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ON FACTORIZATIONS OF FINITE ABELIAN GROUPS WHICH ADMIT REPLACEMENT OF A Z-SET BY A SUBGROUP

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ABSTRACT. A subset A of a finite additive abelian group G is a Z-set if for all $a \in A$, $na \in A$ for all $n \in Z$.

The purpose of this paper is to prove that for a special class of finite abelian groups, whenever the factorization $G = A \oplus B$, where A and B are Z-sets, arises from the series $G = K_1 \supset K_2 \supset \dots \supset K_n \supset <0>$ then there exist subgroups S and T such that the factorization $G = S \oplus T$ also arises from this series. This result is obtained through the introduction of two new concepts: a series admits replacement and the extendability of a subgroup. A generalization of a result of L. Fuchs is given which enables establishment of a necessary and sufficient condition for extendability. This condition is used to show that certain series for finite abelian p-groups admit replacement.

KEY WORDS AND PHRASES. Finite abelian group, factorization, Z-set.
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1. INTRODUCTION.

Let G be a finite abelian additive group and let A and B be subsets of G. If every element $g \in G$ can be uniquely represented in the form $g = a + b$, where $a \in A$, $b \in B$, then we write $G = A \oplus B$ and call this a factorization of G. A subset A is said to be a Z-set if for all $a \in A$, $na \in A$ for all $n \in Z$.

A.D. Sands [1] gave a method which yields all factorizations of a finite abelian good group. His method corrects one given previously by G. Hajos [2].

Our main purpose is to prove that for a special class of finite abelian groups, whenever the factorization $G = A \oplus B$, where A and B are Z-sets, arises from the series $G = K_1 \supset K_2 \supset \dots \supset K_n \supset K_n \supset K_n \supset <0>$ (see [3]), then there exist subgroups S and T such that factorization $G = S \oplus T$ also arises from this series.

In order to achieve this result we introduce two new concepts: a series admits replacement and the extendability of a subgroup. We prove a generalization of a result of L. Fuchs [4] which enables us to derive a necessary and sufficient condition for extendability. This condition is used to show that certain series for finite abelian p-groups admit replacement.

2. PRELIMINARIES.

We shall use the term "Z-factorization" when referring to a factorization of the form $G = A \oplus B$, where A and B are Z-sets.

Our first two lemmas can be readily verified.

LEMMA 1. Let $G = S \oplus A$, where S is a subgroup of G and A is a Z-set. If H and K are subgroups of G with $H = H_S \oplus H_A$, $K = K_S \oplus K_A$, where H_S, K_S are subgroups of S, and H_A, K_A are Z-sets such that $H_A \subseteq A$, $K_A \subseteq A$, then $H \cap K = (H_S \cap K_S) \oplus (H_A \cap K_A)$.

LEMMA 2. Let $G = A \oplus B$ be a Z-factorization of G. If H is a subgroup of G such that $A \subseteq H$ then $H = A \oplus (H \cap B)$.

LEMMA 3. Let $G = G^{(0)} = S \oplus A \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset$ (2.1) $G^{(n+1)} = \langle 0 \rangle$ be a series for G with $S^{(0)} \supset S^{(1)} \supset \ldots \supset S^{(n)} \supset \langle 0 \rangle$ subgroups of G and $A = A^{(0)} \supset A^{(1)} \supset \ldots \supset A^{(n)} \supset \{0\}$ Z-sets. There exists a refinement of (2.1) which is a composition series for G and the subgroups in this refinement have the same properties as (2.1), i.e. any subgroup, H, in the refinement has the form $H = H_S \oplus H_A$, where H_S is a subgroup of S and H_A is a Z-set, $H_A \subseteq A$, and $H \subset K$ implies $H_S \subseteq K_S$ and $H_A \subseteq K_A$. Furthermore, if $H \subset K$ are successive groups in the refinement then either $H_S = K_S$ or $H_A = K_A$.

PROOF. It suffices to show that if there exists $\hat{G} \subset G$ with $G^{(i)} \supset \hat{G} \supset G^{(i+1)}$ then $\hat{G} = \hat{S} \oplus \hat{A}$ with $S^{(i)} \supseteq \hat{S} \supseteq S^{(i+1)}$, $A^{(i)} \supseteq \hat{A} \supseteq A^{(+1)}$, $\hat{A}$ a Z-set, and either $\hat{S} = S^{(i)}$ or $\hat{A} = A^{(i)}$, $0 \leq i \leq n$.

Consider $G^{(i)} = S^{(i)} \oplus A^{(i)} \supset G^{(i+1)} = S^{(i+1)} \oplus A^{(i+1)}$, $0 \leq i \leq n$.

Case 1. Suppose that $A^{(i)} = A^{(i+1)}$. Then for any $\hat{G}$ such that $G^{(i)} \supset \hat{G} \supset G^{(i+1)}$ we have $\hat{G} = A^{(i)} \oplus (\hat{G} \cap S^{(i)})$ by Lemma 2. Clearly $S^{(i)} \supset \hat{G} \cap S^{(i)} \supset S^{(i+1)}$. In this case we have $\hat{A} = A^{(i)}$.

Case 2. Suppose that $A^{(i)} \neq A^{(i+1)}$. We can insert the subgroup $G = \tilde{G}^{(i+1)} + S^{(i)} = S^{(i)} \oplus A^{(i+1)}$ without altering the structure of the series, i.e. we have $G^{(i)} = S^{(i)} \oplus A^{(i)} \supset \tilde{G} = S^{(i)} \oplus A^{(i+1)} \supset G^{(i+1)} = S^{(i+1)} \oplus A^{(i+1)}$.

Let $\hat{G}$ be such that $G^{(i)} \supset \hat{G} \supset G^{(i+1)}$. If $\hat{G} = \tilde{G}$ we are done. If $\tilde{G} \supset \hat{G} \supset G^{(i+1)}$ then by Case 1 $\hat{G}$ has the required form. Finally, if $G^{(i)} \supset \hat{G} \supset \tilde{G}$ then by Lemma 2, $\hat{G} = S^{(i)} \oplus (\hat{G} \cap A^{(i)})$. Clearly $A^{(i)} \supset \hat{G} \cap A^{(i)} \supset A^{(i+1)}$. In this case we have $\hat{S} = S^{(i)}$. This completes the proof.

THEOREM 1. [5] If $G = B^{(1)} \oplus \ldots \oplus B^{(k)}$, where each $B^{(i)}$ is a Z-set, $1 \leq i \leq k$, and if $G = N^{(1)} \oplus \ldots \oplus N^{(r)}$, where each $N^{(j)}$ is a subgroup of G, $1 \leq j \leq r$, such that $(|N^{(i)}|, |N^{(j)}|) = 1$ for $i \neq j$, then

(a) $B^{(i)} = (N^{(1)} \cap B^{(i)}) \oplus \ldots \oplus (N^{(r)} \cap B^{(i)})$, $1 \leq i \leq k$,

and

(b) $N^{(j)} = (N^{(j)} \cap B^{(1)} \oplus \ldots \oplus (N^{(j)} \cap B^{(k)})$, $1 \leq j \leq r$.

The following lemma is a direct consequence of the Second Isomorphism Theorem.

LEMMA 4. Let U, U_1, and K be subgroups of G with $U_1 \subseteq U$. Then $[U \cap K: U_1 \cap K] \leq [U: U_1]$.

Let S be a subgroup of G. We will say that S is homogeneous if S is a direct sum of cyclic groups of the same order.

Theorem 2, which is a generalization of the following result of L. Fuchs [4], p.79), can be readily verified.

(Fuchs) Let S be a pure homogeneous subgroup of G of exponent p^k and let H be a subgroup of G satisfying $p^k G \subseteq H$ and $S \cap H = <0>$. If M is a subgroup of G maximal with respect to the properties $H \subseteq M$ and $M \cap S = <0>$ then $G = S \oplus M$.

THEOREM 2. Let $S = \bigoplus_{i=1}^{n} S_i$ be a pure subgroup of G with S_i, $1 \leq i \leq n$, homogeneous of exponent p^{k_i}, $k_1 > k_2 .. > k_n$, and let $U \subseteq G$. There exists a subgroup, T, of G with $U \subseteq T$ and $G = S \oplus T$ if and only if $[p^{k_j} G + \bigoplus_{i>j} S_i + U] \cap S_j = <0>$, $1 \leq j \leq n$.

