Internat. J. Math. & Math. Sci.
VOL. 12 NO. 4 (1989) 781-786

A CHARACTERIZATION OF THE HALL PLANES BY PLANAR AND NONPLANAR INVOLUTIONS

N.L. JOHNSON

Department of Mathematics
The Univeristy of Iowa
Iowa City, Iowa 52242

(Received May 20, 1988)

ABSTRACT. In this article, the Hall planes of even order q^2 are characterized as translation planes of even order q^2 admitting a Baer group of order q and at least q+1 nontrivial elations.

KEY WORDS AND PHRASES. Translation plane, Baer groups, elations.
1980 AMS SUBJECT CLASSIFICATION CODES. 50D05, 05B25.

1. INTRODUCTION AND BACKGROUND.

Let Σ denote an affine Desarguesian plane of order q^2 coordinatized by a field F isomorphic to $GF(q^2)$. Let $\mathscr{N}$ denote the net defined on the points of Σ whose lines have slopes in $GF(q) \cup (\infty)$. Let σ denote the involution defined by $(x,y) \longrightarrow (x^q,y^q)$ where $x,y \in F$. Let $\hat{F}^*$ denote the kernel homology group of Σ defined by $(x,y) \longrightarrow (ax,ay)$ where $|a| = q+1$, $a \in F$.

Now derive $\mathscr{N}$ to obtain the Hall plane $\bar{\Sigma}$ of order q^2. Then the involutions in $\langle\sigma\rangle\hat{F}^*$ are central collineations in $\bar{\Sigma}$.

If $\mathscr{E}$ denotes an elation group fixing $\mathscr{o} = (0,0)$ with axis $\mathscr{L}$ in $\mathscr{N}$ which acts regularly on the remaining lines of $\mathscr{N}$ incident with $\mathscr{o}$ then $\mathscr{E}$ becomes a collineation group of $\bar{\Sigma}$ of order q which fixes a Baer subplane pointwise.

In [3] and [4], Foulser and Johnson classify the translation planes of order q^2 that admit SL(2,q). In particular, if $q^2 > 16$, the Hall planes are precisely the translation planes admitting SL(2,q) where the Sylow p–subgroups for $q = p^r$ fix Baer subplanes pointwise.

So, the Hall planes of order q^2 admit a Baer group of order q and at least 1+q involutary central collineations.

In this article, we consider translation planes of order q^2 that admit a Baer group of order q and $\geq 1+q$ involutory central collineations. For q odd, it turns out that there are other (i.e. non Hall) translation planes possessing this configuration of groups. For example, the translation planes π corresponding to the Fisher flock of a quadratic cone in PG(3,q) for

$q \equiv 3 \bmod 4$ derive planes $\bar{\pi}$ admitting such groups (see [5]).

However, for q even, we are able to characterize the Hall planes using these planar and non planar involutions.

Our main result is

THEOREM A. Let π be a translation plane of even order q^2 which admits a Baer collineation group $\mathcal{B}$ of order q and at least $1+q$ nontrivial elations (all groups are assumed to be in the translation complement). Then π is the Hall plane of order q^2 and conversely, the Hall plane admits such groups.

The proof of theorem A will be given as a series of lemmas. As a preliminary to the proof, we remind the reader of some results required in the arguments.

RESULT I (JHA, JOHNSON [7] (4.1)). Let π be a translation plane of even order $q^2 \neq 64$. Assume π admits a Baer group of order q and a dihedral group of order $2(1+q)$ which is generated by elations with affine axes. Then π is derivable where the elation axes define a derivable partial spread.

