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ON α-CONVEX FUNCTIONS OF ORDER β OF RUSCHEWEYH TYPE

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Abstract: In this paper we consider functions $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ which are regular in the unit disc $E = \{ z : |z| < 1 \}$ and satisfying the condition

$$\Re e \{ (1-\alpha)(z^n f)^{(n+p)} / [(n+p)(z^{n-1} f)^{(n+p-1)}] + \alpha(z^{n+1} f)^{(n+p+1)} / [(n+p+1)(z^n f)^{(n+p)}] \} > \beta.$$

We obtain a lower bound $\gamma(\alpha, \beta, n, p)$ such that

$$\Re e \{ (z^n f)^{(n+p)} / [(n+p)(z^{n-1} f)^{(n+p-1)}] \} > \gamma(\alpha, \beta, n, p),$$

where p is a positive integer, n is any integer greater than $-p$, $\beta < 1$ and α is real.

KEY WORDS AND PHRASES α-convex functions of order β, Ruscheweyh type, p-valent functions, Hadamard product
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1. Introduction.

Let $A(p)$ be the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, p a positive integer, which are regular in the unit disc E. S RUSCHEWEYH [1] defined the class K_n consisting of all $f \in A(p)$ for $p = 1$ which satisfies

$$\Re e \{ \frac{D^{n+1} f(z)}{D^n f(z)} \} > 1/2$$

for $z \in E$ and for some $n \in N_0 = N \cup \{ 0 \}$, where

$$D^{n+p-1} f(z) = [z^p/(1-z)^{n+p}] * f(z)$$

with the operation $(*)$ stands for the Hadamard product, that is, if $g(z)=\sum_{k=0}^{\infty} a_k z^k$, $h(z)=\sum_{k=0}^{\infty} b_k z^k$, then $(g*h)(z)=\sum_{k=0}^{\infty} a_k b_k z^k$; equivalently,

$$D^{n+p-1} f(z) = [\, z^p (z^{n-1} f(z))^{(n+p-1)} \,] / (n+p-1)!$$

and showed that

$$K_{n+1} \subset K_n \tag{1.1}$$

for all $n \in N_0$. Familiar classes of K_n are $K_0 = S^*(1/2)$, $K_1 = K$, the starlike functions of order 1/2 and the convex functions, respectively. In H. AL-AMIRI [2], H. AL-AMIRI combined two notions of RUSCHEWEYH and MOCANU [5] to introduce the Ruscheweyh-Mocanu α-convex functions of order n(denote it as $MR_n(\alpha)$) as follow: $f \in MR_n(\alpha)$ if

$$\mathcal{R}e\ \{(1-\alpha)\frac{D^{n+1} f(z)}{D^n f(z)} + \alpha \frac{D^{n+2} f(z)}{D^{n+1} f(z)}\} > 1/2$$

$n \in N_0$, is satisfied for all $z \in E$ and for some real α. He showed that

$$MR_n(\alpha) \subset K_n \tag{1.2}$$

for $\alpha \geqq 0$ and $n \in N_0$. Note that $MR_n(0) = K_n$, $MR_n(1) = K_{n+1}$. In GOEL and SOHI [4], GOEL and SOHI introduced the class

$$K_{n+p+1} = \{ f \in A(p) : \mathcal{R}e(D^{n+p} f(z) / D^{n+p-1} f(z)) > 1/2 \}$$

for $n \in N_0$ and $p \in N$ and proved the theorem:

$$K_{n+p} \subset K_{n+p-1} \tag{1.3}$$

In SONI [5], SONI had the generalization of R. SINGH and S. SINGH [6]:

$$R(n+p) \subset R(n+p-1) \tag{1.4}$$

where

$$R(n+p-1) = \{ f \in A(p) : \mathcal{R}e(D^{n+p} f(z) / D^{n+p-1} f(z)) > (n+p-1)/(n+p) \}.$$

In this paper we shall prove that if a function $f \in A(p)$ for some $p \in N$ and satisfies one of the conditions

$$\mathcal{R}e\ \{ (1-\alpha) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + \alpha \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} \} > \beta$$

for all $z \in E$, n is any integer greater than $-p$, then a lower bound $\gamma(\alpha, \beta, n, p)$ is obtained such that for all $z \in E$

$$\mathcal{R}e\ \{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \} > \gamma(\alpha, \beta, n, p).$$