3. REDUCTION TO THE CAUSE OF P-GROUPS.

Consider the series

$$G = G^{(0)} = S \oplus A \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset <0> \qquad (3.1)$$

where $S = S^{(0)} \supseteq S^{(1)} \supseteq \ldots \supseteq S^{(n)} \supseteq <0>$ are subgroups of G and $A = A^{(0)} \supseteq A^{(1)} \supseteq \ldots \supseteq A^{(n)} \supseteq \{0\}$ are Z-sets. We say the series (3 1) admits replacements if there exist subgroups, $T^{(i)}$, such that $T = T^{(0)} \supseteq T^{(1)} \supseteq \ldots \supseteq T^{(n)} \supseteq <0>$ and $G^{(i)} = S^{(i)} \oplus T^{(i)}$, $0 \leq i \leq n$. Let us note that by Proposition 1 [3] there exist subgroups $T^{(i)}$ such that $G^{(i)} = S^{(i)} \oplus T^{(i)}$, $0 \leq i \leq n$. However it is not necessarily the case that $T^{(0)} \supseteq T^{(1)} \supseteq \ldots \supseteq T^{(n)}$. This problem will be treated in the next section.

A group C admits replacement if every series for G of the form (3.1) admits replacements. The following theorem enables us to restrict our investigations in this area to the case of p-groups.

THEOREM 3. Let $G = \bigoplus_p G_p$, where the G_p are the primary components of G. G admits replacement if and only if for each p, G_p admits replacement.

PROOF. Suppose G admits replacement. Let $H = G_p$ for some p and let

$$H = H^{(0)} = S^{(0)} \oplus A^{(0)} \supset H^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset H^{(m)} = S^{(m)} \oplus A^{(m)} \supset <0>$$

be a series for H with $S^{(0)} \supseteq S^{(1)} \supseteq \ldots \supseteq S^{(m)} \supset <0>$ subgroups of H and $A^{(0)} \supseteq A^{(1)} \supseteq \ldots \supseteq A^{(m)} \supset \{0\}$ Z-sets. Define $K = \bigoplus_{p' \neq p} G_{p'}$, so that $G = H \oplus K = (S^{(0)} \oplus K) \oplus A^{(0)}$. Then $G = G^{(0)} = (S^{(0)} \oplus K) \oplus A^{(0)} \supset H^{(0)} = S^{(0)} \oplus A^{(0)} \supset H^{(1)} = S^{(1)} \oplus A^{(1)} \supseteq H^{(m)} = S^{(m)} \oplus A^{(m)} \supseteq <0>$ is a series for G which by hypothesis admits replacements. Consequently the series $H = H^{(0)} \supset H^{(1)} \supset \ldots \supset H^{(m)} \supset <0>$ admits replacements.

Conversely, suppose G_p admits replacement for each p. Let

$$G = G^{(0)} = S^{(0)} \oplus A^{(0)} \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset <0>$$

be a series for G with $S^{(0)} \supseteq S^{(1)} \supseteq \ldots \supseteq S^{(n)} \supseteq <0>$ subgroups of G and $A^{(0)} \supseteq A^{(1)} \supseteq \ldots \supseteq A^{(n)} \supseteq \{0\}$ Z-sets. For each p define $G_p^{(i)}$ to be the p-primary component of $G^{(i)}$, $0 \leq i \leq n$, so that $G_p^{(i)} = G_p \cap G^{(i)}$ and $G^{(i)} = \oplus_p G_p^{(i)}$. By Theorem 1 we have that $G_p^{(i)} = (S^{(i)} \cap G_p^{(i)}) \oplus (A^{(i)} \cap G_p^{(i)})$, $0 \leq i \leq n$.

Define $S_p^{(i)} = S^{(i)} \cap G_p^{(i)}$, $A_p^{(i)} = A^{(i)} \cap G_p^{(i)}$, $0 \leq i \leq n$. Clearly

$S_p^{(0)} \supseteq S_p^{(1)} \supseteq \ldots \supseteq S_p^{(n)} \supseteq <0>$ and $A_p^{(0)} \supseteq A_p^{(1)} \supseteq \ldots \supseteq A_p^{(n)} \supseteq \{0\}$. Thus for each p,

$$G_p = G_p^{(0)} = S_p^{(0)} \oplus A_p^{(0)} \supset G_p^{(1)} = S_p^{(1)} \oplus A_p^{(1)} \supset \ldots \supset G_p^{(n)} = S_p^{(n)} \oplus A_p^{(n)} \supset <0>$$

is a series for G_p which by assumption admits replacements. Thus for each p there exist subgroups $T_p^{(i)}$ such that

(i) $T_p = T_p^{(0)} \supseteq T_p^{(1)} \supseteq \ldots \supseteq T_p^{(n)} \supset <0>$

and

(ii) $G_p^{(i)} = S_p^{(i)} \oplus T_p^{(i)}$, $0 \leq i \leq n$.

Define $T^{(i)} = \sum_p T_p^{(i)}$, $0 \leq i \leq n$. Note that this sum is direct. From (i) we have that $T = T^{(0)} \supset T^{(1)} \supset \ldots \supset T^{(n)} \supset <0>$ and (ii) implies that

$$G^{(i)} = \oplus_p G_p^{(i)} = \oplus_p (S_p^{(i)} \oplus T_p^{(i)}) = (\oplus_p S_p^{(i)}) \oplus (\oplus_p T_p^{(i)}) = S^{(i)} \oplus T^{(i)},$$

$0 \leq i \leq n$. This completes the proof.

4. EXTENDABILITY

Let $G = S \oplus A \supset G' = S' \oplus A' = S' \oplus T'$, where $S' \subseteq S$ are subgroups of F, $A' \subseteq A$ are Z-sets, and T' is a subgroup of G'. We say T' is extendable to G if there exists T, a subgroup of G, such that $T' \subseteq T$ and $G = S \oplus T$.

The following theorem provides a necessary and sufficient condition for extendability of a subgroup T' when G is a p-group.

THEOREM 4. Let G be a finite abelian p-group of exponent p^k and let G' be a subgroup of G. Suppose $G = S \oplus A$, $G' = S' \oplus A' = S' \oplus T'$ with $S' \subseteq S$, subgroups of G, $A' \subseteq A$, Z-sets, and T' a subgroup of G'. T' is extendable to G if and only if there exist subgroups, T_i, such that $T' \supseteq T_1 \supseteq T_2 \supseteq \ldots \supseteq T_{k-1} \supseteq <0>$ and $G' \cap p^iG = S' \cap p^iS) \oplus T_i$, $1 \leq i \leq k-1$.

PROOF. By Lemma 1 we have that $G' \cap p^iG = (S' \cap p^iS) \oplus (A' \cap p^iA)$, $1 \le i \le k-1$.

Assume there exists subgroups, T_i; with $T' \supseteq T_1 \supseteq \ldots \supseteq T_{k-1} \supseteq \langle 0 \rangle$ and $G' \cap p^iG = (S' \cap p^iS) \oplus T_i$, $1 \le i \le k-1$. Let $S = \bigoplus_{i=1}^{k} S_i$, where each S_i is homogeneous of exponent p^i, $1 \le i \le k$. By Theorem 2, to prove the existence of a subgroup, T, of G with $T \supseteq T'$ and $G = S \oplus T$ we must verify

$$(p^kG + \bigoplus_{i<k} S_i + T') \cap S_k = (\bigoplus_{i<k} S_i + T') \cap S_k = \langle 0 \rangle \tag{4.1}$$

and

$$(p^jG + \bigoplus_{i<j} S_i + T') \cap S_j = \langle 0 \rangle,\ 1 \le j \le k-1 \tag{4.2}$$

For (4.1), suppose $s_k = \sum_{i<k} s_i + t'$, $s_i \in S_i$, $1 \le i \le k$, $t' \in T'$. Since $T' \subseteq G'$ we have $t' = s' + a'$, $s' \in S'$, $a' \in A'$. Thus $s_k = \sum_{i<k} s_i + s' + a'$ so that $a' = 0$ by the definition of $S \oplus A$. But then $t' \in S' \cap T' = \langle 0 \rangle$. Consequently, $s_k = \sum_{i<k} s_i$. However, since S is a direct sum of $S_1, S_2, \ldots, S_k$, we have $s_k = 0 + 0 + \ldots + 0 + s_k$. Together these imply that $s_k = 0$. Hence (4.1) is true.

Let $1 \le j < k$. Note that $p^jS_i = \langle 0 \rangle$ if $i \le j$. Thus we have $p^jG = \bigoplus_{i>j} p^jS_i \oplus p^jA$.

