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GENERALIZING THE ARITHMETIC GEOMETRIC MEAN — A HAPLESS COMPUTER EXPERIMENT

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ABSTRACT. The paper discusses the asymptotic behavior of generalizations of the Gauss's arithmetic-geometric mean, associated with the names Meissel (1875) and Borchardt (1876). The "hapless computer experiment" in the title refers to the fact that the author at an earlier stage thought that one had genuine asymptotic formulae but it is now shown that in general "fluctuations" are present. However, no very conclusive results are obtained so the paper ends in a conjecture concerning the special rôle of the algorithms of Gauss and Borchardt.
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KEY WORDS AND PHRASES. Arithmetic-geometric mean, iteration, algorithm, asymptotic formula
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0. INTRODUCTION.

I have now worked on algorithms of the type of Gauss's arithmetic-geometric mean (agM.) for a period of nearly 4 years (starting around the turn of the year 83/84). Strangely enough some of the impetus for getting interested in this field came from the theory of (abtract) interpolation. This connection is described in my talk to the Varna conference in May/June 1984 [P1]. The same year I also prepared an over all survey of "means and their iterations" for the XIXth Nordic Mathematical Congress in Reykjavík [ACJP] (with J. Arazy, T. Claesson and S. Janson as coauthors). I took up the same subject the year after for my address to the A.Haar Memorial Conference in Budapest [P2], which is a collection of "unsolved problems", some of them pertaining to the agM. In particular, I suggested there a certain approximation for a 2-dimensional algorithm derived from the "agM." corresponding to the cyclic group C_3 with three elements analogous to an asymptotic formula in a note by E. Meissel [M] and also the classical asymptotic formula due to Gauss [G]. However, numerical evidence produced later by P. Borwein [Borw] indicates that it here can't be question of a true asymptotic formula. I think now that I fell in the trap of relying too much on information derived from unsufficient numerical data. Hence the second link of the title. (I believe now that the claim in [P1] concerning the asymptotic behavior of iterated power means experiences the same fate.) Thus the question of finding good (asymptotic) approximations for the algorithm of the type in question is by and large open.

The main purpose of this paper is to provide the reader a general background for this problem, and to outline the meager progress I myself have made on it. Maybe, I can thereby inspire other people to continue where I stopped (failed)...

As Grunert's Archiv is nowadays virtually inaccesible for most readers and as the common of knowledge of Western languages, English excluded, among mathematicians is in such a sorrow state, I have included a translation of Meissel's note [M] *in extenso* (see Appendix).

1. THE agM.

G.F. Gauss [1777-1855] studied the agM. from an early age on (some say 14). Let a, b be two real numbers, $0 < b \leq a < \infty$. Taking succesively arithmetic and geometric means we get two sequences $a, a', a'', \ldots$ and $b, b', b'', \ldots$ with

$$a' = \frac{a+b}{2}, \quad b' = \sqrt{ab}$$
$$a'' = \frac{a'+b'}{2}, \quad b'' = \sqrt{a'b'}$$
$$\ldots\ldots$$

It is easy to see that they converge to a common limit,

$$M(a,b) \stackrel{\text{def}}{=} \lim a_n = \lim b_n,$$

called the agM. of a and b. (We have $b \leq b' \leq b'' \leq \ldots \leq a'' \leq a' \leq a$ and

$$a' - b' = (\frac{\sqrt{a}-\sqrt{b}}{2})^2 = \frac{1}{2}(\frac{a-b}{\sqrt{a}-\sqrt{b}})^2 < \frac{1}{2}(a-b),$$

$$a'' - b'' < \frac{1}{2}(a'-b'), \ldots\ldots$$

([G], p. 28).

The agM. and other related "means" are discussed in the survey [ACJP]. (For a more extensive treatment see e.g. the book [BB].) From there we recall only the following.

Basic properties of the Gauss agM.:

1) *Extremely rapid convergence* ("quadratic" in the technical sense).