When $\beta = 1/2$, our Theorem yields

$$\gamma(\alpha, \beta, n, p) = 1/2 \qquad \text{if } \alpha \geqq -(n+p+1)$$

$$\gamma(\alpha, \beta, n, p) > 1/2 \qquad \text{if } \alpha < -(n+p+1).$$

This implies (1.1) and (1.2) for $n \in N_0$, $\alpha = 1$ and for $n \in N_0$, $\alpha \geqq 0$ respectively by taking $p = 1$. On the other hand, let $\alpha = 1$, $\beta = 1/2$ and $p \in N$, then

$$\gamma(\alpha, \beta, n, p) = 1/2$$

which is the result of (1.3). Moreover, if we take $\beta = (n+p)/(n+p+1)$, then

$$\gamma(\alpha, \beta, n, p) > (n+p-1)/(n+p)$$

for $0 < \alpha \leqq 1$, which is (1.4) when $\alpha = 1$. Thus, our Theorem includes or improves all the results of (1.1), (1.2), (1.3) and (1.4).

2. Lemmas.

To prove the main result, we need the following lemmas

1. Jack's Lemma: Let $w(z)$ be regular in the unit disc E, with $w(0) = 0$. Then if $|w|$ attains its maximum value on the circle $|z| = r$ at a point z_1, we can write

$$z_1 w'(z_1) = k w(z_1)$$

where $k \geqq 1$. (JACK [7])

After some calculations we may obtain:

2. Lemma: For $p \in N$, n is any integer greater than $-p$ and α is real,

$$0 \leqq \{2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha - [(2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha)^2 + 16(\alpha-\beta(n+p+1))(n+p+1-\alpha)]^{\frac{1}{2}}\} / [4(n+p+1-\alpha)] \leqq 1/2$$

$$\text{when } \alpha/(n+p+1) \leqq \beta \leqq 1/2 \text{ and } n+p+1 > 2\alpha, \tag{2.1}$$

$$1/2 \leqq \{2\beta(n+p+1) - 3\alpha + [(2\beta(n+p+1) - 3\alpha)^2 + 8\alpha(n+p+1-\alpha)]^{\frac{1}{2}}\} / [4(n+p+1-\alpha)] < 1$$

$$\text{when } 1/2 \leqq \beta < 1 \text{ and } \alpha \neq n+p+1, \tag{2.2}$$

$$1/2 \leqq 1/(3-2\beta) < 1$$

$$\text{when } 1/2 \leqq \beta < 1, \text{ and} \tag{2.3}$$

$$z(D^{n+p-1} f(z))' = (n+p)\, D^{n+p} f(z) - n(D^{n+p-1} f(z)). \tag{2.4}$$

3. Main Result.

Theorem. If $f \in A(p)$ for some $p \in N$ and satisfies the condition

$$\Re e\left\{ (1-\alpha)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + \alpha\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} \right\} > \beta$$

for some n which is any integer greater than $-p$ and α is real, then

$$\Re e\left\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \right\} > \gamma(\alpha, \beta, n, p)$$

where

$$\gamma(\alpha, \beta, n, p) = \{2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha - [(2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha)^2 + 16(\alpha-\beta(n+p+1))(n+p+1-\alpha)]^{\frac{1}{2}}\} / [4(n+p+1-\alpha)]$$

if $\alpha/(n+p+1) \leqq \beta < 1/2$ and $n+p+1 > 2\alpha$,

$$\gamma(\alpha, \beta, n, p) = \{2\beta(n+p+1) - 3\alpha + [(2\beta(n+p+1) - 3\alpha)^2 + 8\alpha(n+p+1-\alpha)]^{\frac{1}{2}}\} / [4(n+p+1-\alpha)]$$

if $1/2 \leqq \beta < 1$ and $\alpha \neq n+p+1$, and

$$\gamma(\alpha, \beta, n, p) = 1/(3-2\beta) \qquad \text{if } 1/2 \leqq \beta < 1 \text{ and } \alpha = n+p+1.$$

Proof: Suppose f satisfies the conditions in the theorem, and let $w(z)$ be a regular function such that

