ON THE HARDY-LITTLEWOOD MAXIMAL THEOREM

SHINJI YAMASHITA

Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya, Tokyo 158
Japan

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ABSTRACT. The Hardy-Littlewood maximal theorem is extended to functions of class PL in the sense of E. F. Beckenbach and T. Radó, with a more precise expression of the absolute constant in the inequality. As applications we deduce some results on hyperbolic Hardy classes in terms of the non-Euclidean hyperbolic distance in the unit disk.

KEY WORDS AND PHRASES. Hardy-Littlewood's Maximal Theorem, Subharmonic Functions of Class PL, Hardy Class, Hyperbolic Hardy Class.

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1. INTRODUCTION.

Let $D = \{ |z| < 1 \}$, let $T = [0, 2\pi)$, and let $u$ be a function subharmonic in $D$. For a function $g$ on $T$ we denote

$$\|g\|_p = \left[ \frac{1}{2\pi} \int_T |g|^p(t) \, dt \right]^{1/p};$$

hereafter always $0 < p < \infty$ unless otherwise specified. Then, although $u$ is not defined on $T$ we customarily denote

$$\|u\|_p = \limsup_{r \to 1^-} \frac{1}{u_r}\|u_r\|_p,$$
where \( u_r(t) = u(re^{it}), \ t \in T, 0 < r < 1. \) For simplicity, \( \|f\|_p = \| |f| \|_p \) for \( f \) holomorphic in \( D. \) Let \( S(t,R) \) be the domain consisting of the interior of the convex hull of the circle \( |z| = R < 1 \) and the point \( e^{it} (t \in T); \) hereafter always \( 0 < R < 1. \) The maximal function \( M_R(u) \) of \( u \) is defined on \( T \) by

\[
M_R(u)(t) = \sup \{ |u(z)|; z \in S(t,R) \}.
\]

Let \( H^p \) be the Hardy class consisting of all functions \( f \) holomorphic in \( D \) such that \( \|f\|_p < \infty. \) Each \( f \in H^p \) has the radial limit \( f^*(t) = \lim_{r \to 1^-} f(re^{it}) \) at \( e^{it} \) for a.e. \( t \in T, \) and \( f^* \in L^p(T). \) We then observe that

\[
\|f\|_p = \|f^*\|_p \quad [1, \text{Theorem 2.6, p. 21}].
\]

In the present paper we introduce the Hardy-Littlewood number \( a(p,R) \) of order \( (p,R) \) by

\[
a(p,R) = \sup \left\{ \frac{\|M_R(|f|)\|_p}{\|f\|_p}; f \in H^p, f \neq 0 \right\}.
\]

The celebrated Hardy-Littlewood theorem \([3, \text{Theorem 27, p. 114}]\) then reads that

\[
a(p,R) < \infty \quad \text{for each pair} \quad (p,R). \quad \text{The main purpose of the present paper is to prove that} \quad a^*(p,R) = a(p,R) = a(1,R)^{1/p}, \quad \text{where} \quad a^*(p,R) \quad \text{is defined in terms of functions of class PL} \ [4, \text{p. 9}].
\]

A function \( u \) defined in \( D \) is said to be of class PL, or \( u \in \text{PL}, \) if \( u \geq 0 \) and if \( \log u \) is subharmonic in \( D; \) we regard \(-\infty\) as a subharmonic function. For \( u \in \text{PL}, \) the function \( u^p \) is subharmonic in \( D, \) and for \( f \) holomorphic in \( D, \) the modulus \( |f| \in \text{PL}. \) Let \( \text{PL}^p \) be the family of all \( u \in \text{PL} \) such that \( \|u\|_p < \infty. \) It will be observed that \( u \in \text{PL}^p \) has the radial limit \( u^*(t) \) at \( e^{it} \) for a.e. \( t \in T \) and that \( \|u\|_p = \|u^*\|_p. \) Apparently, \( |f| \in \text{PL}^p \) if \( f \in H^p. \) Set

\[
a^*(p,R) = \sup \left\{ \frac{\|M_R(|f|)\|_p}{\|f\|_p}; u \in \text{PL}^p, \ u \neq 0 \right\}.
\]