2) *Integral representation.* As Gauss discovered, one has

$$\frac{\frac{\pi}{2}}{M(a,b)} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}},$$

where the integral to the left is a complete elliptic integral of the first kind. In standard notation, writing $k = b/a$ ("modulus"), $k' = \sqrt{1 - b^2/a^2}$ ("complementary modulus"), the latter is just $K(k')/a$ (cf. [BB], theorem 1.1). Thus the formula can also be written

$$M(a,b) = a\frac{\frac{\pi}{2}}{K(k')}.$$

The importance of the agM. therefore stems in part from the fact that it connects with "elliptic" theory (elliptic integrals, differentials, functions, curves).

3) *Differential equation.* The function $y = \dfrac{1}{M(1,k)}$ satisfies a differential equation with algebraic coefficients (known as Legendre's differential equation)

$$k(k^2-1)y'' + (3k^2-1)y' + ky = 0,$$

which after a change of variable becomes a special case of the hypergeometric equation, likewise considered by Gauss.

4) *Uniformization.* Writing a and b in the form

$$a = Mp^2(x), b = Mq^2(x) \quad (M = M(a,b)),$$

where p and q are, what Gauss calls, summatoric functions (= theta values; in conventional notation $p(x) = \vartheta_{00}(0,t)$, $q(x) = \vartheta_{01}(0,t)$ where $x = e^{\pi i t}$), the algorithm reduces to $x \mapsto x^2$.

5) *Complex values.* Using the uniformization it is possible to extend the algoritm to the case of complex values of a and b. Then the limit $M(a,b)$ is not any longer unique and the values t corresponding to different "determinations" of it are related by a modular transformation. This is also one of the historic roots of the theory of modular functions.

6) *Asymptotic formula.* Central in Gauss's treatment of the agM. is the asymptotic formula

$$M(1,k) \sim \frac{\frac{\pi}{2}}{\log\frac{1}{k}} \quad (k \to 0).$$

It can be readily derived from the following formula for the uniformizing parameter, likewise due to Gauss (see again the discussion in [BB]):

$$x = e^{-\frac{\pi K(k)}{K(k')}}$$

2. BORCHARDT'S GENERALIZATION OF THE agM. WITH FOUR "ELEMENTS".

C.W. Borchardt [1817-1880] was a student and close friend of Jacobi's. He also edited volume one of Jacobi's collected works and became the editor of "Crelle's Journal" after Crelle's death; therefore this journal was for some time known as "Borchardt's Journal". He proposed as a generalization of the agM. the following scheme based on the iteration of the transformation

$$\begin{cases} a' = & \dfrac{a+b+c+e}{4} \\ b' = & \dfrac{\sqrt{ab}+\sqrt{ce}}{2} \\ c' = & \dfrac{\sqrt{ac}+\sqrt{ce}}{2} \\ e' = & \dfrac{\sqrt{ae}+\sqrt{bc}}{2} \end{cases}$$

It turns out that this algorithm has a theory entirely parallel to Gauss's. In particular, one has proper counterparts of all the properties 1)-6) mentioned in Sec. 1, with the possible exception of 6). The "uniformization" is now obtained by theta functions in two variables and in place of the elliptic curve now enters Kummer's quartic surface. Borchardt also briefly indicates a 2^n-dimensional generalization, but for various reasons this algorithm is far more defective, largely because theta functions in n variables do not suffice for purposes of uniformization. We quote ([Borc], p. 621): "... but of which kind these transcendental functions are, in terms of which the limit can be represented, is a question whose answer must be left for the future".

3. THE "MONSTER" ALGORITHM.

In [ACJP] it is pointed out that both Gauss's and Borchardt's algorithm are special cases of the following very general construction. Let G be any compact group. Let f be a positive measurable function in the Lebesgue space $L^{1/2}(G)$. Then $f' = \sqrt{f} * \sqrt{f}$, where $*$ stands for convolution (with Haar measure so normalized that $1 * 1 = 1$), is a function of the same type (in fact, in $L^1(G)$). Similarly, $f'' = \sqrt{f'} * \sqrt{f'}$ is a continuous function (in $C(G)$). Continuing we obtain a sequence of functions which, as is proved in [ACJP], tends pointwise, in fact uniformly, to a *constant* functions, denoted $M(f)$ or $M_G(f)$ (and identified with the corresponding number).

Properties:

0) $M(1) = 1$.

