta < |z| < 1$ implies

$$\frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| < \frac{\epsilon}{2}. \quad (2.25)$$

Let $K = \{z \in D : |z| \leq \delta\}$. Note that K is a compact subset of D and ϕ is normal, we have

$$\begin{aligned} \|J_g f_n\|_{\mathcal{B}^\alpha} &= \sup_{z \in D} (1 - |z|^2)^\alpha |(J_g f_n)'(z)| \\ &\leq \sup_{z \in K} (1 - |z|^2)^\alpha |g'(z) f_n(z)| + \sup_{z \in D \setminus K} \frac{(1 - |z|^2)^\alpha}{\phi(|z|)(1 - |z|^2)^{1/p}} |g'(z)| \|f_n\|_{H(p, p, \phi)} \\ &\leq CN \sup_{z \in K} (1 - |z|^2)^{s+1/p} |f_n(z)| + \frac{C\epsilon}{2}, \end{aligned} \quad (2.26)$$

where

$$N = \sup_{z \in D} \frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)|. \quad (2.27)$$

By the assumption and Theorem 2.6, we obtain $\|J_g f_n\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact.

Conversely, suppose $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact. Let $\{z_n\}$ be a sequence in D such that $|z_n| \rightarrow 1$ as $n \rightarrow \infty$. Let

$$f_n(z) = \frac{(1 - |z_n|^2)^{t+1}}{\phi(|z_n|)(1 - \bar{z}_n z)^{1/p+t+1}}. \quad (2.28)$$

Then $f_n \in H(p, p, \phi)$ and f_n converges to 0 uniformly on compact subsets of D (see [7]). Since J_g is compact, by Lemma 2.5, $\|J_g f_n\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $n \rightarrow \infty$. In addition,

$$\|J_g f_n\|_{\mathcal{B}^\alpha} = \sup_{z \in D} (1 - |z|^2)^\alpha |(J_g f_n)'(z)| \geq \frac{(1 - |z_n|^2)^\alpha |g'(z_n)|}{\phi(|z_n|)(1 - |z_n|^2)^{1/p}}, \quad (2.29)$$

from which the result follows. $\square$

THEOREM 2.9. Assume that $0 < p < \infty$, $\alpha > 0$, and ϕ is normal on $[0, 1)$. Then $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^{\alpha-1/p-1}}{\phi(|z|)} |g(z)| = 0. \quad (2.30)$$

Proof. Suppose that $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact. Let $\{z_n\}$ be a sequence in D such that $|z_n| \rightarrow 1$ as $n \rightarrow \infty$. Let $f_n(z)$ be defined by (2.28). Then from the proof of Theorem 2.8 and the compactness of I_g , $\|I_g f_n\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $n \rightarrow \infty$. In addition,

$$\|I_g f_n\|_{\mathcal{B}^\alpha} = \sup_{z \in D} (1 - |z|^2)^\alpha |(I_g f_n)'(z)| \geq \left(\frac{1}{p} + t + 1\right) \frac{(1 - |z_n|^2)^\alpha |g(z_n) \overline{z_n}|}{\phi(|z_n|)(1 - |z_n|^2)^{1+1/p}}, \quad (2.31)$$

from which we get the desired result. $\square$

Assume (2.30) holds, in order to prove that I_g is compact, it suffices to show that if $\{f_n\}$ is a bounded sequence in $H(p, p, \phi)$ that converges to 0 uniformly on compact subsets of D , then $\|I_g f_n\|_{\mathcal{B}^\alpha} \rightarrow 0$. Let $\{f_n\}$ be a sequence in $H(p, p, \phi)$ with $\|f_n\|_{H(p, p, \phi)} \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of D . By the assumption, for any $\epsilon > 0$, there is a constant δ , $0 < \delta < 1$, such that $\delta < |z| < 1$ implies

$$\frac{(1 - |z|^2)^\alpha |g(z)|}{\phi(|z|)(1 - |z|^2)^{1+1/p}} < \frac{\epsilon}{2}. \quad (2.32)$$

Similar to the proof of Theorem 2.8, we have

$$\|I_g f_n\|_{\mathcal{B}^\alpha} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.33)$$

Therefore, $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is compact.

