NOTE ON A ROLE FOR ENTIRE FUNCTIONS
OF THE CLASSES P AND P*

C.L. PRATHER
Department of Mathematics
Virginia Tech
Blacksburg, Virginia 24061 U.S.A.

ABSTRACT. We use the B and B* operators of Levin on the Classes P and P* and a

comparison principle to prove a Gauss-Lucas Theorem for differential operators.

The connection with the determination of final sets for differential operators is

then clarified.

KEY WORDS AND PHRASES. Final set, differential operators, comparison principle,
Gauss-Lucas Theorem.

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1. INTRODUCTION.

The point of this note is to indicate another value of the classes P and P*

of the entire functions as introduced by B. Ya. Levin (see e.g. Levin [1, Chap. 9]

or Boas [2, Chap. 11]). As we shall see, they are the appropriate classes with

which to obtain Gauss-Lucas and final set results for differential operators of

the types B and B*. In writing this, the author is prompted in part by the remark

of Efimov, Krein and Ostrowski in a recent tribute to Levin [3, pg. 142]:

"Apparently the role of the class P in the theory of entire functions is not com-

pletely clarified". While the author has no claims as to what "the" role is,

the comparison principle presented here has new applications, some of which have

been recently worked out [4, 5, 6, 7].

First, we recall some definitions [2, Chap. 11].

Definition 1. A function ϕ is in the class P if it is entire of exponential type, has no zeros in $y < 0$, or satisfies the equivalent condition $h(-\pi/2) \geq h(\pi/2)$, where h is the indicator function of ϕ

Definition 2. A B-operator is an additive, homogeneous operator that transforms functions of exponential type into functions of exponential type and leaves the class P invariant.

The class of B-operators is fairly large [2], some examples are: $D = d/dz$; $D - \beta I$, for $\text{Im}(\beta) \leq 0$; $P(D)$, where P is a polynomial with roots in the lower half plane $\text{Im}(z) \leq 0$ and more generally $\phi(D) = \Sigma_k C_k D^k$, where $\bar{\phi}(z) = \Sigma_k \bar{C}_k z^k$ is in the class P; $\exp(-cD^2) \phi(D)$ with $c > 0$ and $\bar{\phi} \in P$; the operator $(\phi(D)f)(z)$ $= \int_{z-i\lambda}^{z+i\lambda} f(t)dt$ for $\lambda > 0$.

Functions of the class P are characterized by their infinite product expansions [2, Theorem 7.8.3]: A function $\omega(z)$ of exponential type belongs to the class P if and only if it has the form $\omega(z) = Az^m \exp(cz)\prod_{n=1}^{\infty}(1 - z/z_n)\exp(z\text{Re}(1/z_n))$, where $\text{Im}(a_n) \geq 0$ and $2\,\text{Im}(c) = h(-\pi/2) - h(\pi/2) \geq 0$, and h is the indicator function of ω.

Well known is the fact that Bernstein's theorem has been generalized in various directions using B-operators on the class P[1,2]. In particular, inequalities of Bernstein type have been demonstrated for the class of so called asymmetric entire functions as given in [9,10,11,12].

The new applications [4,6,7] of B-operators and the class P are based on an extension of an idea due to Pólya [13,14].

Definition 3. Let L be a differential operator and f a function analytic on a domain D. We say that a point z_0 lies in the final set $S = S(L,f)$ of f relative to L when, by considering the iterates $L^n f$, $n = 0, 1, 2, 3, \ldots$, $L^0 = I$, every neighborhood of z_0 contains zeros of infinitely many of the iterates $L^n f$.

Final sets for derivatives of meromorphic functions and certain restricted classes of entire functions have been given by Pólya [13,14] and Edrei [15,16].

Definition 4. a) Let $f(z)$ be a trigonometric sum $f(z) = \Sigma_k a_k \exp(\lambda_k z)$, where $\lambda_k \in C$, lie in a bounded set, and $\Sigma_k |a_k| < \infty$. Let H be the convex hull of the λ_k and suppose that H is either a line segment or is polygonal. We say that f is balanced when the following conditions hold:

Letting $\delta = \sup_k |\lambda_k|$, the circle $|z| = \delta$ contains at least two exponents $\lambda_{k_1}, \ldots, \lambda_{k_p}$ and the coefficients corresponding to these exponents are all nonzero.

b) We say that the finite Fourier integral $f(z) = \int_{-\delta}^{\delta} e^{izt} \psi(t) dt$ is strongly balanced when $\psi(t)$ is bounded and measurable and $\psi(t) \to L_1$, L_2 as $t \to -\delta$, $+\delta$, respectively with $L_1 L_2 \neq 0$.