RESULT II (FOULSER [2] THEOREM 2 AND COROLLARY 3 (2)). Let π be a translation plane of order q^2 that admits a Baer group $\mathcal{B}$ of order q. (1) Then the Baer subplane $\pi_0 = \text{Fix}\,\mathcal{B}$ pointwise fixed by $\mathcal{B}$ is Desarguesian. (2) Furthermore, if the collineation group $\mathcal{G}_{[\pi_0]}$ fixing π_0 pointwise has order $> q$ then the net $\mathcal{N}$ defined by the lines of π_0 is a derivable net. (3) In the general case, $\mathcal{G}_{[\pi_0]}$ is a subgroup of $AG(1,q)$, the 1–dimensional affine group over $GF(q)$.

RESULT III (JHA, JOHNSON [7]). Let π be a translation plane of even order q^2 that admits a Baer 2–group of order $\geq 2\sqrt{q}$. Then an elation group with fixed affine axis has order ≤ 2.

RESULT IV (A MODIFIED VERSION OF THE MAIN RESULTS OF HERING [6], OSTROM [10]). Let π be a translation plane of even order. Let $\mathcal{G}$ denote the collineation group generated by all elations in the translation complement. If $\mathcal{G}$ is solvable then either $\mathcal{G}$ is an elementary abelian 2–group or has order $2 \cdot t$ where t is odd.

RESULT V (JHA–JOHNSON [8]). Let π be a translation plane of even order q^2 which admits collineation groups $\mathcal{B}_1$, $\mathcal{B}_2$ of orders $\geq 2\sqrt{q}$ such that $\mathcal{B}_i$ fixes a Baer subplane π_i $i = 1,2$ pointwise. If $\pi_1 \neq \pi_2$ then π is Hall or a known plane of order 16.

2. THE CHARACTERIZATION.

Assume for this section, the assumptions of Theorem A and assume π is not Hall.

(2.1) LEMMA. Result I is valid for $q^2 = 64$.

PROOF. π is a translation plane of order 64 that admits a Baer group $\mathcal{B}$ of order 8 and $\geq 1+8$ affine elations. If π is not Hall then $\mathcal{D}$ still becomes dihedral of order $2 \cdot 9$ and centralizes $\mathcal{B}$. Let $\mathcal{C}$ denote the cyclic stem of $\mathcal{D}$. Let $\mathcal{C} = \langle g \rangle$. There are $8 \cdot 7$ components of π not in $\mathcal{N}$ so that g must fix at least two of these components $\mathcal{L}_1$, $\mathcal{L}_2$. Now g leaves invariant $\pi_0 = \text{Fix}\,\mathcal{B}$, $\mathcal{L}_1$ and $\mathcal{L}_2$. Thus, g fixes ≥ 3 mutually disjoint 2m–spaces (if $q = 2^m$) over GF(2). Now the argument given by Jha–Johnson [7] for result I will be valid for $q^2 = 64$. This proves (2.1).

Now assume the order of the plane is 16. The translation planes of order 16 are either semifield planes or derived from semifield planes (see Johnson [9] and Dempwolff and Riefart [1]). In any case, the non Hall planes admitting Baer groups of order 4 do not admit ≥ 5 elations.

So, we may assume $q \neq 4$.

(2.2) LEMMA. Let $\mathcal{D}$ denote the collineation group generated by the affine elations. Then $\mathcal{D}$ is dihedral of order $2(q+1)$, acts faithfully on π_0 and centralizes $\mathcal{B}$.

PROOF. By result IV, no two of the elations can have a common axis. Hence, it follows that $\mathcal{D}$ is solvable by result IV, $|\mathcal{D}| = 2 \cdot t$ where t is odd.

By result II, $\mathcal{D}$ must normalize $\mathcal{B}$. Clearly, the elations must have axes nontrivially intersecting π_0 and leaving π_0 invariant. Since a central collineation is uniquely determined by its axis (co axis) and one specified nontrivial image point, it follows that $\mathcal{D}$ centralizes $\mathcal{B}$. Hence, if $y \in \mathcal{D} \cap \mathcal{B} - \langle 1 \rangle$ then the Sylow 2–subgroups of $\mathcal{D}$ would have order ≥ 4. So $\mathcal{D} \cap \mathcal{B} = \langle 1 \rangle$.