From the introduction, we can easily get the following corollary.

COROLLARY 2.10. Let $0 < p < \infty$, $\alpha \geq 1 + 2/p$. Then

(1) $J_g : A^p \rightarrow \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z \in D} (1 - |z|^2)^{\alpha-2/p} |g'(z)| < \infty, \quad (2.34)$$

(2) $I_g : A^p \rightarrow \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z \in D} (1 - |z|^2)^{\alpha-2/p-1} |g(z)| < \infty, \quad (2.35)$$

(3) $J_g : A^p \rightarrow \mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-2/p} |g'(z)| = 0, \quad (2.36)$$

(4) $I_g : A^p \rightarrow \mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-2/p-1} |g(z)| = 0. \quad (2.37)$$

3. $J_g, I_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$

In this section, we characterize the boundedness and compactness of $J_g, I_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$. For this purpose, we need Lemma 3.1. When $\alpha = 1$, Lemma 3.1 was proved in [8]. For the general case, the proof is similar to the proof of the case $\alpha = 1$. We omit the details.

LEMMA 3.1. *A closed set K in $\mathcal{B}_0^\alpha$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z| \rightarrow 1} \sup_{f \in K} (1 - |z|^2)^\alpha |f'(z)| = 0. \quad (3.1)$$

THEOREM 3.2. *Assume that $0 < p < \infty$, $\alpha > 0$, and ϕ is normal on $[0, 1)$. Then the following statements hold.*

- (i) $J_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is bounded if and only if $g \in \mathcal{B}_0^\alpha$ and $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded.
- (ii) $J_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| = 0. \quad (3.2)$$

Proof. (i) It is clear to see that $g \in \mathcal{B}_0^\alpha$ and $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded if $J_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is bounded.

Conversely, suppose that $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded and $g \in \mathcal{B}_0^\alpha$. For any polynomial $p(z)$, since $g \in \mathcal{B}_0^\alpha$ and

$$(1 - |z|^2)^\alpha (J_g p)'(z) = (1 - |z|^2)^\alpha p(z) g'(z), \quad (3.3)$$

we know that $J_g p \in \mathcal{B}_0^\alpha$. For any $f \in H(p, p, \phi)$, there exists a sequence of polynomials $\{p_n\}$ such that $\|f - p_n\|_{H(p, p, \phi)} \rightarrow 0$ as $n \rightarrow \infty$. Since $\mathcal{B}_0^\alpha$ is closed, we get

$$J_g f = \lim_{n \rightarrow \infty} J_g p_n \in \mathcal{B}_0^\alpha. \quad (3.4)$$

In addition, $J_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded. Therefore, $J_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is bounded.

- (ii) If $J_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is compact, then by Theorem 2.8, we get (3.2).

Conversely, assume that (3.2) holds. It follows from Lemma 3.1 that $J_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{H(p, p, \phi)} \leq 1} (1 - |z|^2)^\alpha |(J_g f)'(z)| = 0. \quad (3.5)$$

In fact,

$$(1 - |z|^2)^\alpha |(J_g f)'(z)| = \frac{(1 - |z|^2)^\alpha |g'(z)|}{\phi(|z|)(1 - |z|^2)^{1/p}} \phi(|z|)(1 - |z|^2)^{1/p} |f(z)|. \quad (3.6)$$

By Lemma 2.1, the result follows. $\square$

Similarly, we have the following results.