Suppose

$$s_j = p^jg + \sum_{i<j} s_i + t' = \sum_{i>j} p^js_i + p^ja + \sum_{i<j} s_i + t',$$

where $s_i \in S_i$, $1 \le i \le k$, $a \in A$, $t' \in T'$. Since $T' \subseteq G' = S' \oplus A'$ we have

$$t' = s' + a',\ s' \in S',\ a' \in A' \tag{4.3}$$

Thus $s_j = \sum_{i>j} p^js_i + \sum_{i<j} s_i + s' + p^ja + a'$. Therefore

$$p^ja = -a' \in p^jA \cap A' \subseteq G' \cap p^jG \tag{4.4}$$

$$s_j = \sum_{i>j} p^js_i + \sum_{i>j} s_i + s' \tag{4.5}$$

Since $p^jA \cap A'$ is clearly a Z-set we have from (4.4) that $a' \in p^jA \cap A'$. By hypothesis, $G' \cap p^jG = (S' \cap p^jS) \oplus T_j$ with $T_j \subseteq T'$. Therefore (4.4) implies that $a' = \sum_{i>j} p^j\tilde{s}_i + t_j$, where $\sum_{i>j} p^j\tilde{s}_i \in S' \cap p^jS = S' \cap \bigoplus_{i>j} p^jS_i$ and $t_j \in T_j \subseteq T'$. But then (4.3) becomes

$t' = s' + \sum_{i>j} p^j\tilde{s}_i + t_j$, where $s' + \sum_{i>j} p^j\tilde{s}_i \in S'$ and $t_j \in S'$ and $t_j \in T'$. Consequently

$s' + \sum_{i>j} p^j\tilde{s}_i = 0$ by the definition of $S' \oplus T'$, and we have $s' = -\sum_{i>j} p^j\tilde{s}_i$.

Substituting this expression for s' into (4.5) we obtain

$s_j = \sum_{i>j} p^j s_i + \sum_{i>j} s_i - \sum_{i>j} p^j\tilde{s}_i$ so that $s_j = 0$. Hence (4.2) is true.

Conversely, suppose there exists a subgroup $T \supseteq T'$ such that $G = S \oplus T$. By Lemma 1, $G' \cap p^iG = (S' \cap p^iS) \oplus (T' \cap P^iT)$, $1 \le i \le k-1$. Clearly $T' \supseteq T' \cap pT \supseteq T' \cap p^2T \supset \ldots \supset T' \cap p^{k-1}T$. Thus we can complete the proof by choosing $T_i = T' \cap p^iT$, $1 \le i \le k-1$.

Let us note that if $G = S \oplus A$, where S is a subgroup of G and A is a Z-set, is an elementary abelian p-group, and G' is a subgroup of G such that $G' = S' \oplus A' = S' \oplus T'$, where $S' \subseteq S$, $T' \subseteq G'$, and A' is a Z-set contained in A, then T' is always extendable to G.

LEMMA 5. Let $G = S \oplus A \supseteq G' = S' \oplus A$ be a series for G with $S' \subseteq S$ subgroups of G and A a Z-set. If T is a subgroup of G' such that $G' = S' \oplus T$ then $G = S \oplus T$.

PROOF. Let $\tilde{S}$ be a set of coset representatives for S modulo S'. Then $G = S \oplus A = \tilde{S} \oplus S' \oplus A = \tilde{S} \oplus S' \oplus T = S \oplus T$.

5. SOME GROUPS WHICH ADMIT REPLACEMENT.

We noted in Section 4 that given the series

$$G = G^{(0)} = S^{(0)} \oplus A^{(0)} \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset \langle 0\rangle \tag{5.1}$$

where $S = S^{(0)} \supseteq S^{(1)} \supseteq \ldots \supseteq S^{(n)} \supseteq \langle 0\rangle$ are subgroups and $A = A^{(0)} \supseteq A^{(1)} \supseteq \ldots \supseteq A^{(n)} \supseteq \{0\}$ are Z-sets, one can always find subgroups $T^{(i)}$ such that $G^{(i)} = S^{(i)} \oplus T^{(i)}$, $0 \le i \le n$, although it need not be the case that $T = T^{(0)} \supseteq T^{(1)} \supseteq \ldots \supseteq T^{(n)} \supseteq \langle 0\rangle$. However, by applying the extendability criterion of Theorem 4, we can ensure that for each i, $0 \le i \le n$, our choice of the subgroup $T^{(i)}$ will be extendable to each $G^{(\alpha)}$, $\alpha \le i$, and consequently we will have $T = T^{(0)} \supseteq T^{(1)} \supseteq \ldots \supseteq T^{(n)}$.

We will briefly illustrate how successive applications of Theorem 4 when G is a finite abelian p-group of exponent p^3 results in Figure 1 since this lattice-type structure clarifies the proof of the major theorem in this section. By Lemma 3 we may assume that (5.1) is a composition series for G.

We introduce the following notation to simplify the discussion:

$$G^{(i)}_{j,\ell} = G^{(i)} \cap p^jG^{(\ell)},$$

$$S^{(i)}_{j,\ell} = S^{(i)} \cap p^jS^{(\ell)},$$

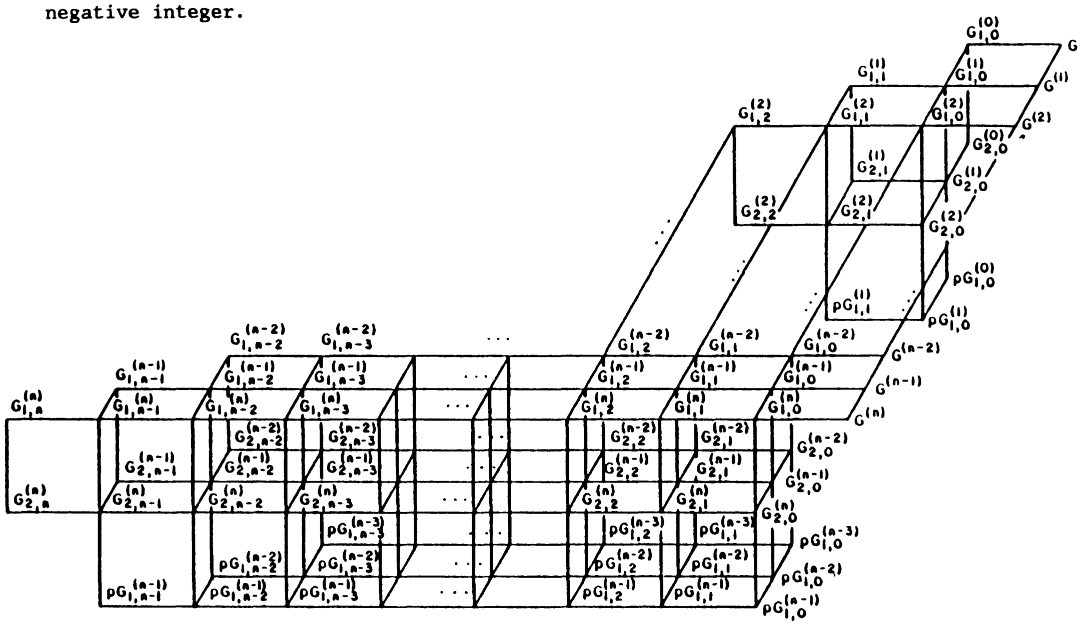
$$A^{(i)}_{j,\ell} = A^{(i)} \cap p^jA^{(\ell)},$$

where $0 \leq i \leq n$, $0 \leq \ell \leq n$, $j = 1,2$. By Lemma 1 we have $G^{(i)}_{j,\ell} = S^{(i)}_{j,\ell} \oplus A^{(i)}_{j,\ell}$, $0 \leq i \leq n$, $0 \leq \ell \leq n$, $j = 1, 2$.

In addition, $T^{(i)}_{j,\ell}$ will denote any subgroup of $G^{(i)}_{j,\ell}$ such that

$G^{(i)}_{j,\ell} = S^{(i)}_{j,\ell} \oplus A^{(i)}_{j,\ell} = S^{(i)}_{j,\ell} \oplus T^{(i)}_{j,\ell}$, and $T^{(i)}_{h,1,\ell}$ will denote any subgroup of $p^hG^{(i)}_{1,\ell}$ such that

$p^hG^{(i)}_{1,\ell} = p^hS^{(i)}_{1,\ell} \oplus p^hA^{(i)}_{1,\ell} = p^hS^{(i)}_{1,\ell} \oplus T^{(i)}_{h,1,\ell}$ $0 \leq i \leq n$, $0 \leq \ell \leq i$, $j = 1, 2$, h a non-negative integer.