Since \( 1 \in H^p, \) it follows that \( 1 \leq a(p,R) \leq a^*(p,R). \) We first observe

**THEOREM 1.** \( a^*(p,R) = a(p,R) = a(1,R)^{1/p}. \)

**REMARK.** Let \( S^p \) be the family of subharmonic functions \( u \geq 0 \) in \( D \) such that \( \|u\|_p < \infty, \) where \( p > 1. \) Then
\[ b(p,R) = \sup \left\{ \frac{\| M_R(u) \|_p}{\| u \|_p}; u \in S^p, u \neq 0 \right\} \]

is finite for \( p > 1 \) by [3, Theorem 26, p. 113]. Obviously,
\[ a^*(p,R) \leq b(p,R) \text{ for } p > 1. \]

To propose an application to the hyperbolic Hardy class \( H^p_\sigma \) we let
\[ \sigma(z,w) = \tanh^{-1}(\frac{|z - w|}{1 - \overline{z}w}) \]
be the non-Euclidean hyperbolic distance between \( z \) and \( w \) in \( D \). Set \( \sigma(z,0) = \frac{1}{2} \log[(1 + |z|)/(1 - |z|)] \), \( z \in D \). For \( f \) holomorphic and bounded, \( |f| < 1 \), in \( D \), the hyperbolic counterpart of \( |f| \) is \( \sigma(f) \). We thus define \( H^p_\sigma \) as the family of such \( f \) with \( \| \sigma(f) \|_p < \infty \). The subharmonicity of \( \sigma(f)^P = \exp[p \log \sigma(f)] \) follows from that of \( \log \sigma(f) \) (or, \( \sigma(f) \in PL \)) observed in [6].

Therefore \( \sigma(f) \in PL^p \) for all \( f \in H^p_\sigma \). A few modifications of the proof of [6, Theorem 4], with \( H^1_h = H^1_\sigma \), show that \( H^p_\sigma \) is a semigroup with respect to the multiplication, and is convex. Since each \( f \in H^p_\sigma \) is bounded, \( f \) has the radial limit \( f^*(t) \) at \( e^{it} \) for a.e. \( t \in T \). We then propose

**THEOREM 2.** For each \( f \in H^p_\sigma \), the function \( \sigma(f^*) \) is a member of \( L^p(T) \), and
\[ \int_T \sigma(f(re^{it}), f^*(t))^p dt \to 0 \text{ as } r \to 1-0. \]

The inequality
\[ \int_T \sup \left\{ \sigma(f)^P(z); z \in S(t,R) \right\} dt \leq a(1,R) \int_T \sigma(f^*)^P(t) dt \]
holds for all \( f \in H^p_\sigma \).

The first assertion, a consequence of the second, is the hyperbolic counterpart of the F. Riesz theorem [1, Theorem 2.6].

**2. PROOFS.**

For the proof of Theorem 1 it suffices to show that
\[ a^*(p,R) = a(1,R)^{1/p} \leq a(p,R). \]
Since \( a(p,R) \leq a^*(p,R) \), the identities in Theorem 1 follow.

To prove that \( a(1,R)^{1/p} \leq a(p,R) \) we let \( f \in H^1 \) with \( f \neq 0 \). Then \( f \) admits an inner-outer factorization, \( f = IF \), where \( I \) and \( F \) are an inner and an outer function, respectively, such that the radial limits satisfy \( |I^*| = 1 \) and \( |F^*| = |f^*| \) a.e. on \( T \). Since \( F \) is zero-free in \( D \), \( g = F^{1/p} \in H^P \), so that \( |f^*| = |g^*|^P \) a.e. on \( T \). Therefore,

\[
\|M_R(|f|)\|_1 \leq \|M_R(|F|)\|_1 = \|M_R(|g|)\|_P^P \leq a(p,R)^P \|g\|^P = a(p,R)^P f\|_1,
\]

whence \( a(1,R) \leq a(p,R)^P \).