THEOREM 3.3. *Assume that $0 < p < \infty$, $\alpha > 0$, and ϕ is normal on $[0, 1)$. Then the following statements hold.*

(i) $I_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is bounded if and only if $I_g : H(p, p, \phi) \rightarrow \mathcal{B}^\alpha$ is bounded and

$$\lim_{|z| \rightarrow 1} |g(z)| (1 - |z|^2)^\alpha = 0. \quad (3.7)$$

(ii) $I_g : H(p, p, \phi) \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^{\alpha-1/p-1}}{\phi(|z|)} |g(z)| = 0. \quad (3.8)$$

COROLLARY 3.4. *Let $0 < p < \infty$, $\alpha > 0$. Then the following statements hold.*

(i) $J_g : A^p \rightarrow \mathcal{B}_0^\alpha$ is bounded if and only if $J_g : A^p \rightarrow \mathcal{B}^\alpha$ is bounded and $g \in \mathcal{B}_0^\alpha$.

(ii) $J_g : A^p \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-2/p} |g'(z)| = 0. \quad (3.9)$$

COROLLARY 3.5. *Let $0 < p < \infty$, $\alpha > 0$. Then the following statements hold.*

(i) $I_g : A^p \rightarrow \mathcal{B}_0^\alpha$ is bounded if and only if $I_g : A^p \rightarrow \mathcal{B}^\alpha$ is bounded and

$$\lim_{|z| \rightarrow 1} |g(z)| (1 - |z|^2)^\alpha = 0. \quad (3.10)$$

(ii) $I_g : A^p \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-2/p-1} |g(z)| = 0. \quad (3.11)$$

Remark 3.6. In Corollary 3.5, if $\alpha \leq 2/p + 1$, then by the maximum modulus principle, it is easy to see that $g \equiv 0$.

4. $J_g, I_g : \mathcal{B}^\alpha \rightarrow H(p, p, \phi)$

The following lemma is well known (e.g., see [19]).

LEMMA 4.1. *Let $f \in \mathcal{B}^\alpha(D)$, $0 < \alpha < \infty$. Then*

$$|f(z)| \leq \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^\alpha} \frac{1 - (1 - |z|)^{1-\alpha}}{1 - \alpha}, & \alpha \neq 1, \\ |f(0)| + \|f\|_{\mathcal{B}^\alpha} \ln \frac{2}{1 - |z|}, & \alpha = 1. \end{cases} \quad (4.1)$$

The following lemma can be found in [10].

LEMMA 4.2. *Let μ be a positive measure on D and $0 < p < \infty$. Let either $0 < \alpha < \infty$ and $n \in \mathbb{N}$, or $1 < \alpha < \infty$ and $n = 0$. Then*

$$\int_D \frac{d\mu(z)}{(1 - |z|^2)^{\alpha p}} < \infty \quad (4.2)$$

if and only if there is a positive constant C such that

$$\left(\int_D |f^{(n)}(z)|^p (1 - |z|^2)^{p(n-1)} d\mu(z) \right)^{1/p} \leq C \|f\|_{\mathcal{B}^\alpha} \quad (4.3)$$

for all analytic functions f in D , in particular, for all $f \in \mathcal{B}^\alpha$.

Let $0 < p < \infty$, let μ be a positive Borel measure on D . Define

$$D_p(\mu) = \left\{ f \in H(D), \|f\|_{D_p(\mu)}^p = \int_D |f'(z)|^p d\mu(z) < \infty \right\}. \quad (4.4)$$

LEMMA 4.3. Let μ be a positive measure on D and $0 < p, \alpha < \infty$. Then the following statements are equivalent.

- (1) $i : \mathcal{B}^\alpha \rightarrow D_p(\mu)$ is bounded.
- (2) $i : \mathcal{B}^\alpha \rightarrow D_p(\mu)$ is compact.
- (3) $i : \mathcal{B}_0^\alpha \rightarrow D_p(\mu)$ is bounded.
- (4) $i : \mathcal{B}_0^\alpha \rightarrow D_p(\mu)$ is compact.
- (5)

$$\int_D \frac{d\mu(z)}{(1 - |z|^2)^{\alpha p}} < \infty. \quad (4.5)$$

Remark 4.4. The above lemma was obtained by Zhao when $0 < \alpha \leq 1$ (see [24]). In fact, his proof implies that the result also holds for $\alpha > 1$. Partial results can also be found in [4] when $\alpha = 1$.