1) $M(\lambda f) = \lambda M(f)$ $(\lambda > 0)$ (homogeneity).

2) $\inf f \leq M(f) \leq \sup f$.

3) $M(f) \leq M(g)$ if $f \leq g$ a.e. (monotonicity).

4) $M(f+g) \geq M(f) + M(g)$ (concavity).

It is likewise mentioned in [ACJP] that if we pass to complex valued functions f, assuming that $\mathrm{Re} f \geq 0$, the same convergence holds true, but this is much harder to prove.

EXAMPLE. For instance, if $G = \mathbb{T}$ and if f is analytic inside the unit disk D (we make the identification $\partial D = \mathbb{T}$) then $M(f) = f(0)$.

We are however mostly interested in the case of finite groups.

EXAMPLE. If $G = C_2$ (the cyclic group with two elements) we get back Gauss's algorithm and if $G = C_2 \times C_2$ we get Borchardt's algorithm. Borchardt's generalization with 2^n elements similarly corresponds to $G = C_2^n$. The first non-classical case is thus $G = C_3$ (the cyclic group with three elements). Spelled out explicitly it is thus question of iterating the map

$$\begin{cases} a' = & \dfrac{a+b+c}{3} \\ b' = & \dfrac{2\sqrt{ab}+c}{3} \\ c' = & \dfrac{2\sqrt{ac}+b}{3} \end{cases}.$$

4. ALGORITHMS DERIVED FROM THE "MONSTER" ALGORITHM.

Let us return to the case of a general group G. For simplicity we take G finite. Let H be any subgroup of G. We restrict attention to functions f which are constant on H and on the complement $H^c = G \backslash H$, i.e.

$$f = \begin{cases} a & H \\ b & H^c \end{cases}$$

Then the iterates are of the same type:

$$f' = \begin{cases} \frac{1}{N}a + (1-\frac{1}{N})b & \overset{\text{fed}}{=} a' \\ \frac{2}{N}\sqrt{ab} + (1-\frac{2}{N})b & \overset{\text{fed}}{=} b' \end{cases},$$

$$f'' = \begin{cases} \frac{1}{N}a' + (1-\frac{1}{N})b' \\ \frac{2}{N}\sqrt{a'b'} + (1-\frac{2}{N})b' \end{cases},$$

.....

where $N = |G:H| = |G|/|H|$ is the *index* of H in G. (By Lagrange's theorem we know that N is an integer.) Thus we are lead to consider, quite generally, the 2-dimensional algorithm

$$\text{(1)} \qquad \begin{cases} a' = & \theta(1-a) + \theta b \\ b' = & 2\theta\sqrt{ab} + (1-2\theta)b = \theta(1-a) + \theta b - \theta(\sqrt{a}-\sqrt{b})^2 \end{cases},$$

where θ is any number in the interval $(0, \frac{1}{2}]$.

REMARK. The same game can be played with a tower of semigroups, say,

$$G = G_r \supset G_{r-1} \supset \dots \supset G_1 \supset G_0 = E.$$

Restricting attention to functions f which are constant on the sets $G_i \backslash G_{i-1}$ ($i = 1, ..., r$), that is, $f|_{G_1} = a_1$, $f|_{G_2\backslash G_1} = a_2$, ..., $f|_{G_r\backslash G_{r-1}} = a_r$ with $|G_1| = \alpha_1$, $|G_2\backslash G_1| = \alpha_2$, ..., $|G_r\backslash G_{r-1}| = \alpha_r$, $\alpha_1+...+\alpha_r = |G|$, we are faced with the r-dimensional algorithm ($\theta_i = \alpha_i/|G|$)

$$(1') \quad \begin{cases} a_1' = & \theta_1 a_1 + ... + \theta_r a_r \\ a_2' = & 2\theta_1\sqrt{a_1a_2} + (\theta_2 - \theta_1)a_2 + \theta_3 a_3 + ... + \theta_r a_r \\ a_3' = & 2\theta_1\sqrt{a_1a_2} + 2\theta_2\sqrt{a_2a_3} + (\theta_3 - \theta_1 - \theta_2)a_3 + \theta_4 a_4 + ... + \theta_r a_r \\ ... & \\ a_r' = & 2\theta_1\sqrt{a_1a_2} + 2\theta_2\sqrt{a_2a_3} + ... + 2\theta_{r-1}\sqrt{a_{r-1}a_r} + (\theta_r - \theta_1 - ... - \theta_{r-1})a_r \end{cases}$$

