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A CHARACTERIZATION OF THE ALGEBRA OF HOLOMORPHIC FUNCTIONS ON A SIMPLY CONNECTED DOMAIN

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ABSTRACT: Let A be a singly-generated $\mathcal{F}$-algebra. It is shown that A is isomorphic to $H(\Omega)$ where Ω is a simply connected domain in $\mathbb{C}$ if and only if A has no topological divisors of zero. It follows from this that there are exactly three $\mathcal{F}$-algebras (up to isomorphism) which are singly generated and have no topological divisors of zero.

KEY WORDS AND PHRASES. F-algebras, holomorphic functions, topological divisors of zero
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1. INTRODUCTION.

The algebra $H(\Omega)$ of holomorphic functions on a domain $\Omega\subset\mathbb{C}$ with pointwise operations and compact-open topology is an interesting example of an $\mathcal{F}$-algebra. This algebra has been characterized in terms of some of the special properties it enjoys that are derived from the fact that it consists of holomorphic functions. (See for example [1], [2], [3], [4] and [5] for characterizations in terms of the local maximum modulus principle, the Cauchy estimate, Montel's theorem, the existence of derivations, and Taylor's theorem.) In [6] a characterization of the algebra of entire functions in terms of Liouville's theorem is given.

Watson [5] shows that an $\mathcal{F}$-algebra A which has a Schauder basis that is generated by an element $z\in A$ with open spectrum is algebraically and topologically isomorphic to $H(\Omega)$ where Ω is an open disk in $\mathbb{C}$. In this paper we study $\mathcal{F}$-algebras that are generated by a single element z (without requiring that z generate a basis for A). Of course, this condition alone is not enough to completely describe the algebra $H(\Omega)$ among $\mathcal{F}$-algebras. We will show, however, that this together with the condition that A has no topological divisors of zero, completely characterizes $H(\Omega)$ for a simply connected domain Ω. It follows from this that there are exactly three singly generated $\mathcal{F}$-algebras (up to isomorphism) which have no topological divisors of zero.

2. PRELIMINARIES.

An $\mathcal{F}$-algebra is a complete metrizable locally m-convex algebra. (All the algebras we consider are assumed to be commutative algebras over $\mathbb{C}$.) The topology of such an algebra is given by an increasing sequence of seminorms $\{p_n|n\in\mathbb{N}\}$. Each p_n determines a Banach algebra A_n which is the completion of $A/ker(p_n)$. If $n\leq m$ then the natural homomorphism from $A/ker(p_m)$ to $A/ker(p_n)$ induces a norm decreasing homomorphism $\pi_{nm}: A_m \to A_n$ whose range is a dense subalgebra of A_n. The Banach algebras A_n with maps π_{nm} form an inverse limit system and $\varprojlim (A_n, \pi_{nm})$ is topologically and algebraically isomorphic to A.

The maximal ideal space of A is the space $\mathcal{M}(A)$ consisting of all non-zero continuous multiplicative linear functionals on A endowed with the Gelfand topology. This topology is the weak topology on $\mathcal{M}(A)$ generated by the Gelfand transforms $\hat{x}: \mathcal{M}(A) \to \mathbb{C}$ defined by $\hat{x}(f) = f(x)$. The map $\gamma: A \to \hat{A}$ is a continuous homomorphism onto the algebra $\hat{A} \subset C(\mathcal{M}(A))$ of Gelfand transforms. For each $n \in \mathbb{N}$ the quotient map π_n from A onto $A/ker(p_n)$ induces a homeomorphism π_n^* of the maximal ideal space $\mathcal{M}(A_n)$ of A_n onto a compact subset M_n of $\mathcal{M}(A)$. For $n \leq m$ we have $M_n \subset M_m$ and $\mathcal{M}(A) = \bigcup M_n$.

The spectrum of $z \in A$ is the set $\sigma = \sigma(z) = \{f(z) \mid f \in \mathcal{M}(A)\}$. For each $n \in \mathbb{N}$ the set $\sigma_n = \sigma_n(z) = \{ f(z) \mid f \in M_n\}$ and $\sigma = \cup \sigma_n$. The element $z \in A$ generates A if A is the smallest closed subalgebra containing z and e (the identity of A). In this case the spectrum map $\varphi: \mathcal{M}(A) \to \sigma(z)$ defined by $f \mapsto f(z)$ is a continuous bijection [7].

An element z in a Banach algebra B is a topological divisor of zero if the multiplication map $M_z: A \to zA$ is not an isomorphism (i.e., does not have a continuous inverse). In an $\mathcal{F}$-algebra A, z is called a topological divisor of zero if for each sequence $\{p_n : n \in \mathbb{N}\}$ of seminorms defining the topology of A there exists $k \in \mathbb{N}$ such that $\pi_k(z)$ is a topological divisor of zero in the Banach algebra A_k [8, pp. 46–47].