THEOREM 4.5. Assume that $0 < \alpha < \infty$, $0 < p < \infty$, and ϕ is normal on $[0, 1)$. Then the following statements are equivalent.

- (1) $I_g : \mathcal{B}^\alpha \rightarrow H(p, p, \phi)$ is bounded.
- (2) $I_g : \mathcal{B}^\alpha \rightarrow H(p, p, \phi)$ is compact.
- (3) $I_g : \mathcal{B}_0^\alpha \rightarrow H(p, p, \phi)$ is bounded.
- (4) $I_g : \mathcal{B}_0^\alpha \rightarrow H(p, p, \phi)$ is compact.
- (5)

$$\int_D |g(z)|^p (1 - |z|^2)^{p-p\alpha} \frac{\phi^p(|z|)}{1 - |z|} dA(z). \quad (4.6)$$

Proof. Since

$$\begin{aligned} \|I_g f\|_{H(p, p, \phi)}^p &\asymp \int_D |(I_g f)'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &= \int_D |g(z)|^p |f'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) = \int_D |f'(z)|^p d\mu(z), \end{aligned} \quad (4.7)$$

where

$$d\mu(z) = |g(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z). \quad (4.8)$$

By Lemma 4.3, we know that $I_g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow H(p, p, \phi)$ is bounded (or compact) if and only if

$$\infty > \int_D \frac{d\mu}{(1 - |z|^2)^{\alpha p}} = \int_D |g(z)|^p (1 - |z|^2)^{p-p\alpha} \frac{\phi^p(|z|)}{1 - |z|} dA(z). \quad (4.9)$$

□

THEOREM 4.6. *Assume that $\alpha > 1$, $0 < p < \infty$, and ϕ is normal on $[0, 1)$. Then the following statements are equivalent.*

- (i) $J_g : \mathcal{B}^\alpha \rightarrow H(p, p, \phi)$ is bounded.
- (ii) $J_g : \mathcal{B}^\alpha \rightarrow H(p, p, \phi)$ is compact.
- (iii)

$$\int_D |g'(z)|^p (1 - |z|^2)^{(2-\alpha)p} \frac{\phi^p(|z|)}{1 - |z|} dA(z) < \infty. \quad (4.10)$$

Proof. Since

$$\begin{aligned} \|J_g f\|_{H(p, p, \phi)}^p &\asymp \int_D |(J_g f)'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &= \int_D |g'(z)|^p |f(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &= \int_D |f(z)|^p (1 - |z|^2)^{-p} d\mu(z), \end{aligned} \quad (4.11)$$

where

$$d\mu = |g'(z)|^p (1 - |z|^2)^{2p} \frac{\phi^p(|z|)}{1 - |z|} dA(z). \quad (4.12)$$

Similar to the proof of Theorem 4.5, we get (i) $\Leftrightarrow$ (iii) by Lemma 4.2.

(ii) $\Rightarrow$ (i) is clear. Next we prove that (iii) $\Rightarrow$ (ii). Assume (iii) holds, we obtain that J_g is bounded and so $g \in H(p, p, \phi)$. In addition to this, we also find that for any $\varepsilon > 0$, there is an $r \in (0, 1)$ such that

$$\int_{|z|>r} |g'(z)|^p (1 - |z|^2)^{(2-\alpha)p} \frac{\phi^p(|z|)}{1 - |z|} dA(z) < \varepsilon. \quad (4.13)$$