Notice that one can also write

$$\begin{cases} a_2' = & \theta_1 a_1 + ... + \theta_r a_r - \theta_1(\sqrt{a_1} - \sqrt{a_2})^2 \\ a_3' = & \theta_1 a_1 + ... + \theta_r a_r - \theta_1(\sqrt{a_1} - \sqrt{a_3})^2 - \theta_2(\sqrt{a_2} - \sqrt{a_3})^2 \\ & ..... \\ a_r' = & \theta_1 a_1 + ... + \theta_r a_r - \theta_1(\sqrt{a_1} - \sqrt{a_r})^2 - \theta_2(\sqrt{a_2} - \sqrt{a_r})^2 - ...- \\ & -\theta_{r-1}(\sqrt{a_{r-1}} - \sqrt{a_r})^2 \end{cases}$$

PROOF OF (1′). Indeed consider the equation $i = jk$.
Assume first that $i \in G_1$. Then we have the following possibilities:
$j \in G_1$, $k \in G_1$ corresponds to a term $\alpha_1 a_1$,
$j \in G_2\backslash G_1$, $k \in G_2\backslash G_1$ corresponds to a term $\alpha_2 a_2$
.....
$j \in G_r\backslash G_{r-1}$, $k \in G_r\backslash G_{r-1}$ corresponds to a term $\alpha_r a_r$.
This accounts for the formula for a_2'.
Next assume that $i \in G_2\backslash G_1$. Then we have the following possibilities:
$j \in G_2$, $k \in G_2 \in G_1$ or $j \in G_2\backslash G_1$, $k \in G_2\backslash G_1$ corresponds to a term $2\alpha_1\sqrt{a_1a_2}$
$j \in G_2\backslash G_1$, $k \in G_2\backslash G_1$ corresponds to a term $(\alpha_2 - \alpha_1)a_2$
$j \in G_3\backslash G_2$, $k \in G_3\backslash G_2$ corresponds to a term $\alpha_3 a_3$
....
$j \in G_r\backslash G_{r-1}$, $k \in G_r\backslash G_{r-1}$ corresponds to a term $\alpha_r a_r$.
This accounts for the formula for a_2'.
In the same way one treats the case $i \in G_3\backslash G_2$. The proof is concluded by an induction argument.#

5. MEISSEL.

In a short note [M] (cf. Appendix to this paper) published in 1874 in Grunert's Archiv (D.E.)F. Meissel [1826-1895], who was headmaster of a secondary school in Kiel, Germany, toward the end of the last century, considers the iteration

$$\begin{cases} a' = & \dfrac{a+b+c}{3} \\ b' = & \sqrt{\dfrac{ab+ac+bc}{3}} \\ c' = & {}^3\sqrt{abc} \end{cases}.$$

In particular, he states without proof the following asymptotic formula for the corresponding limit $M(a, b, c)$:

$$(2) \qquad M(1,1,c) \asymp \Big(\frac{A}{\log \frac{B}{c}}\Big)^{\lambda} \quad c \to 0$$

where $\lambda = 0.43331485...$ is obtained, upon eliminating of k, from the equations:

$$3^{\lambda} = \frac{3}{1+k} = \sqrt{\frac{k}{3}}$$

and A and B are certains constants

$$A = 0.3951642..., \quad B = 10^{0.2997049...}.$$

(The exact meaning of $\asymp$ is not clear from the context; actually, Meissel himself writes simply $=$ in formula (2).)

In Sec. 8 I will give an attempt to justify Meissel's formula (1).

REMARK. In the excellent book [BB], p. 268-269, this algorithm is called *Schlömilch's algorithm* and in this context reference is made to a paper by Schoenberg's [Scho], which however has been inaccessible to me. I have checked with the index of *Jahrbuch* up to and including the year 1903 (the year of my mother's birth) and found only one paper of Schlömilch's dealing with iteration of means, namely [Schlö], but it does not seem to be very relevant in the present context.