If $1 \neq h \in \mathcal{D}$ fixes π_0 pointwise then the collineation fixing π_0 pointwise has order $> q$ so that by result II(2), the net $\mathcal{N}$ (see notation in II(2)) is derivable.

Let π_1 be a Baer subplane of $\mathcal{N}$ incident with the zero vector $\mathit{0}$. The infinite points of π_1 are exactly those of π_0. If σ is any elation in $\mathcal{D}$ then the axis of σ is in π_1 and σ permutes the infinite points of π_1. Hence, σ leaves π_1 invariant and since $\mathcal{D}$ is generated by elations, it follows that $\mathcal{D}$ must fix each of the q+1 Baer subplanes of $\mathcal{N}$ incident with $\mathit{0}$. However, this means that h cannot fix π_0 pointwise.

Thus, $\mathcal{D}$ acts faithfully on π_0. Now π_0 is Desarguesian by result II(1) and since $\mathcal{D}$ is generated by elations of π_0, $\mathcal{D} \leq SL(2,q) \cong PSL(2,q)$ and $|\mathcal{D}| = 2 \cdot t$ where t is odd. Thus, $\mathcal{D}$ is dihedral and admits $\geq 1+q$ involutions. This proves (2.1).

(2.3) LEMMA. π is derivable with derivable net $\mathcal{N}$ (in the above notation).

PROOF. (2.2 and result I).

(2.4) LEMMA. Let σ be any elation in $\mathcal{D}$. Then for any $\tau \in \mathcal{B} - \langle 1 \rangle$, $\tau\sigma$ is a Baer involution. Furthermore, if $\rho \in \mathcal{B} - \langle 1 \rangle$, $\rho \neq \tau$ then the set of components of π not in $\mathcal{N}$ fixed by $\rho\sigma$ is disjoint from the set of components not in $\mathcal{N}$ fixed by $\tau\sigma$.

PROOF. If $\tau\sigma$ is an elation then $(\tau\sigma)\sigma \in \mathcal{D}$. But $\mathcal{D} \cap \mathcal{B} = \langle 1 \rangle$. Hence, $\tau\sigma$ is a Baer involution.

Let $\mathcal{L}$ be a component fixed by both $\tau\sigma$ and $\rho\sigma$. Then $(\tau\sigma)(\rho\sigma)$ also fixes $\mathcal{L}$ and

$(\tau\sigma)\rho\sigma = \tau\rho\sigma^2 = \tau\rho \in \mathscr{B}$ fixes $\mathscr{L}$. Thus $\mathscr{L}$ is a component of $\mathscr{N}$.

(2.5) LEMMA. Let σ be any elation in $\mathscr{D}$. Then each component of π not in $\mathscr{N}$ is fixed by exactly one Baer involution in $\sigma(\mathscr{B} - \langle 1 \rangle)$.

PROOF. By (2.4), there are q(q–1) distinct components fixed by some involution in $\sigma(\mathscr{B}-\langle 1 \rangle)$. Since there are exactly q(q–1) components not in $\mathscr{N}$, (2.5) is proved.

(2.6) LEMMA. Let $\mathscr{D} = \langle \sigma,\chi \rangle$ where σ,χ are distinct elations. Each component $\mathscr{L}$ of π not in $\mathscr{N}$ is fixed by $\sigma\chi$.

PROOF. By (2.5), there exists a Baer involution $\rho\sigma \in (\mathscr{B} - \langle 1 \rangle)\sigma$ which fixes $\mathscr{L}$ and similarly, there is a Baer involution $\tau\chi$ in $(\mathscr{B} - \langle 1 \rangle)\chi$ which fixes $\mathscr{L}$.