Let $\{f_k\}$ be any sequence in the unit ball of $\mathcal{B}^\alpha$ and converges to 0 uniformly on compact subsets of D . For the above ε , there exists a $k_0 > 0$ such that $\sup_{|z|\leq r} |f_k(z)| < \varepsilon$ as $k > k_0$. Hence we have

$$\begin{aligned} \|J_g f_k\|_{H(p, p, \phi)}^p &\asymp \left(\int_{|z|\leq r} + \int_{|z|>r} \right) |(J_g f_k)'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &\leq C\varepsilon \|g\|_{H(p, p, \phi)}^p + \|f_k\|_{\mathcal{B}^\alpha}^p \int_{|z|>r} |g'(z)|^p (1 - |z|^2)^{(2-\alpha)p} \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &\leq C\varepsilon \|g\|_{H(p, p, \phi)}^p + \varepsilon \|f_k\|_{\mathcal{B}^\alpha}^p. \end{aligned} \quad (4.14)$$

In other words, we obtain $\lim_{k \rightarrow \infty} \|J_g f\|_{H(p,p,\phi)} = 0$ and so $J_p : \mathcal{B}^\alpha \rightarrow H(p,p,\phi)$ is compact. $\square$

THEOREM 4.7. *Assume that $0 < \alpha < 1$, $0 < p < \infty$, and ϕ is normal on $[0, 1)$. Then the following statements are equivalent.*

- (i) $J_g : \mathcal{B}^\alpha \rightarrow H(p,p,\phi)$ is bounded.
- (ii) $J_g : \mathcal{B}^\alpha \rightarrow H(p,p,\phi)$ is compact.
- (iii)

$$\int_D |g'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) < \infty. \quad (4.15)$$

Proof. (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (iii). Assume that $J_g : \mathcal{B}^\alpha \rightarrow H(p,p,\phi)$ is bounded. Hence

$$\|J_g f\|_{H(p,p,\phi)}^p \asymp \int_D |g'(z)|^p |f(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z). \quad (4.16)$$

Taking $f = 1$, we get (iii).

Conversely, we assume that (iii) holds. Let $f \in \mathcal{B}^\alpha$, then $|f(z)| \leq C\|f\|_{\mathcal{B}^\alpha}$. Therefore, by (4.16), we see that $J_g : \mathcal{B}^\alpha \rightarrow H(p,p,\phi)$ is bounded. Similar to the proof of Theorem 4.6, we obtain that (iii) $\Rightarrow$ (ii). $\square$

THEOREM 4.8. *Assume that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. Then the following statements hold.*

- (i) If the operator $J_g : \mathcal{B} \rightarrow H(p,p,\phi)$ is bounded, then

$$\sup_{z \in D} \phi(|z|) (1 - |z|^2)^{1+1/p} |g'(z)| \ln \frac{2}{1 - |z|^2} < \infty. \quad (4.17)$$

- (ii) If

$$\sup_{z \in D} \phi(|z|) (1 - |z|^2)^{1-1/p} |g'(z)| \ln \frac{2}{1 - |z|^2} < \infty, \quad (4.18)$$

then $J_g : \mathcal{B} \rightarrow H(p,p,\phi)$ is bounded.

Proof. Assume that (4.18) holds. For any $f \in \mathcal{B}$, by Lemmas 2.2, 4.1 and the fact $(J_g f)'(z) = f(z)g'(z)$, $(J_g f)(0) = 0$, we have

$$\begin{aligned} \|J_g f\|_{H(p,p,\phi)}^p &\asymp \int_D |(J_g f)'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &= \int_D |g'(z)|^p |f(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &\leq C\|f\|_{\mathcal{B}}^p \int_D |g'(z)|^p \ln^p \frac{2}{1 - |z|^2} (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \end{aligned}$$

$$\begin{aligned}
&\leq C \|f\|_{\mathcal{B}}^p \sup_{z \in D} |g'(z)|^p |\ln^p \frac{2}{1-|z|^2} (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} \int_D dA(z) \\
&\leq C \|f\|_{\mathcal{B}}^p \sup_{z \in D} |g'(z)|^p |\ln^p \frac{2}{1-|z|^2} (1-|z|^2)^{p-1} \phi^p(|z|).
\end{aligned} \tag{4.19}$$

Therefore, $J_g : \mathcal{B} \rightarrow H(p, p, \phi)$ is bounded.