Consider a subgroup $T^{(n)}$ such that in the series (5.1) we have $G^{(n)} = S^{(n)} \oplus T^{(n)}$.

By Theorem 4, $T^{(n)}$ will be extendable to $G^{(i)}$, $0 \leq i \leq n-1$, if an only if there exist subgroups $T^{(n)}_{j,1}$ such that $T^{(n)} \supseteq T^{(n)}_{1,i} \supseteq T^{(n)}_{2,i}$ with $G^{(n)}_{j,i} = S^{(n)}_{j,i}$, $0 \leq i \leq n-1$, $j = 1,2$.

We have the following array for the containments of the subgroups $G^{(n)}_{j,i}$, $0 \leq i \leq n$, $j=1,2$:

$$\begin{array}{ccccccccccc} G^{(n)}_{1,n} = pG^{(n)} & \supseteq & G^{(n)}_{1,n-1} & \supseteq & G^{(n)}_{1,n-2} & \supseteq \cdots \supseteq & G^{(n)}_{1,1} & \supseteq & G^{(n)}_{1,0} & \supseteq & G^{(n)} \\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \\ G^{(n)}_{2,n} = p^2G^{(n)} & \supseteq & G^{(n)}_{2,n-1} & \supseteq & G^{(n)}_{2,n-2} & \subseteq \cdots \subseteq & G^{(n)}_{2,1} & \supseteq & G^{(n)}_{2,0} & & \end{array}$$

Thus if we can find subgroups $T^{(n)}_{j,i}$, $0 \leq i \leq n$, $j = 1,2$, $T^{(n)}$, such that

$$\begin{array}{ccccccccccc} T^{(n)}_{1,n} & \subseteq & T^{(n)}_{1,n-1} & \subseteq & T^{(n)}_{1,n-2} & \subseteq \cdots \subseteq & T^{(n)}_{1,1} & \subseteq & T^{(n)}_{1,0} & \subseteq & T^{(n)} \\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \\ T^{(n)}_{2,n} & \subseteq & T^{(n)}_{2,n-1} & \subseteq & T^{(n)}_{2,n-2} & \subseteq \cdots \subseteq & T^{(n)}_{2,1} & \subseteq & T^{(n)}_{2,0} & & \end{array}$$

and $G^{(n)}_{j,i} = S^{(n)}_{j,i} \oplus T^{(n)}_{j,i}$, $0 \leq i \leq n$, $j = 1,2$, $G^{(n)} = S^{(n)} \oplus T^{(n)}$, we would have $T^{(n)}$ extendable to $G^{(i)}$, $0 \leq i \leq n-1$.

Later it will become apparent that we need $T^{(n)}_{1,i}$ extendable to $G^{(n-1)}_{1,i}$, $0 \leq i \leq n-1$. We will show how this can be incorporated in our discussion on $T^{(n)}$.

Using Lemma 4 with $U = G^{(n-1)}$, $U_1 = G^{(n)}$, $K = pG^{(i)}$, $0 \leq i \leq n-1$, we have $\left[G^{(n-1)}_{1,i} : G^{(n)}_{1,i}\right] \leq p$. Thus $pG^{(n-1)}_{1,i} \subseteq G^{(n)}_{1,i}$ so that $G^{(n)}_{1,i} \cap pG^{(n-1)}_{1,i} = pG^{(n-1)}_{1,i}$, $0 \leq i \leq n-1$. Note that $G^{(n-1)}_{1,i}$ has exponent at most p^2. By Theorem 4, $T^{(n)}_{1,i}$ is extendable to $G^{(n-1)}_{1,i}$, $0 \leq i \leq n-1$, if and only if there exists a subgroup $T^{(n-1)}_{1,1,i}$ such that $T^{(n)}_{1,i} \supseteq T^{(n-1)}_{1,1,i}$ and $pG^{(n-1)}_{1,i} = pS^{(n-1)}_{1,i} \oplus T^{(n-1)}_{1,1,i}$, $0 \leq i \leq n-1$. Since $pG^{(n-1)}_{1,i} = p(G^{(n-1)} \cap pG^{(i)})$ $\subseteq pG^{(n-1)} \cap p^2G^{(i)} \subseteq G^{(n)} \cap p^2G^{(i)} = G^{(n)}_{2,i}$, $0 \leq i \leq n-1$, we have the following array for the subgroups $G^{(n)}$, $G^{(n)}_{j,i}$, $pG^{(n-1)}_{1,i}$, $0 \leq i \leq n-1$, $j = 1,2$:

$$\begin{array}{ccccccccccc}
G^{(n)}_{1,n} & \subseteq & G^{(n)}_{1,n-1} & \subseteq & G^{(n)}_{1,n-2} & \subseteq \cdots \subseteq & G^{(n)}_{1,1} & \subseteq & G^{(n)}_{1,0} & \subseteq & G^{(n)} \\
\cup\!| & & \cup\!| & & \cup\!| & & \cup\!| & & \cup\!| & & \\
G^{(n)}_{2,n} & \subseteq & G^{(n)}_{2,n-1} & \subseteq & G^{(n)}_{2,n-2} & \subseteq \cdots \subseteq & G^{(n)}_{2,1} & \subseteq & G^{(n)}_{2,0} & & \\
 & & \cup\!| & & \cup\!| & & \cup\!| & & \cup\!| & & \\
 & & pG^{(n-1)}_{1,n-1} & \subseteq & pG^{(n-1)}_{1,n-2} & \subseteq \cdots \subseteq & pG^{(n-1)}_{1,1} & \subseteq & pG^{(n-1)}_{1,0} & &
\end{array}$$

Thus if we can find subgroups $T^{(n)}_{j,i}$, $T^{(n-1)}_{1,1,i'}$, $0 \leq i \leq n$, $0 \leq i' \leq n-1$, $j = 1,2$, $T^{(n)}$ such that

$$\begin{array}{ccccccccccc}
T^{(n)}_{1,n} & \subseteq & T^{(n)}_{1,n-1} & \subseteq & T^{(n)}_{1,n-2} & \subseteq \cdots \subseteq & T^{(n)}_{1,1} & \subseteq & T^{(n)}_{1,0} & \subseteq & T^{(n)} \\
\cup\!| & & \cup\!| & & \cup\!| & & \cup\!| & & \cup\!| & & \\
T^{(n)}_{2,n} & \subseteq & T^{(n)}_{2,n-1} & \subseteq & T^{(n)}_{2,n-2} & \subseteq \cdots \subseteq & T^{(n)}_{2,1} & \subseteq & T^{(n)}_{2,0} & & \\
 & & \cup\!| & & \cup\!| & & \cup\!| & & \cup\!| & & \\
 & & T^{(n-1)}_{1,1,n-1} & \subseteq & T^{(n-1)}_{1,1,n-2} & \subseteq \cdots \subseteq & T^{(n-1)}_{1,1,1} & \subseteq & T^{(n-1)}_{1,1,0} & &
\end{array}$$

with $G^{(n)}_{j,i} = S^{(n)}_{j,i} \oplus T^{(n)}_{j,i}$, $pG^{(n-1)}_{1,i'} = pS^{(n-1)}_{1,i'} \oplus T^{(n-1)}_{1,1,i'}$, $0 \leq i \leq n$, $0 \leq i' \leq n-1$, $j = 1,2$ $G^{(n)} = S^{(n)} \oplus T^{(n)}$, we would have $T^{(n)}_{1,i}$ extendable to $G^{(n-1)}_{1,i}$, $0 \leq i \leq n-1$, and $T^{(n)}$ extendable to $G^{(i)}$, $0 \leq i \leq n-1$. In particular, we would know there exists a subgroup $T^{(n-1)}$ with $T^{(n-1)} \supseteq T^{(n)}$ and $G^{(n-1)} = S^{(n-1)} \oplus T^{(n-1)}$. However, we must ensure that our choice for $T^{(n-1)}$ is extendable to $G^{(i)}$, $0 \leq i \leq n-2$. Appyling the previous argument to $T^{(n-1)}$ and then to $T^{(i)}$, $0 \leq i \leq n-3$, we obtain Figure 1. We remark that lattice-type structures similar to Figure 1 can be obtained for finite abelian p-groups of exponent p^k, where k is any non-negative integer. Such structures become rather complicated when the exponent of the group exceeds p^3.