When $H = [-i\delta, +i\delta]$, for $\delta > 0$, the class of trigonometric sums described in a) coincides with the class $[\delta]$ as defined by Levin [1, Chapter 6] and accordingly its zeros lie in some strip $|y| < h$ [1, Chap. 6, Theorem 3]. As we shall see, not only is it true that the zeros of the successive derivatives lie in $|y| < h$ as discussed in [9,10], but the zeros of more general B-operators on these functions lie in this strip.

Final sets for the B-operator $\phi(D)$, where $D = d/dz$ and $\phi(z)$ is an entire function of genus ≤ 1 having only real zeros times the factor $\exp(-az^2)$, $a \geq 0$ were determined in [4,5,6] on both balanced exponential sums $\Sigma_k C_k \exp(i\lambda_k z)$, where the λ_k are real, and strongly balanced finite Fourier transforms. For the case of exponential sums, the exponents are allowed to accumulate to the endpoints $\pm i\delta$, say, provided they do so at a certain rate. The result is that the final set consists of a discrete set of points on the horizontal line $\mathrm{Im}(z) = -(2\delta)^{-1} \ell n |C_{-\delta}/C_{\delta}|$ or the whole line, the conditions by which each occurs being given. A similar result is give for balanced exponential sums $\Sigma_k C_k \exp(\lambda_k z)$, $\lambda_k \in C$, for derivatives [6].

The fact that the functions we are considering are balanced is important when ϕ has real zeros, for Boas [8] has shown that for an asymmetric entire function f bounded on the real axis with $h_f(-\pi/2) = 0$, every half plane $y > a \geq 0$ contains zeros of infinitely many derivatives of f, making a final set result unlikely. However, as shown in [6], it is possible to obtain a nonempty final set with

asymmetric exponential sums for operators $\phi(D)$, where ϕ has nonreal zeros. Another reason to start with balanced exponential sums is that the extremal functions for theorems of Bernstein type are balanced.

We also consider B*-operators on functions of the class P* [1, Chap. 9]. It is well known that a B*-operator is a B-operator and $P \subset P*$. Moreover, f belongs to the class P* if and only if it is a uniform limit on compact sets of a sequence of polynomials whose zeros lie in the upper half plane [1, Chap. 8].

The elementary yet useful comparison principle [9,10] is the key to obtaining final sets for B*-operators.

COMPARISON PRINCIPLE: Let K be a closed subset of the complex plane C and M a complex linear space of meromorphic functions with poles in K. Let N be that part of M consisting of functions having no zeros outside K and L a linear operator from M to M. Then

    a)   the inequality $|f(z)| \leq |g(z)|$, $z \in C/K$, $f,g \in M$ implies the

         inequality $|(Lf)(z)| \leq |(Lg)(z)|$

    if and only if

    b) $L(N) \subset N$.

This principle is, in essence, a Gauss-Lucas Theorem for B* (and hence B) operators, simple cases of which were given by Genchev [9,10,17]. Wide applications have been made of Gauss-Lucas Theorems for derivatives; e.g., Marden [18] and Rubel [19]. Gauss-Lucas Theorems for operators are the point of Pólya's papers [20,21] the lengthy proofs of which occur in the paper of Benz [22]. However, the above principle yields a very simple proof of this type theorem. For example, we demonstrate the following:

THEOREM. Let $\phi(D)$ be a B*-operator, where $\phi(z)$ is an entire function of genus ≤ 1 with real zeros times the factor $\exp(-az^2)$, $a \geq 0$. Let f be an entire function of order < 2 with zeros in the strip $T = \{z: A \leq \mathrm{Im}(z) \leq B\}$. Then $(\phi(D)f)(z)$ also has its zeros in T.

PROOF. If $A \geq 0$, then $f \in P*$. If $A < 0$, then $f(z+Ai) \in P*$, so it suffices to

let $A \geq 0$. As $\phi(z)$ belongs to the class P*, $\phi(z)$ can be uniformly approximated on compact sets by polynomials $P_n(z)$ whose zeros lie on the real axis, by a characterization of the class P* stated earlier. Letting $L_n = P_n(D)$, L_n is a B*-operator. By taking M to be the linear span of the class P* and N the sub-class of M consisting of those functions in M whose zeros lie in T, since by definition B*-operators preserve the class P*, $L_n(N) \subset N$. By the comparison principle, $|f(z)| > 0$ for $z \in C/T$ (by hypothesis) implies $|(L_n f)(z)| > 0$. By Hurwitz's Theorem, $|(\phi(D)f)(z)| > 0$ for $z \in C/T$. Hence $\phi(D)f$ has all its zeros in T.

Using this idea, final sets have been determined for certain B*-operators on classes of entire functions of order < 2 representable by a Fourier integral having only real zeros [7]. One is lead to suspect that other final set results are obtainable for B*-operators by asymptotic methods. A study of the zeros of the successive iterates of multiplier sequence operators (which includes B-operators) on analytic functions has been submitted for publication.

Under not too restrictive assumptions related to the continuity of the operator, Levin [1, Chap. 9] obtains the general form of B and B*-operators.

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