6. DISCUSSION (THE COMPUTER EXPERIMENT).

Guided by the asymptotic formulae by Gauss and Meissel I suggested in [P2] the following approximtion of the limit $M(a,b)$ of the iterations of the transformation (1) in Sec. 4:

$$M(1,b) \asymp A \frac{1}{(\log \frac{4}{b})^{\frac{\log N}{\log 2}}} \quad (b \to 0) \tag{3}$$

I made also numerical experiments on a minicomputer, which to some extent seemed to support my guess. Here are some values for the constant A obtained:

$$\begin{array}{ll} N=2 & A = 1.570796327... \\ N=3 & A = 2.3410... \\ N=4 & A = 3.289868133... \\ N=5 & A = 4.420... \\ N=6 & A = 5.738... \\ N=7 & A = 7.251... \\ N=8 & A = 8.96... \\ ..... & \end{array}$$

I thought at the time that I had a rigorous proof of (3) interpreted as a genuine asymptotic formula so I did not pay much attention to the fluctuations in the numerical data, which seemed to increase with N. Of course, as (3) reduces to Gauss's formula if $N = 2$ there are no such fluctuations in this case and also not in Borchardt's case ($N = 4$). Therefore Borwein's letter [Borw] came as a surprise, if not a shock. However, I now quickly realized that the fluctuations really were part of the picture (in all other cases but $N = 2,4$) and not something connecting with the insufficiency of the numerical device available to me. This will be explained in the next Section.

The numerical experiment thus suggests only the following: *There is an exact asymptotic formula only if $N = 2,4$ and there can be an integral formula for the limit of the Gauss-Borchardt type only in these two cases.* But, I emphasize, this is something that I have not proved so it is therefore just question of another *conjecture*. In the former case $A = \pi/2$ exactly and the latter case *probably* $A = \pi^2/3$.

7. EXPLANATION OF THE FLUCTUATION.

In inhomogeneous notation ($M(b) = M(1,b)$) the functional equation for the limit M of the algorithm (1) can be written

$$M(b) = g(b)M(b'), \tag{4}$$

where we for convenience have put

$$g(b) = \theta + (1-\theta)b.$$

It is easy to see that the only solution of the functional equation (4) which is analytic near $b = 1$ normalized by the requirement $M(1) = 1$ comes as an infinite product

$$M(b) = g(b)g(b')g(b'')... \tag{5}$$

This is connected with the fact that $b = 1$ is an attractive (even hyperattractive) fixed point of the map

$$b \mapsto b' = \frac{2\theta\sqrt{b} + (1-2\theta)}{\theta + (1-\theta)b}$$

and that $g(1) = 1$, which guarantees the convergence of the product in (5). (In (5) $b', b'', ...$, of course, are the iterates of b under this map.)

Next, consider instead the functional equation

$$Z(b) = h(b)Z(b'), \tag{6}$$

where now

$$h(b) = \frac{g(b)}{\theta} = \frac{\theta + (1-\theta)b}{\theta}.$$

$b = 0$ is a repulsive fixed point for the map $b \mapsto b'$, hence an attractive (again even hyperattractive) one for the inverse map $b \mapsto {}'b$. (If b is small then $b' \approx 2\sqrt{b}$ so that for the inverse holds $b \mapsto {}'b \approx b^2/4$.) Also $h(0) = 1$. It follows, by the same reason as before, that there is a unique solution of (5), analytic at $b = 0$ and normalized by $Z(0) = 1$. One can thus write

$$Z(b) = 1 + z_1 b + z_2 b^2 + ...$$

and it is possible to write down a recursion for the coefficients z_n, which generalizes the one used by Gauss [G].

Finally, consider

$$N(b) = (N(b'))^2.$$

This is essentially *Böttcher's equation* for the inverse map (see e.g. [ACJP]). It follows that there exists a unique solution which admits the expansion

$$N(b) = \frac{4}{b} + n_0 + n_1 b + n_2 b^2 + ...,$$

thus in partcular satisfying

$$N(b) \sim \frac{4}{b} \quad (b \to 0).$$

REMARK. The letters Z and N are picked in honor of Gauss (for German *Zähler*, nominator, and *Nenner*, denominator, respectively). Do not confuse the function $N = N(b)$ and the previous integer valued parameter N (= the index of the subgroup H the finite group G); see Sec. 4.