3. CHARACTERIZING H(Ω).

Let $\Omega \subset \mathbb{C}$ be a simply connected domain. The algebra $H(\Omega)$ of holomorphic functions on Ω is an $\mathcal{F}$-algebra in the compact-open topology. It is well known that $H(\Omega)$ has no (nonzero) topological divisors of zero [9], and is singly-generated. We will show that these last two properties of $H(\Omega)$ completely characterize it among $\mathcal{F}$-algebras.

For the rest of this paper A will denote an $\mathcal{F}$-algebra with identity e which is generated by z, where z is not a scalar multiple of e, and which has no nonzero topological divisors of zero.

LEMMA 1. A is semisimple and so the Gelfand transform is a bijection.

PROOF: Suppose $y \in Rad(A)$, $y \neq 0$. Then $\sigma(y) = \{0\}$ and by [8, Propositon 11.8] y is a topological divisor of zero.

LEMMA 2. The spectrum $\sigma(z)$ is a domain in $\mathbb{C}$.

PROOF: If $\lambda \in \sigma(z)$ is a boundary point of $\sigma(z)$, then again by [8, Proposition 11.8], $z - \lambda e$ is a topological divisor of zero. Thus $\sigma(z)$ is open.

If $\sigma(z)$ includes the two components U_1 and U_2, then the characteristic functions h_1 of U_1 and h_2 of U_2 are analytic on $\sigma(z)$. By the functional calculus there exist $x_1, x_2 \in A$ with $\hat{x}_1 = h_1(\hat{z})$ and $\hat{x}_2 = h_2(\hat{z})$. Clearly $\hat{x}_1 \hat{x}_2 = 0$ so by Lemma 1 $x_1 x_2 = 0$ and thus these elements are nonzero (topological) divisors of zero.

LEMMA 3. The domain $\sigma(z)$ is simply connected.

PROOF: Let $\varphi: \mathcal{M}(A) \to \sigma(z)$ be the spectrum map and for $t \in \sigma(z)$ we use the notation $f_t = \varphi^{-1}(t)$. For each $x \in A$ define $\tilde{x}: \sigma(z) \to \mathbb{C}$ by $\tilde{x}(t) = \hat{x}(\varphi^{-1}(t)) = \hat{x}(f_t)$.

We show that $\tilde{x}$ is analytic on $\sigma(z)$. Since A is generated by z there exists a sequence of polynomials $p_n = p_n(z)$ converging to x in A and so $f(p_n(z)) \to f(x)$, for every $f \in \mathcal{M}(A)$. For $t \in \sigma(z)$, $p_n(t) = p_n[(f_t(z)] = f_t[p_n(z)]$ converges to $f_t(x) = \hat{x}(f_t) = \tilde{x}(t)$, so each $\tilde{x}$ is a pointwise limit of polynomials on $\sigma(z)$. We now show that this convergence is uniform on compact subsets of $\sigma(z)$ and so each $\tilde{x}$ is analytic on this spectrum. Since A has no topological divisors of zero, for each $n\in\mathbb{N}$ there exists $m>n$ such that $\sigma_n\subset\operatorname{int}\sigma_m$ (see Arens [9]). So without loss of generality, we may assume that for $n=1,2,\ldots$,

$$\sigma_n\subset\operatorname{int}\sigma_{n+1}\subset\sigma_{n+1} \quad \text{and} \quad \sigma=\bigcup\operatorname{int}\sigma_n=\bigcup\sigma_n.$$

Since $\varphi|M_n$ is a homeomorphism onto its image $\varphi(M_n)=\sigma_n$, it follows that if K is a compact subset of σ there exists $n\in\mathbb{N}$ such that $K\subset\operatorname{int}\sigma_n\subset\sigma_n$. Thus $\varphi^{-1}(K)\subset\varphi^{-1}(\sigma_n)=M_n$ and so $\varphi^{-1}(K)$ is a compact subset of $\mathcal{M}(A)$. Now $p_n\to x$ so by the continuity of the Gelfand map γ, $\hat{p}_n\to\hat{x}$ in $\hat{A}$, i.e., the convergence is uniform on compact subsets of $\mathcal{M}(A)$. Thus for $\epsilon>0$ and sufficiently large n,

$$|\hat{p}_n(f_t)-\hat{x}(f_t)|<\epsilon$$

for $f_t\in\varphi^{-1}(K)$, which is the same as

$$|p_n(t)-\tilde{x}(t)|<\epsilon$$

for $t\in K$. Thus each $\tilde{x}$ is the limit of polynomials, uniformly on compact subsets of $\sigma(z)$, and hence is analytic there.