Assume that $J_g : \mathcal{B} \rightarrow H(p, p, \phi)$ is bounded. For $w \in D$, put $f_w(z) = \ln 2 / (1 - \bar{w}z)$. Since

$$(1 - |z|^2) |f'_w(z)| \leq (1 - |z|^2) \frac{|w|}{|1 - \bar{w}z|} \leq \frac{1 - |z|^2}{|1 - \bar{w}z|} \leq 2, \tag{4.20}$$

we have $\|f_w\|_{\mathcal{B}} \leq \ln 2 + 2$. By the subharmonicity, we have

$$\begin{aligned}
C \|J_g\|^p &\geq C \|J_g\|^p \|f_w\|_{\mathcal{B}}^p \geq C \|J_g f_w\|_{H(p, p, \phi)}^p \\
&\geq C \int_D |(J_g f_w)'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\
&\geq C \int_{D(w, r)} |g'(z)|^p |f_w(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\
&\geq C |g'(w)|^p |f_w(w)|^p (1 - |w|^2)^{p+1} \phi^p(|w|) \\
&\geq C (1 - |w|)^{p+1} \phi^p(|w|) |g'(w)|^p \left(\ln \frac{2}{1 - |w|^2} \right)^p.
\end{aligned} \tag{4.21}$$

Therefore, we get the desired result. $\square$

Remark 4.9. We use another method to prove the necessary condition of the boundedness of $J_g : \mathcal{B} \rightarrow H(p, p, \phi)$.

Since $J_g f \in H(p, p, \phi)$, by Lemma 2.3, we have

$$|(J_g f)'(z)| \leq C \frac{\|J_g f\|_{H(p, p, \phi)}}{(1 - |z|^2)^{1/p+1} \phi(|z|)} \leq C \frac{\|J_g\|_{\mathcal{B} \rightarrow H(p, p, \phi)} \|f\|_{\mathcal{B}}}{(1 - |z|^2)^{1/p+1} \phi(|z|)}. \tag{4.22}$$

For any $w \in D$, let $f_w(z) = \ln 2 / (1 - z\bar{w})$. Then we get

$$|g'(z)| |f_w(z)| \phi(|z|) (1 - |z|^2)^{1/p+1} \leq C \|J_g\|_{\mathcal{B} \rightarrow H(p, p, \phi)} \|f_w\|_{\mathcal{B}}. \tag{4.23}$$

Let $z = w$, we have

$$|g'(w)| \ln \frac{2}{1 - |w|^2} \phi(|w|) (1 - |w|^2)^{1/p+1} \leq C \|J_g\|_{\mathcal{B} \rightarrow H(p, p, \phi)} \|f_w\|_{\mathcal{B}}. \tag{4.24}$$

Therefore, we get the desired result.

THEOREM 4.10. Assume that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. Then the following statements hold.

(i) If the operator $J_g : \mathcal{B} \rightarrow H(p, p, \phi)$ is compact, then

$$\lim_{|z| \rightarrow 1} \phi(|z|)(1 - |z|^2)^{1+1/p} |g'(z)| \ln \frac{2}{1 - |z|^2} = 0. \quad (4.25)$$

(ii) If

$$\lim_{|z| \rightarrow 1} \phi(|z|)(1 - |z|^2)^{1-1/p} |g'(z)| \ln \frac{2}{1 - |z|^2} = 0, \quad (4.26)$$

then $J_g : \mathcal{B} \rightarrow H(p, p, \phi)$ is compact.