The following definitions will facilitate references to Figure 1.

DEFINITION 1. The row for $G^{(i)}$, $0 \leq i \leq n$, is the series

$$G^{(i)} \supseteq G^{(i)}_{1,0} \supseteq G^{(i)}_{1,1} \supseteq \cdots \supseteq G^{(i)}_{1,i} .$$

DEFINITION 2. The row for $G^{(i)}_{2,0}$, $0 \leq i \leq n$, is the series

$$G^{(i)}_{2,0} \supseteq G^{(i)}_{2,1} \supseteq G^{(i)}_{2,2} \supseteq \cdots \supseteq G^{(i)}_{2,i} .$$

DEFINITION 3. The row for $pG^{(i)}_{1,0}$, $0 \leq i \leq n-1$, is the series

$$pG^{(i)}_{1,0} \supseteq pG^{(i)}_{1,1} \supseteq pG^{(i)}_{1,2} \supseteq \cdots \supseteq pG^{(i)}_{1,i} .$$

DEFINITION 4. The sub-figure for $G^{(i)}$, $0 \leq i \leq n$, consists of the rows for $G^{(\ell)}$, $G^{(\ell)}_{2,0}$, and $pG^{(\ell)}_{1,0}$, $i \leq \ell \leq n$, $i-1 \leq \ell' \leq n-1$.

DEFINITION 5. We say the sub-figure for $G^{(i)} = S^{(i)} \oplus A^{(i)}$, $0 \leq i \leq n$, is complete if

(i) for every subgroup H in the sub-figure for $G^{(i)}$, $H = S_H \oplus A_H$, S_H a subgroup of $S^{(i)}$, A_H a Z-set, $A_H \subseteq A^{(i)}$, there exists a subgroup T_H such that $H = S_H \oplus T_H$,

(ii) for all sub-groups H, K in the sub-figure for $G^{(i)}$ with $H \subseteq K$ we have $T_H \subseteq T_K$.

Let us note that, by the construction of Figure 1, if the sub-figure for $G^{(i)}$, $1 \leq i \leq n$, is complete then $T^{(i)}$ is extendable to $G^{(i-1)}$ and $T^{(i)}_{1,\ell}$ is extendable to $G^{(i-1)}_{1,\ell}$, $0 \leq \ell \leq i$.

DEFINITION 6. The row for $G^{(i)}$, $1 \leq i \leq n$, is complete if there exist subgroups $T^{(i)}$, $T^{(i)}_{1,\ell}$, $0 \leq \ell \leq i$, such that

(i) $T^{(i)} \supseteq T^{(i)}_{1,0} \supseteq T^{(i)}_{1,1} \supseteq \cdots \supseteq T^{(i)}_{1,i} \supseteq \langle 0 \rangle$,

(ii) $G^{(i)} = S^{(i)} \oplus T^{(i)}$, $G^{(i)}_{1,\ell} = S^{(i)}_{1,\ell} \oplus T^{(i)}_{1,\ell}$, $0 \leq \ell \leq i$,

(iii) $T^{(i)}$ is extendable to $G^{(i-1)}$ and $T^{(i)}_{1,\ell}$ is extendable to $G^{(i-1)}_{1,\ell}$, $0 \leq \ell \leq i$.

We will say that the sub-figure (row) for $G^{(i)}$ can be completed if we can prove the existance of the subgroups T_H ($T^{(i)}$, $T^{(i)}_{1,\ell}$ discussed in the definition for "The sub-figure (row) for $G^{(i)}$ is complete."

PROPOSITION 1. Let G be a finite abelian p-group and let $G = G^{(0)} = S \oplus A \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \dots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset \langle 0\rangle$ be a composition series for G with $S = S^{(0)} \supseteq S^{(1)} \supseteq \dots \supseteq S^{(n)} \supseteq \langle 0\rangle$ subgroups of G and $A = A^{(0)} \supseteq A^{(1)} \supseteq \dots \supseteq A^{(n)} \supseteq \{0\}$ Z-sets. The following statements are true for $0 \le i \le n-1$, $0 \le i' \le n$:

(a) $\left[G^{(i)}_{1,i} : G^{(i+1)}_{1,i+1}\right] \le p$

(b) $\left[G^{(i)}_{1,\ell} : G^{(i+1)}_{1,\ell}\right] \le p$, $0 \le \ell \le i+1$

(c) $\left[G^{(i)}_{1,\ell'+1} : G^{(i+1)}_{1,\ell'+1}\right] \le \left[G^{(i)}_{1,\ell'} : G^{(i+1)}_{1,\ell}\right]$, $0 \le \ell' \le i$

(d) $\left[G^{(i')}_{1,\ell} : G^{(i')}_{1,\ell'+1}\right] \le p$, $0 \le \ell' \le i'$.

(e) $\left[G^{(i+1)}_{1,\ell'} : G^{(i+1)}_{1,\ell'+1}\right] \le \left[G^{(i)}_{1,\ell'} : G^{(i)}_{1,\ell'+1}\right]$, $0 \le \ell' \le i$.

(f) $\left[G^{(i)}_{2,\ell} : G^{(i+1)}_{2,\ell}\right] \le \left[G^{(i)}_{1,\ell} : G^{(i+1)}_{1,\ell}\right]$, $0 \le \ell \le i+1$

PROOF. Let $G^{(i)} = \bigoplus_{k=1}^{r} \langle g_k\rangle$. Since $\left[G^{(i)} : G^{(i+)}\right] = p$ we have $G^{(i+1)} = \langle g_1\rangle \oplus \langle g_2\rangle \oplus \dots \oplus \langle pg_j\rangle \oplus \dots \oplus \langle g_k\rangle$ for some j, $1 \le j \le r$.

Then $G^{(i)}_{1,i} = pG^{(i)} = \bigoplus_{k=1}^{r} \langle pg_k\rangle$ and $G^{(i+1)}_{1,i+1} = pG^{(i+1)} = \langle pg_1\rangle \oplus \langle pg_2\rangle \oplus \dots \oplus \langle p^2 g_j \oplus \dots \oplus \langle pg_k\rangle$.

If $o(g_j) = p$, then $G^{(i)}_{1,i} = G^{(i+1)}_{1,i+1}$. If $o(g_j) > p$, then $\left[G^{(i)}_{1,i} : G^{(i+1)}_{1,i+1}\right] = p$.

This proves (a).

Each of properties (b) through (f) can be deduced from Lemma 4 by choosing U, U_1, and K appropriately as follows:

(b) $U = G^{(i)}$, $U_1 = G^{(i+1)}$, $K = G^{(\ell)}_{1,\ell}$, $0 \le \ell \le i+1$.

(c) $U = G^{(i)}_{1,\ell'}$, $U_1 = G^{(i+1)}_{1,\ell'}$, $K = G^{(\ell'+1)}_{1,\ell'+1}$, $0 \le \ell' \le i$.

(d) $U = G^{(\ell')}_{1,\ell'}$, $U_1 = G^{(\ell'+1)}_{1,\ell'+1}$, $K = G^{(i')}$, $0 \le \ell \le i'$.

(e) $U = G^{(i)}_{1,\ell'}$, $U_1 = G^{(i)}_{1,\ell'+1}$, $K = G^{(i+1)}$, $0 \le \ell' \le i$.

(f) $U = G^{(i)}_{1,\ell}$, $U_1 = G^{(i+1)}_{1,\ell}$, $K = G^{(\ell)}_{2,\ell}$, $0 \le \ell \le i+1$.

Observe that $\left[G^{(i)}: G^{(i+1)}\right] = p$ implies that $pG^{(i)} \subseteq G^{(i+1)}$ and $p^2G^{(i)} \subseteq pG^{(i+1)} \subseteq G^{(i+2)}$. Thus $\left[G^{(i)}_{1,i}: G^{(i+1)}_{1,i}\right] = 1$, $0 \le i \le n-1$, $\left[G^{(i)}_{2,i}: G^{(i+1)}_{2,i}\right] = 1$, $0 \le i \le n-1$, and $\left[G^{(i+1)}_{2,i}: G^{(i+2)}_{2,i}\right] = 1$, $0 \le i \le n-2$.