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{1-(2\gamma-1)w(z)}{1-w(z)}, \tag{3.1}$$

where $\gamma = \gamma(\alpha, \beta, n, p)$, then $w(0) = 0$. The Theorem will follow if we can show that $|w(z)| < 1$ in E. Now by differentiating (3.1) logarithmically and applying (2.4), we get

$$-(n+p)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (n+p+1)\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} - 1 = \frac{2(1-\gamma)zw'(z)}{(1-w(z))(1-(2\gamma-1)w(z))}$$

Put

$$H(z) = (1-\alpha)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + \alpha\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} - \beta$$

$$= \frac{\alpha}{n+p+1} - \beta + \left(1 - \frac{\alpha}{n+p+1}\right)\frac{1-(2\gamma-1)w(z)}{1-w(z)} + \frac{2(1-\gamma)\alpha}{n+p+1}\,\frac{zw'(z)}{(1-w(z))(1-(2\gamma-1)w(z))}.$$

If $|w| \not< 1$ in E, there exists, by Jack's Lemma, $z_1 \in E$ such that $|w(z_1)| = 1 \geqq |w(z)|$ for all $|z| \leqq |z_1|$ and $z_1 w'(z_1) = kw(z_1)$ where $k \geqq 1$.

Let $w(z_1) = u + iv$. After simple computation, we get

$$\Re e \frac{z_1 w'(z_1)}{(1-w(z_1))(1-(2\gamma-1)w(z_1))}$$

$$= \Re e \frac{z_1 w'(z_1)}{w(z_1)} \cdot \frac{w(z_1)}{1-w(z_1)} \cdot \frac{1}{1-(2\gamma-1)w(z_1)}$$

$$k\gamma \,/\, [2(2\gamma^2 - 2\gamma + 1 - (2\gamma - 1)u)]$$

Let $g(u) = 1 \,/\, (2\gamma^2 - 2\gamma + 1 - (2\gamma - 1)u)$, (3.2)

$$g'(u) = (2\gamma-1) \,/\, (2\gamma^2 - 2\gamma + 1 - (2\gamma - 1)u)^2 \qquad (3.3)$$

and $g(-1) = 1/(2\gamma^2)$, $g(1) = 1/(2(1-\gamma)^2)$.

From (2.1), $\alpha/(n+p+1) \leqq \beta \leqq 1/2$ and $n+p+1 > 2\alpha$ implies that

$$0 \leqq \gamma(\alpha, \beta, n, p) = \gamma \leqq 1/2, \text{ then } g'(u) \leqq 0.$$

Therefore we have $g(1) \leqq g(u) \leqq g(-1)$. (3.4)

Taking the real part of $\mathcal{H}(z_1)$ and applying (3.2), (3.3) and (3.4), we obtain

$$\Re e\, \mathcal{H}(z) = \frac{\alpha}{n+p+1} - \beta + \left(1 - \frac{\alpha}{n+p+1}\right)\gamma - \frac{\gamma(1-\gamma)\alpha k}{n+p+1} g(u)$$

$$\leqq \frac{\alpha-\beta(n+p+1)}{n+p+1} + \left(1 - \frac{\alpha}{n+p+1}\right)\gamma + \frac{-\gamma(1-\gamma)\alpha k}{n+p+1} \frac{1}{2(1-\gamma)^2}$$

$$\leqq -\{ 2(n+p+1-\alpha)\gamma^2 - [2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha]\gamma$$

$$- 2(\alpha - \beta(n+p+1))\} \,/\, [2(n+p+1)(1-\gamma)].$$

Since $\gamma(\alpha, \beta, n, p)$ is a root of the polynomial

$2(n+p+1-\alpha)x^2 - [2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha]x - 2(\alpha - \beta(n+p+1)) = 0$,

then $\Re e\, \mathcal{H}(z) \leqq 0$, which contradicts to $\Re e\, \mathcal{H}(z) > 0$ for all $z \in E$. Thus, $|w(z)| < 1$, for all $z \in E$. Therefore Theorem is proved in the case of $\alpha/(n+p+1) \leqq \beta < 1/2$ and $n+p+1 > 2\alpha$. Similarly, the other cases of Theorem can be proved by using (2.2) and (2.3). Hence the proof is completed.

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