Proof. Suppose the operator $J_g : \mathcal{B} \rightarrow H(p, p, \phi)$ is compact. Let z_n be a sequence in D such that $|z_n| \rightarrow 1$ as $n \rightarrow \infty$. Take

$$f_n(z) = \left(\ln \frac{2}{1 - |z_n|^2} \right)^{-1} \left(\ln \frac{2}{1 - \overline{z_n} z} \right)^2. \quad (4.27)$$

Then

$$f'_n(z) = 2 \left(\ln \frac{2}{1 - |z_n|^2} \right)^{-1} \left(\ln \frac{2}{1 - \overline{z_n} z} \right) \frac{\overline{z_n}}{1 - \overline{z_n} z}. \quad (4.28)$$

Thus for any $z \in D$,

$$(1 - |z|^2) |f'_n(z)| \leq 2(1 - |z|^2) \left| \frac{\ln 2/(1 - \overline{z_n} z)}{\ln 2/(1 - |z_n|^2)} \right| \frac{1}{1 - |z|} \leq 4 \frac{C + \ln 2/(1 - |z_n|)}{\ln 2/(1 - |z_n|^2)} \leq C. \quad (4.29)$$

On the other hand,

$$|f_n(0)| \leq \left(\ln \frac{2}{1 - |z_n|^2} \right)^{-1} (\ln 2)^2 \leq \ln 2. \quad (4.30)$$

Thus $\|f_n\|_{\mathcal{B}} \leq M$, where M is a constant independent of n . Since for $|z| = r < 1$, we have

$$|f_n(z)| = \frac{|\ln 2/(1 - \overline{z_n} z)|^2}{\ln 2/(1 - |z_n|^2)} \leq \frac{(\ln 2/(1 - r) + C)^2}{\ln 2/(1 - |z_n|^2)} \rightarrow 0 \quad (n \rightarrow \infty), \quad (4.31)$$

that is, $f_n \rightarrow 0$ uniformly on compact subsets of D as $n \rightarrow \infty$. By the proof of Theorem 4.8, we obtain

$$\phi(|z_n|)(1 - |z_n|^2)^{1+1/p} |g'(z_n)| \ln \frac{2}{1 - |z_n|^2} \leq \|J_g f_n\| \rightarrow 0 \quad (4.32)$$

as $n \rightarrow \infty$. Therefore, we get (4.25).

From (4.26), for any $\varepsilon > 0$, there exists an r , $0 < r < 1$, such that

$$\phi(|z|)(1 - |z|^2)^{1-1/p} |g'(z)| \ln \frac{2}{1 - |z|^2} < \varepsilon, \quad (4.33)$$

when $|z| > r$. Also, from (4.26), we see that there exist $C > 0$ such that

$$\sup_{|z| \leq r} \phi(|z|)(1 - |z|^2)^{1-1/p} |g'(z)| < C. \quad (4.34)$$

Let $\{f_k\}$ be any sequence in the unit ball of $\mathcal{B}$ and converges to 0 uniformly on compact subsets of D . For the above ε , there exists a $k_0 > 0$ such that $\sup_{|z| \leq r} |f_k(z)| < \varepsilon$ as $k > k_0$. Hence we have

$$\begin{aligned} \|J_g f_k\|_{H(p,p,\phi)}^p &\asymp \left(\int_{|z| \leq r} + \int_{|z| > r} \right) |(J_g f_k)'(z)|^p (1 - |z|^2)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &\leq C\varepsilon + \|f_k\|_{\mathcal{B}}^p \int_{|z| > r} |g'(z)|^p (1 - |z|^2)^p \left(\ln \frac{2}{1 - |z|^2} \right)^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) \\ &\leq C\varepsilon + \varepsilon \|f_k\|_{\mathcal{B}}^p, \end{aligned} \quad (4.35)$$

as $k > k_0$, from which we get the desired result. $\square$

Acknowledgments

The author thanks the referee for his (her) useful comments. The author also would like to express his sincere thanks to Professor Stevo Stević and Professor Kehe Zhu for their help during the preparation of this paper. The author is partially supported by NNSF (no. 10371051, no. 10371069).

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As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods:** artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields:** asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects:** decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

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Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

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