PROPOSITION 2. Let G be a finite abelian p-group and let $G = G^{(0)} = S \oplus A \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \dots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset \langle 0\rangle$ be a composition series for G with $S = S^{(0)} \supseteq S^{(1)} \supseteq \dots \supseteq S^{(n)} \supset \langle 0\rangle$ subgroups of G and $A = A^{(0)} \supseteq A^{(1)} \supseteq \dots \supseteq A^{(n)} \supseteq \{0\}$ Z-sets. The following statements are true for $0 \le i \le n-1$, $0 \le \ell \le i$, $0 \le \ell' \le i-1$.

(a) If $\left[G^{(i)}_{1,\ell} : G^{(i+1)}_{1,\ell+1}\right] = 1$ and $\left[G^{(i)}_{1,\ell}: G^{(i+1)}_{1,\ell}\right] = p$, then $\left[G^{(i)}_{2,\ell} : G^{(i+1)}_{2,\ell}\right] = 1$.

(b) If $\left[G^{(i)}_{1,\ell'+1}: G^{(i+1)}_{1,\ell'+1}\right] = \left[G^{(i-1)}_{1,\ell'+1}: G^{(i)}_{1,\ell'+1}\right] = 1$ and $\left[G^{(i)}_{1,\ell'}: G^{(i+1)}_{1,\ell'}\right] = p$, then $\left[G^{(i-1)}_{1,\ell'}: G^{(i)}_{1,\ell'}\right] = 1$.

PROOF. We have $G^{(i)}_{2,\ell} \subseteq G^{(i)}_{1,\ell+1} = G^{(i)}_{1,\ell+1} \subseteq G^{(i+1)}$. Therefore $G^{(i)}_{2,\ell} = G^{(i)}_{2,\ell} \cap G^{(i+1)} = G^{(i+1)}_{2,\ell}$. Hence (a) is true. $\left[G^{(i)}_{1,\ell'+1}: G^{(i+1)}_{1,\ell'+1}\right] = 1$ and $\left[G^{(i)}_{1,\ell'}: G^{(i+1)}_{1,\ell'}\right] = 1$ imply that $\left[G^{(i)}_{1,\ell'}: G^{(i)}_{1,\ell'+1}\right] = p$ and $\left[G^{(i+1)}_{1,\ell'}: G^{(i+1)}_{1,\ell'+1}\right] = 1$. By (d) and (e) of Proposition 1 we have that $\left[G^{(i-1)}_{1,\ell'}: G^{(i+1)}_{1,\ell'+1}\right] = p$. Consequently $\left[G^{(i-1)}_{1,\ell'}: G^{(i)}_{1,\ell'}\right] = 1$.

We can eliminate from consideration several combinations of indices in Figure 1 since by Proposition 2 it is impossible for them to occur.

LEMMA 6. Let W, X, Y and Z be subgroups with the following properties:

(i) $W \subseteq X$, $Y \subseteq X$, $Z = W \cap Y$

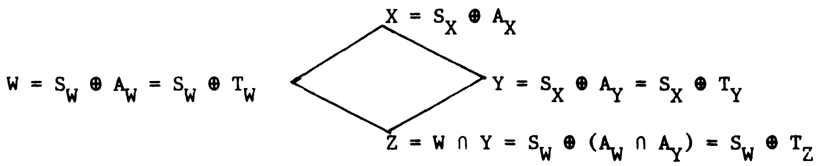
(ii) $[W:Z] = [X:Y] = p$

(iii) $X = S_X \oplus A_X$, $W = S_W \oplus A_W = S_W \oplus T_W$, $Y = S_X \oplus A_Y = S_X \oplus T_Y$, where S_W, S_X, T_W, T_Y are subgroups with $S_W \subset S_X$ and A_W, A_Y, A_X are Z-sets with $A_W \subseteq A_X$, $A_Y \subseteq A_X$.

(iv) $Z = S_W \oplus T_Z$ with $T_Z \subseteq T_W$ and $T_Z \subseteq T_Y$.

Then $X = S_X \oplus T_X$, where $T_X = T_W + T_Y$.

PROOF. By Lemma 1 we have $Z = W \cap Y = (S_W \cap S_X) \oplus (A_W \cap A_Y) = S_W \oplus (A_W \cap A_Y)$. The following diagram illustrates the relations between the subgroups W, X, Y, and Z.



We will first show that $X = W + Y$. We have $Y \subseteq W+Y \subseteq X$ and $[W:Y] = p$. Since $[W:Z] = p$ we must have $X = W+Y = S_X + (T_W + T_Y)$.

We will complete the proof by showing that $S_X \cap (T_W + T_Y) = \langle 0\rangle$.

Let

$$s_X = t_W + t_Y,\ s_X \in S_X,\ t_W \in T_W,\ t_Y \in T_Y \tag{5.2}$$

We can write $t_W = s_W + a_W$, $t_Y = s'_X + a_Y$, $s_W \in S_W$, $s'_X \in S_X$, $a_W \in A_W$, $a_Y \in A_Y$. Thus (5.2) becomes $s_X = s_W + a_W + s'_X + a_Y$ and we have $s_X - s'_X = s_W$ and $-a_W = a_Y \in A_W \cap A_Y \subseteq Z$. Consequently we can write $a_W = s'_W + t_Z$, $s'_W \in S_W$, $t_Z \in T_Z$. But then $t_W = s_W + s'_W + t_Z$. Since $T_Z \subseteq T_W$ and $T_W \cap S_W = \langle 0\rangle$ we must have $s_W + s'_W = 0$. Similarly, $t_Y = s'_X - s'_W - t_Z$ so that $s'_X - s'_W = 0$. Hence $s_X = s_W + s'_W - s'_W - s'_X = 0$.

THEOREM 5. Let G be a finite abelian p-group of exponent p^k, $k \geq 1$, and let

$$G = G^{(0)} = S \oplus A \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset \langle 0\rangle \tag{5.3}$$

be a series for G with $S = S^{(0)} \supseteq S^{(1)} \supseteq \ldots \supseteq S^{(n)} \supseteq \langle 0\rangle$ subgroups of G and $A = A^{(0)} \supseteq A^{(1)} \supseteq \ldots \supseteq A^{(n)} \supseteq \{0\}$ Z-sets. If $p^2A = \{0\}$ then the series (5.3) admits replacements.

PROOF. By Lemma 3 we can assume that (5.3) is a composition series for G. Note that for $0 \leq i \leq n$, $0 \leq \ell \leq i$, $j \geq 2$, $h \geq 1$,

$$A^{(i)}_{j,\ell} = A^{(i)} \cap p^jA^{(\ell)} \subseteq p^2A = \{0\},$$

and

$$p^hA^{(i)}_{1,\ell} \subseteq p^{h+1}A^{(\ell)} \subseteq p^2A = \{0\}.$$

Thus we have $G^{(i)}_{j,\ell} = S^{(i)}_{j,\ell}$ and $p^hG^{(i)}_{1,\ell} = p^hS^{(i)}_{1,\ell}$, $0 \leq i \leq n$, $0 \leq \ell \leq i$, $j \geq 2$, $h \geq 1$. Consequently we must have $T^{(i)}_{j,\ell} = T^{(i)}_{h,1,\ell} = \langle 0\rangle$, $0 \leq i \leq n$, $0 \leq \ell \leq i$, $j \geq 2$, $h \geq 1$.

We use a "backward induction" on i to show that the sub-figure for each $G^{(i)}$, $0 \leq i \leq n$, can be completed.

For $i = n$, $|G^{(n)}| = p$ implies that $G^{(n)}$ is cyclic of forder p. Thus the sub-figure for $G^{(n)}$ is trivially complete.

Now assume the sub-figure for $G^{(i+1)}$ is complete. In view of the preceeding comments on the subgroups $T^{(i)}_{j,\ell}$, $T^{(i)}_{h,1,\ell}$, $0 \leq \ell \leq i$, $j \geq 2$, $h \geq 1$, if we can complete the row for $G^{(i)}$ in such a way that $T^{(i+1)}_{1,\ell} \subseteq T^{(i)}_{1,\ell}$, $0 \leq \ell \leq i$, $T^{(i+1)} \subseteq T^{(i)}$, then the sub-figure for $G^{(i)}$ will be complete as well.

By Lemma 3 we must consider two cases, (1) $A^{(i)} = A^{(i+1)}$, and (2) $S^{(i)} = S^{(i+1)}$.