Now it is clear that the function

$$M' \stackrel{\text{def}}{=} \frac{Z}{(\log N)^{\lambda}}$$

satisfies the same functional equation (4) as our mean M. The quotient is a function which is invariant under $b \mapsto b'$. It follows that we have

$$M = M' \cdot O,$$

where O is an "oscillatory" function, that is, invariant under the transformation $b \mapsto b'$ ($O(b) = O(b')$). *Now the presence of the fluctuations is completely explained.* Our above conjecture (end of Sec. 6) amounts therefore to $O = \text{const}$ iff $N = 2, 4$.

REMARK. A natural parameter near $b = 0$ seems to be $1/N(b)$ or even better

$$t = 2\pi \frac{\log\log \dfrac{1}{N(b)}}{\log 2}.$$

(If $b \mapsto b'$ then $t \mapsto t + 2\pi$.) Therefore one can go a step further an ask for the *Fourier development* of the function O:

$$O = A_0 + A_1 \cos t + B_1 \sin t + A_2 \cos 2t + B_2 \sin 2t + \ldots$$

We see thus that, in a way, what we have observed in the numerical experiment is just the 0-th coefficient $A = A_0$ in this expansion.

PROBLEM: To account for the "higher" terms!

8. ATTEMPTS TO JUSTIFY MEISSEL'S FORMULA.

We return to Meissel's algorithm (Sec. 5), denoting its limit by $M(a, b, c)$. The following properties of $M(a, b, c)$ are obvious:

1) $M(\lambda a, \lambda b, \lambda c) = \lambda M(a, b, c)$ (homogeneity),
2) $M(1, 1, 1) = 1$ (normalization),
3) $M(a, b, c) = M(a', b', c')$ (invariance).

Now we make the "Ansatz":

$$M(a, b, c) \asymp A(\log \frac{B}{c})^{-\lambda},$$

where $A = A(a, b)$ is supposed to be homogeneous of degree 1 and $B = B(a, b)$ homogeneous of degree 0. Plugging this into the functional equation we find

$$A(\log \frac{B}{c})^{-\lambda} \asymp A(\frac{a+b}{3}, \sqrt{\frac{ab}{3}}) \cdot (\log \frac{B(\frac{a+b}{3}, \sqrt{\frac{ab}{3}})}{\sqrt[3]{abc}})^{-\lambda}.$$

Write

$$A' = A(a', b'), \quad B' = B(a', b').$$

The log-factors to the left and to the right can be written as

$$(\log \frac{1}{c} + \log B)^{-\lambda}$$

respectively

$$(\frac{1}{3} \log \frac{1}{c} + \frac{1}{3} \log \frac{1}{ab} + \log B')^{-\lambda} = 3^{\lambda} \left(\log \frac{1}{c} + \log \frac{1}{ab} + 3 \log B' \right)^{-\lambda}.$$

This gives

$$\boxed{A = A' 3^{\lambda}}.$$

and

$$\log B = \log \frac{1}{ab} + 3 \log B' \quad \text{or} \quad \boxed{B = \frac{(B')^3}{ab}}.$$

This seems to reduce the problem to the iteration of a two dimensional homogeneous map:

$$\begin{cases} a' = & \dfrac{a+b}{3} \\ b' = & \sqrt{\dfrac{ab}{3}} \end{cases},$$

which in the base ($t = a/b$) induces the map

$$t' = \frac{t+1}{\sqrt{3t}},$$

having one repulsive fixed point $t = k^{-1}$ determined (as in [M]) by the equation

$$k(1+k)^2 = 3.$$

The b-equation can be written

$$b' = \sqrt{\frac{t}{3}}b.$$

Hence the A-equation is satisfied by

$$A \stackrel{\text{def}}{=} b\sqrt{kt}\sqrt{k't}\sqrt{k''t}...$$

with

$$3^\lambda = \frac{3}{1+k} = \sqrt{\frac{k}{3}},$$

whence (as in [M])