Let $h\in H(\sigma(z))$. We show that h is the limit of polynomials in $H(\sigma(z))$, then it follows that $\sigma(z)$ is simply connected. Using the functional calculus for $\mathfrak{F}$-algebras we find $x\in A$ such that $\hat{x}(f)=h(\hat{z}(f))$, $f\in\mathcal{M}(A)$. Therefore $h=\tilde{x}$. This together with the preceding paragraph completes the proof.

LEMMA 4. $\sigma(z)$ is homeomorphic with $\mathcal{M}(A)$.

PROOF: The map φ is a continuous bijection. But $\tilde{x}\circ\varphi^{-1}=\hat{x}$ is continuous for each $\hat{x}\in\hat{A}$ and so the continuity of φ^{-1} follows from the fact that the topology of $\mathcal{M}(A)$ is the weak topology generated by $\hat{A}$.

Lemma 4 may also be derived from [7, Theorem 1.3]. Notice that Lemmas 3 and 4 imply that $\mathcal{M}(A)$ is homeomorphic to the open unit disc. We now prove our main result.

THEOREM 1. An $\mathfrak{F}$-algebra A is algebraically and topologically isomorphic to $H(\Omega)$ for a simply connected domain Ω if and only if A is singly-generated and has no nonzero topological divisors of zero.

PROOF: That the $\mathfrak{F}$-algebra $H(\Omega)$ has these properties is discussed at the beginning of this section.

Conversely, let $\tilde{A}=\{\tilde{x}|x\in A\}$ and equip $\tilde{A}$ with the compact open topology. From the proof of Lemma 3, $\tilde{A}=H(\sigma(z))$ algebraically and topologically. Also, $\hat{A}$ and $\tilde{A}$ are isomorphic as $\mathfrak{F}$-algebras via the map $\delta:\hat{A}\to\tilde{A}$ by $\hat{x}\mapsto\tilde{x}=\hat{x}\circ\varphi^{-1}$. Since the Gelfand map $\gamma:A\to\hat{A}$ is bijective by Lemma 1, it follows that the map $\delta\circ\gamma$ is a continuous bijection of A onto $\tilde{A}=H(\sigma(z))$. The open mapping theorem now yields the result.

The notion of topological divisor of zero we used above is that due to Michael [8, p. 47]. Our Theorem 1 does not remain valid if that notion is replaced by the stronger definition of Arens [10] (called *strong topological divisor of zero* by Michael). In fact, the $\mathfrak{F}$-algebra $\mathbb{C}[[X]]$ of formal power series (with the topology of pointwise convergence in the coefficients) is singly generated and has no strong topological divisors of zero [11]. But this algebra is not isomorphic to $H(\Omega)$ for any domain Ω.

The Riemann mapping theorem yields the following corollary:

COROLLARY 1. There are (up to isomorphism) exactly three $\mathfrak{F}$-algebras which are singly generated and have no nonzero topological divisors of zero. Namely, $\mathbb{C}$, the algebra $H(\mathbb{D})$ where $\mathbb{D}$ is the open unit disk, and the algebra $\mathcal{E}$ of entire functions.

Birtel [6] (see also [12] and [13]) gave a characterization of the algebra of entire functions as a singly-generated Liouville algebra without topological divisors of zero. A *Liouville* algebra is an $\mathcal{F}$-algebra in which every element with bounded spectrum is a scalar multiple of the identity. We give another proof of Birtel's theorem based on our Theorem 1.

THEOREM 2. (Birtel) An $\mathcal{F}$-algebra A is topologically and algebraically isomorphic to the algebra $\mathcal{E}$ of entire functions if and only if A is a singly-generated Liouville algebra with no nonzero topological divisors of zero.

PROOF: By Theorem 1, $\sigma(z)$ is simply connected and A is isomorphic to $H(\sigma(z))$. If $\sigma(z) \neq \mathbb{C}$ then there is a one-to-one analytic function ψ from $\sigma(z)$ onto $\mathbb{D}$. Thus there exists $x \in A$ such that $\hat{x} = \psi \circ \hat{z}$. Clearly x is not a scalar multiple of e and $\sigma(x) = \mathbb{D}$, contradicting the assumption that A is Liouville.

A natural extension of the notion of a simply connected domain to $\mathbb{C}^n$ is that of a Runge domain. If Ω is a Runge domain in $\mathbb{C}^n$ then $H(\Omega)$ is n-generated and has no nonzero topological divisors of zero. We pose the question of whether a finitely-generated $\mathcal{F}$-algebra A with no nonzero topological divisors of zero is isomorphic to $H(\Omega)$ for a Runge domain Ω. In the case that A has a finitely-generated Schauder basis in which the joint spectrum of the generators is an open set in $\mathbb{C}^n$, it is shown in [14] that A is isomorphic to $H(\Omega)$ for a complete logarithmically convex Reinhardt domain Ω.

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