Case (1). By hypothesis, the sub-figure for $G^{(i+1)}$ is complete so that there exist subgroups $T^{(i+1)} \supseteq T^{(i+1)}_{1,0} \supseteq T^{(i+1)}_{1,1} \supseteq \ldots \supseteq T^{(i+1)}_{1,i+1}$ such that

$G^{(i+1)} = S^{(i+1)} \oplus T^{(i+1)}$, $G^{(i+1)}_{1,\ell} = S^{(i+1)}_{1,\ell} \oplus T^{(i+1)}_{1,\ell}$, $0 \leq \ell \leq i+1$, and $T^{(i+1)}$ is extendable to $G^{(i)}$, $T^{(i+1)}_{1,\ell}$ is extendable to $G^{(i)}_{1,\ell}$, $0 \leq \ell \leq i$. Setting $H = G^{(i)}$, $K = pG^{(\ell)}$, $0 \leq \ell \leq i$, in Lemma 1, we conclude that $A^{(i)}_{1,\ell} = A^{(i+1)}_{1,\ell}$, $0 \leq \ell \leq i$. Applying Lemma 5 with $G = G^{(i)}_{1,\ell}$, $G' = G^{(i+1)}_{1,\ell}$, $T = T^{(i+1)}_{1,\ell}$, $0 \leq \ell \leq i$, we have $G^{(i)}_{1,\ell} = S^{(i)}_{1,\ell} \oplus T^{(i+1)}_{1,\ell}$, $0 \leq \ell \leq i$. Again applying Lemma 5, this time with $G = G^{(i)}$, $G' = G^{(i+1)}$, $T = T^{(i+1)}$, we have $G^{(i)} = S^{(i)} \oplus T^{(i+1)}$. Thus we can complete the sub-figure for $G^{(i)}$ by choosing $T^{(i)}_{1,\ell} = T^{(i+1)}_{1,\ell}$, $0 \leq \ell \leq i$, $T^{(i)} = T^{(i+1)}$.

Case (2). We have $\left[G^{(i)}_{1,\ell} : G^{(i+1)}_{1,\ell}\right] \leq p$, $0 \leq \ell \leq i$, by (b) of Proposition 1. If $\left[G^{(i)}_{1,\ell} : G^{(i+1)}_{1,\ell}\right] = 1$, $0 \leq \ell \leq i$, we can complete the sub-figure for $G^{(i)}$ by choosing $T^{(i)}_{1,\ell} = T^{(i+1)}_{1,\ell}$, $0 \leq \ell \leq i$, and extending $T^{(i+1)}$ to $G^{(i)}$. ($T^{(i+1)}$ is extendable to $G^{(i)}$ by the hypothesis that the sub-figure for $G^{(i+1)}$ is complete).

Now suppose there exists ℓ_o such that

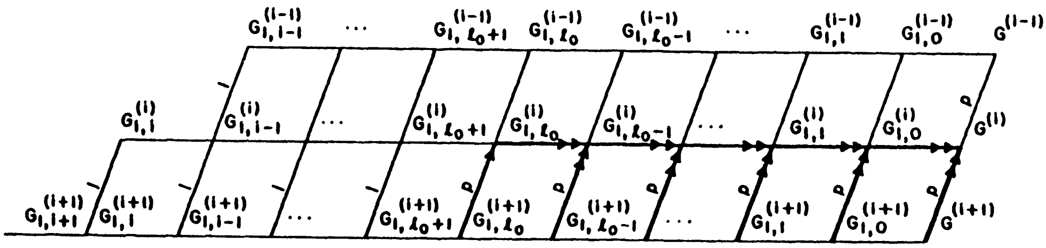
$$\left[G^{(i)}_{1,\ell} : G^{(i+1)}_{1,\ell}\right] = \begin{cases} 1 \text{ for } \ell_o < \ell \leq i \\ p \text{ for } 0 \leq \ell \leq \ell_o \end{cases}$$

This situation is illustrated in Figure 2, where, for simplicity, we have omitted the subgroups $G^{(r)}_{j,\ell}$ and $G^{(r)}_{h,1,\ell}$ since $T^{(r)}_{j,\ell} = T^{(r)}_{h,1,\ell} = \langle 0 \rangle$, $0 \leq \ell \leq n$, $1 \leq h \leq k-2$, $2 \leq j \leq k-1$, $r = i+1$, i, $i-1$. The numbers 1 and p in Figure 2 represent indices.

We can choose $T^{(i)}_{1,\ell} = T^{(i+1)}_{1,\ell}$ for $\ell_o < \ell \leq i$. As remarked previously, the hypothesis that the sub-figure for $G^{(i+1)}$ is complete implies that $T^{(i+1)}_{1,\ell_o}$ can be extended to $G^{(i)}_{1,\ell_o}$. This extension is indicated in Figure 2 by a single arrow.

Setting $X = G^{(i)}_{1,\ell}$, $Y = G^{(i+1)}_{1,\ell}$, $W = G^{(i)}_{1,\ell+1}$, $0 \leq \ell \leq \ell_o - 1$, we have $Z = W \cap Y = G^{(i+1)}_{1,\ell+1}$, $0 \leq \ell \leq \ell_o - 1$, so that W,X,Y,Z satisfy the conditions of Lemma 6.

Thus we can apply Lemma 6 to obtain $G_{1,\ell}^{(i)} = S_{1,\ell}^{(i+1)} \oplus T_{1,\ell}^{(i)}$, where $T_{1,\ell}^{(i)} = T_{1,\ell+1}^{(i)} + T_{1,\ell}^{(i+1)}$, $0 \leq \ell \leq \ell_o - 1$. We can again apply Lemma 6, taking $X = G^{(i)}$, $W = G_{1,0}^{(i)}$, $Y = G^{(i+1)}$, $Z = W \cap Y = G_{1,0}^{(i+1)}$, to obtain $G^{(i)} = S^{(i+1)} \oplus T^{(i)}$, where $T^{(i)} = T_{1,0}^{(i)} + T^{(i+1)}$. These sums are indicated in Figure 2 by double arrows. Clearly, $T^{(i)} \supseteq T_{1,0}^{(i)} \supseteq \ldots \supseteq T_{1,\ell}^{(i)}$ and $T_{1,\ell}^{(i+1)} \subset T_{1,\ell}^{(i)}$, $0 \leq \ell \leq i$. Thus the sub-figure for $G^{(i)}$ is complete.



COROLLARY 1. If G is a finite abelian p-group of exponent less than or equal to p^2 then G admits replacement.

PROOF. Let $G = G^{(0)} = S \oplus A \supset G^{(1)} = S^{(1)} \oplus A^{(1)} \supset \ldots \supset G^{(n)} = S^{(n)} \oplus A^{(n)} \supset \langle 0 \rangle$ (5.4) be a series for G with $S = S^{(0)} \supseteq S^{(1)} \supseteq \ldots \supseteq S^{(n)} \supseteq \langle 0 \rangle$ subgroups of G and $A = A^{(0)} \supseteq A^{(1)} \supseteq \ldots \supseteq A^{(n)} \supseteq \{0\}$ Z-sets. We have $p^2A = \{0\}$ since by hypothesis $p^2G = \langle 0 \rangle$. By Theorem 5 the series (5.4) admits replacements. Hence G admits replacement.

6. RELATION TO A VARIATION OF A METHOD OF A.D. SANDS.

Our terminology will be the same as in [3] when referring to factorizations which are obtained by the variation of Sands' method.

The following Proposition can be readily verified.

PROPOSITION 3. Let $G = K_1 \supset K_2 \supset \ldots \supset K_n \supset \langle 0 \rangle$ be a series for G with coset representatives H_i, $1 \leq i \leq n$, $K_n = H_n$. If $H_i \oplus H_{i+2} \oplus \ldots$ is a subgroup (Z-set) then $H_{i+2} \oplus H_{i+4} \oplus \ldots$ is also a subgroup (Z-set).

THEOREM 6. Let G be a finite abelian group which admits replacement and let

$$G = K_1 \supset K_2 \supset \ldots \supset K_n \supset \langle 0 \rangle \tag{6.1}$$

be a series for G. If $G = A \oplus B$ is a Z-factorization of G arising from (6.1) then there exist subgroups S, T such that the factorization $G = S \oplus T$ arises from the series (6.1).

PROOF. We will assume that n is odd since the proof for n even is similar. We will proceed by induction on the order of the group G.

If $|G|=p$, then $G = G \oplus \langle 0 \rangle$ is the only Z-factorization of G. Thus in any series from which this factorization arises we must have $G = K_1 = S$, $T = K_i = \langle 0 \rangle$, $i \geq 2$.