$$\lambda = 1 - \frac{\log(1+t)}{\log 3}.$$

The B-equation can be written

$$bB(\frac{a}{b}, 1) = \frac{(b')^3}{ab} \cdot (B(\frac{a'}{b'}, 1))^3$$

or, in inhomogeneous notation,

$$B(t) = \frac{t^{1/2}}{3^{3/2}} \cdot (B(t'))^3.$$

In particular,

$$B(k^{-1}) = \sqrt{3^{3/2}}k^{1/2}.$$

That is,

$$\frac{B(t)}{B(k^{-1})} = \sqrt{kt}(\frac{B(t')}{B(k^{-1})})^3$$

or iterating

$$B(t) = B(k^{-1})\frac{1}{(kt)^{\frac{1}{2}\frac{}{3}}} \cdot \frac{1}{(k \cdot {}'t)^{\frac{1}{2}\frac{}{9}}} \cdot \frac{1}{(k \cdot {}''t)^{\frac{1}{2}\frac{}{27}}} \cdot ...$$

Thus A and B do "exist". It is however not at all clear to me in what sense M is approximated in terms of the functions A and B.

NOTE (added Sept. 1988).Recently J. and P. Borwein sent me a truly marvelous paper entitled "On the mean iteration $(a, b) \to (\frac{a+3b}{4}, \frac{\sqrt{ab}+b}{2})$", where in particular the conjecture about the asymptotic behavior of the Borchardt mean at the end of Sec. 6 is fully established. Even more, the authors show that the mean in question shares with the agM. practically all of its remarkable properties (see Sec. 1), in particular thus that the algorithm can be "uniformized" in terms of theta constants.

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APPENDIX. TRANSLATION OF MEISSELS'S NOTE [M].

REMARKS ON THE HYPERGEOMETRIC SERIES.

Recently I communicated to Professor Rümker in Hamburg the following relation:

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\left(\frac{1-(1-k^2)\sin^2\phi}{k}\right)^{1-\alpha}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\left(\frac{1-(1-k^2)\sin^2\phi}{k}\right)^{\alpha}}$$

which I incidentally found as a byproduct in investigations concerning the Gauss hypergeometric series. Another relation is

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1+\alpha}{2}}} = \cos(\frac{\pi\alpha}{2}) \int_0^{\frac{\pi}{2}} \frac{\mathrm{tg}^\alpha\phi}{\sqrt{1-k^2\sin^2\phi}};$$

I do not remember having seen it anywhere.

What I really aimed at was really an extension of the investigation, originating from Lagrange and further amplified by Gauss, of the limit to which the sequence of (successive) arithmetic and geometric means of two numbers a and b converges, and I considered the equations

$$\begin{aligned} 3a_{n+1} &= a_n + b_n + c_n \\ 3b_{n+1}^2 &= a_n b_n + a_n c_n + b_n c_n \\ c_{n+1}^3 &= a_n b_n c_n \end{aligned}$$

writing $a_\infty = b_\infty = c_\infty = M(a_0, b_0, c_0)$.

In particular, I was interested in the extremal case

$$a_0 = 1; \quad b_0 = 1; \quad c_0 \text{ very small;}$$

and I found then as a limit of $a_\infty = b_\infty = c_\infty$

$$M(1,1,w) = \Big(\frac{A}{\log \frac{B}{w}}\Big)^\lambda \tag{1}$$

where A, B, λ are constants and in fact

$$\lambda = 1 - \frac{\log(1+k)}{\log 3}$$

where k is the root of the equation

$$k(1+k)^3 = 3.$$

The value of λ is

$$\lambda = 0,4333\,1485.$$

For simplicity we take the log to be Briggian and then the values of A and B become approximtely

$$A = 0,395\,1642$$

$$\log \text{brigg} B = 0,299\,7049$$

I have carried out the computations for very small w and, for example, found directly that

$$\text{for } w = \frac{1}{10^{24}}, \quad M = 0,1678\,3257$$
$$\text{for } w = \frac{1}{10^{6561}}, \quad M = 0,0148\,3609$$

Kiel, April 8, 1874. E. Meissel