To prove that \( a^*(p,R) \leq a(1,R)^{1/p} \), we let \( v \in PL^P \) with \( v \neq 0 \). Setting \( u = p \log v \) and \( \varphi(x) = e^x \), one finds that \( v^P = \varphi(u) \). Since \( \varphi(u) \) admits a harmonic majorant in \( D \), there exists a positive harmonic majorant of \( u \) in \( D \) [5, p. 65]. The F. Riesz decomposition then yields that \( u = u^* - P \), where \( P \geq 0 \) is the Green potential in \( D \), and

\[
u^*(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|e^{it} - z|^2} d\mu(t) \quad (z \in D)
\]
is the Poisson integral of the measure

\[
d\mu(t) = u^*(t)dt + d\mu_S(t).
\]

The radial limit \( u^* \) of \( u \) is of \( L^1(T) \) and \( d\mu_S(t) \) is singular with respect to \( dt \). It follows from a general theorem [2, Theorem], applied to the present \( u \) and \( \varphi \), that \( d\mu_S(t) \leq 0 \) a.e. on \( T \) and that \( \varphi(u^*) \in L^1(T) \). Consequently,

\[
u(z) \leq h(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|e^{it} - z|^2} u^*(t)dt \quad (z \in D),
\]

and the Jensen inequality yields that

\[
\varphi(u) \leq \varphi(h) \leq V,
\]

where \( V \) is the Poisson integral of \( \varphi(u^*) \). Set \( f = e^{h+ik} \), where \( k \) is a conju-
gate of \( h \) in \( D \). Then, \(|f| = \varphi(h) \leq v\), so that \( f \in H^1 \) with \(|f^*| = \varphi(h^*) = \varphi(u^*) = v^*p\). On the other hand, \( v^p = \varphi(u) \leq \varphi(h) = |f| \) in \( D \), whence

\[
\|M^p_R(v)\|_p \leq \|M^p_R(|f|)\|_1 \leq a(1,R) \|f\|_1.
\]

The Lebesgue dominated convergence theorem, together with

\[
v_r^p(t) \leq M^p_R(v)(t) \quad (t \in T),
\]

yields that \( \|v_r^p\|_p \to \|v\|_p = \|v^*\|_p = \|f\|_1 \) as \( r \to 1-0 \). Therefore \( a^*(p,R) \leq a(1,R)^{1/p} \) follows from

\[
\|M^p_R(v)\|_p \leq a(1,R) \|v\|_p.
\]

We next prove Theorem 2. Set

\[
a_\sigma(p,R) = \sup \{\|M^p_R(\sigma(f))\|_p / \|\sigma(f)\|_p : f \in H^p_\sigma, f \not= 0\}.
\]

Since \( \sigma(f) \in PL^p \) for all \( f \in H^p_\sigma \), it follows that \( a_\sigma(p,R) \leq a^*(p,R) = a(1,R)^{1/p} \), so that

\[
\|M^p_R(\sigma(f))\|_p \leq a(1,R)^{1/p} \|\sigma(f)\|_p.
\]

As is observed in the proof of Theorem 1, \( \sigma(f)^* = \sigma(f^*) \) a.e. on \( T \) because \( \sigma(f) \in PL^p \), and \( \|\sigma(f)\|_p = \|\sigma(f^*)\|_p \). Thus, the second assertion holds with \( \sigma(f^*) \in L^p(T) \). The Lebesgue dominated convergence theorem with the estimate

\[
\sigma(f(t),f^*(t))^p \leq 2^p \sigma(f)^p(f(t)) + 2^p \sigma(f^*)^p(t) \leq 2^{p+1}M^p_R(\sigma(f)^p)(t) = 2^{p+1}M^p_R(\sigma(f))^p(t),
\]

again yields that

\[
\int_T \sigma(f(t),f^*(t))^p dt \to 0 \quad \text{as } r \to 1-0.
\]

**REMARK.** Since \( \sigma(f) \equiv 1 \) for \( f \equiv (e^2 - 1)/(e^2 + 1) \in H^p_\sigma \), it follows that

\( 1 \leq a_\sigma(p,R) \).
REFERENCES


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