Thus $(\rho\sigma)(\tau\chi)$ also fixes $\mathscr{L}$. However, $(\rho\sigma)(\tau\chi) = (\rho\tau)(\sigma\chi)$ by (2.1). Further, $((\rho\tau)(\sigma\chi))^2 = (\rho\tau)^2(\sigma\chi)^2$ again by (2.1)) $= (\sigma\chi)^2$. Since $|\langle \sigma\chi \rangle| = q+1$ and q+1 is odd, then $\langle \sigma\chi \rangle = \langle (\sigma\chi)^2 \rangle$. Thus, $(\sigma\chi)^{2j}$ and thus $\sigma\chi$ fixes $\mathscr{L}$.

(2.7) LEMMA. Let $\bar{\pi}$ denote the translation plane obtained from π by deriving $\mathscr{N}$. Then $\mathscr{D}\mathscr{B}$ is a collineation group of $\bar{\pi}$.

PROOF. $\mathscr{D}\mathscr{B}$ leaves $\mathscr{N}$ invariant.

(2.8) LEMMA. Let $\mathscr{C}$ denote the cyclic stem of $\mathscr{D}$ of order q+1. Then $\mathscr{C}$ is a kernel homology group of $\bar{\pi}$.

PROOF. It was noted in the proof to (2.1) that $\mathscr{D}$ must fix each Baer subplane incident with a in $\mathscr{N}$. Hence, the stem $\mathscr{C}$ of $\mathscr{D}$ must fix each such Baer subplane. The components of $\bar{\pi}$ are the components of π not on $\mathscr{N}$ and the Baer subplanes of $\mathscr{N}$ which are incident with a. By (2.6), if $\mathscr{D} = \langle \sigma,\chi \rangle$ then $\mathscr{C} = \langle \sigma\chi \rangle$ so that $\mathscr{C}$ fixes each component of π not in $\mathscr{N}$. Thus, $\mathscr{C}$ must induce a kernel homology group in $\bar{\pi}$.

Let the kernel of $\bar{\pi}$ be isomorphic to $GF(2^r) \leq GF(q^2)$. Let $q = 2^m$ so that $r|2m$. then $1+q \mid 2^r-1$ by (2.7). Thus, $r > m$ so that $r = 2m$.

Thus, the kernel of $\bar{\pi}$ is isomorphic to $GF(q^2)$ so that $\bar{\pi}$ is Desarguesian. Thus, π must be Hall and we obtain the proof to theorem A.

REFERENCES

1. V. DEMPWOLFF and A. RIEFART. The Classification of the Translation planes of Order 16 (I), *Geom. Ded.* 15 (1983), 137–153.

2. D.A. FOULSER. Subplanes of Partial Spreads in Translation Planes, *Bull. London Math. Soc.* 41 (1972), 32–38.

3. D.A. FOULSER and N.L. JOHNSON. The Translation Planes of Order q^2 that Admit SL(2,q). I. Even Order, *J. Alg.* 82, No. 2 (1984), 385–406.

4. D.A. FOULSER and N.L. JOHNSON. The Translation Planes of Order q^2 that Admit SL(2,q). II. Odd Order., *J. Geom.* 18 (1982), 122–139.

5. H. GEVAERT and N.L. JOHNSON. Flocks of Quadratic Cones, Generalized Quadrangles and Translation Planes, *Geom. Ded.* (to appear).

6. Ch. HERING. On Shears of Translation Planes, *Abh. Math. Sem. Hamburg*, 37 (1972), 258-268.

7. V. JHA and N.L. JOHNSON. Coexistence of Elations and Large Baer Groups in Translation Planes, *J. London Math. Soc.* 2 (32) (1985), 297–304.

8. V. JHA and N.L. JOHNSON. Solution to Dempwolff's Nonsolvable B–group Problem, *European J. Comb.*, Vol. 7, No. 3, July (1986), 227–235.

9. N.L. JOHNSON. The Translation Planes of Order 16 that Admit Nonsolvable Collineation Groups, *Math. Z.* 185 (1984), 355–372.

10. T.G. OSTROM. Linear Transformations and Collineations of Translation Planes, *J. Alg.*, 14 (1970), 405–416.