Assume the theorem is true for groups of order less than G. Let $G = A \oplus B$ be a Z-factorization of G arising from (6.1). By Lemma 5 of [3] we may assume

$$A = H_1 \oplus H_3 \oplus H_5 \oplus \dots \oplus H_n,$$

$$B = H_2 \oplus H_4 \oplus H_6 \oplus \dots \oplus H_{n-1},$$

where $0 \in H_i$, $1 \leq i \leq n$.

We have the $H_3 + H_5 + \dots + H_n$ is a Z-set by Proposition 3.
Thus

$$K_2 = (H_3 \oplus H_5 \oplus \dots \oplus H_n) \oplus (H_2 \oplus H_4 \oplus \dots \oplus H_{n-1})$$

is a Z-factorization of K_2 arising from the series

$$K_2 \supset K_3 \supset \dots \supset K_n \supset \langle 0 \rangle \tag{6.2}$$

$|K_2| < |G|$ implies, by the induction hypothesis, that there exist subgroups, S', T' such that the factorization $K_2 = S' \oplus T'$ arises from the series (6.2). Thus, by Lemma 5 [3] there exist transversals H_i', $2 \leq i \leq n$, such that

$$S' = H_2' \oplus H_4' \oplus \dots \oplus H_{n-1}',$$

$$T' = H_3' \oplus H_5' \oplus \dots \oplus H_n',$$

where $0 \in H_i'$, $2 \leq i \leq n$.

Note that $K_n = H_n = H_n'$ and

$$K_i = H_i' \oplus K_{i+1} = H_i \oplus K_{i+1},\ 2 \leq i \leq n-1 \tag{6.3}$$

since both H_i and H_i' are coset representatives for K_i modulo K_{i+1}, $2 \leq i \leq n$. Using (6.3) successively, starting with $i = n-1$, we see that we can choose H_1, H_3, H_5, ..., H_n, H_2', H_4', H_6', ..., H_{n-1}' as coset representatives for the series (6.1) to obtain the factorization

$$G = (H_1 \oplus H_3 \oplus \dots \oplus H_n) \oplus (H_2' \oplus H_4' \oplus \dots \oplus H_{n-1}') = A \oplus S' \tag{6.4}$$

By Proposition 3, $H_i \oplus H_{i+2} \oplus \dots \oplus H_n$ is a Z-set, $i = 1,3,\dots n$, and $H_i' \oplus H_{i+2}' \oplus \dots \oplus H_{n-1}'$ is a subgroup, $i = 2,4,\dots, n-1$. Set

$$S^{(i)} = H_i' \oplus H_{i+2}' \oplus \dots \oplus H_{n-1}',\ i = 2,4,\dots, n-3$$

$$S^{(n-1)} = H_{n-1}'$$

$$A^{(i)} = H_i \oplus H_{i+2} \oplus \dots \oplus H_n,\ i = 1,3,\dots, n-2$$

$$A^{(n)} = H_n = K_n.$$

Then the series (6.1) can be written as

$$G=K_1=S^{(2)} \oplus A^{(1)} \supset K_2=S^{(2)}\oplus A^{(3)}\supset K_3=S^{(4)} \oplus A^{(3)} \supset\ldots\supset K_{n-1}=S^{(n-1)}\oplus A^{(n)}\supset K_n=A^{(n)}\supset <0>$$

where $S^{(2)} = S'$ and $A^{(1)} = A$. In general,

$$K_i = \begin{cases} S^{(i)} \oplus A^{(i+1)} \text{ for i even, } 2 \le i \le n-1 \\ S^{(i+1)} \oplus A^{(i)} \text{ for i odd, } 1 \le i \le n-2, \quad K_n = A^{(n)} = H_n. \end{cases}$$

By hypothesis G admits replacement. Thus there exist subgroups $T^{(1)} \supseteq T^{(2)}\supseteq\ldots\supseteq T^{(n)}$ such that

$$K_i = \begin{cases} S^{(i)} \oplus T^{(i)} \text{ for i even, } 2 \le i \le n-1, \\ S^{(i+1)} \oplus T^{(i)} \text{ for i odd, } 1 \le i \le n-2, \ K_n = T^{(n)} = H_n = A^{(n)}. \end{cases}$$

Define $H''_n = T^{(n)} = A^{(n)}$. We have

$$K_{n-1} = S^{(n-1)} \oplus T^{(n-1)}= S^{(n-1)} \oplus A^{(n)} = S^{(n-1)} \oplus T^{(n)}$$

so that $|T^{(n-1)}| = |T^{(n)}|$. But $T^{(n)} \subseteq T^{(n-1)}$. Therefore $T^{(n-1)} = T^{(n)}$.

Next,

$$K_{n-2} = S^{(n-1)} \oplus T^{(n-2)}, \quad T^{(n-1)} \subseteq T^{(n-2)}.$$

If we choose H''_{n-2} a set of coset representatives for $T^{(n-2)}$ modulo $T^{(n-1)}$ we have $T^{(n-2)} = H''_{n-2} \oplus T^{(n-1)} = H''_{n-2} \oplus T^{(n)} = H''_{n-2} + H''_n$, and $K_{n-2} = S^{(n-1)} \oplus H''_{n-2} \oplus T^{(n)}$

$= H''_{n-2} \oplus K_{n-1}$. Thus H''_{n-2} is also a set of coset representatives for K_{n-2} modulo K_{n-1}.

$$K_{n-3} = S^{(n-3)} \oplus T^{(n-3)} = S^{(n-3)} \oplus A^{(n-2)} \text{ and}$$

$K_{n-2} = S^{(n-1)} \oplus T^{(n-2)} = S^{(n-1)} \oplus A^{(n-2)}$ imply that $|T^{(n-3)}| = |A^{(n-2)}| = |T^{(n-2)}|$.
But $T^{(n-2)} \subset T^{(n-3)}$. Hence we have that $T^{(n-2)} = T^{(n-3)}$ and

$K_{n-3} = S^{(n-3)} \oplus T^{(n-2)} = (H'_{n-3} \oplus H'_{n-1}) \oplus T^{(n-2)} = H'_{n-3} \oplus K_{n-2}$.

In general, given $T^{(i)}$ for i odd we have $T^{(i-1)} = T^{(i)}$ and $T^{(i-2)} = H''_{i-2} \oplus T^{(i)}$ so that the factorizations of K_i, $1 \le i \le n$, are as follows:

$K_n = T^{(n)} = H''_n = H_n$

$K_{n-1} = S^{(n-1)} \oplus T^{(n)} = H'_{n-1} \oplus H''_n$

$K_{n-2} = S^{(n-1)} \oplus T^{(n-2)} = H'_{n-1} \oplus H''_{n-2} \oplus T^{(n)} = H'_{n-1} \oplus (H''_{n-2} \oplus H''_n)$

$K_{n-3} = S^{(n-3)} \oplus T^{(n-2)} = (H'_{n-3} \oplus H'_{n-1}) \oplus (H''_{n-2} \oplus H''_{n})$

$K_{n-4} = S^{(n-3)} \oplus T^{(n-4)} = S^{(n-3)} \oplus H''_{n-4} \oplus T^{(n-2)} = (H'_{n-3} \oplus H'_{n-1}) + (H''_{n-4} \oplus H''_{n-2} \oplus H''_{n})$

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$K_3 = S^{(4)} \oplus T^{(3)} = S^{(4)} \oplus H''_3 \oplus T^{(5)} = (H'_4 \oplus H'_6 \oplus \ldots \oplus H'_{n-1}) + (H''_3 \oplus H''_5 \oplus \ldots \oplus H''_n)$

$K_2 = S^{(2)} \oplus T^{(3)} = (H'_2 \oplus H'_4 \oplus \ldots \oplus H'_{n-1}) + (H''_3 \oplus H''_5 \oplus \ldots \oplus H''_n)$

$K_1 = S^{(2)} \oplus T^{(1)} = S^{(2)} \oplus H''_1 \oplus T^{(3)} = (H'_2 \oplus H'_4 \oplus \ldots \oplus H'_{n-1}) \oplus (H''_1 \oplus H''_3 \oplus \ldots \oplus H''_n)$.

We can complete the proof by defining $S = S^{(2)} = H'_2 \oplus H'_4 \oplus H'_6 \oplus \ldots \oplus H'_{n-1}$ and $T = T^{(1)} = H''_1 \oplus H''_3 \oplus H''_5 \oplus \ldots \oplus H''_n$ to obtain the factorization $G = S \oplus T$, $S \subseteq G$, $T \subseteq G$, which arises from the series (6.1).

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