Internat. J. Math. & Math. Sci.
VOL. 13 NO. 1 (1990) 145-150

TIME — PERIODIC WEAK SOLUTIONS

ELIANA HENRIQUES DE BRITO

Instituto de Matemática
Universidade Federal do Rio de Janeiro
C.P. 68530, Rio de Janeiro, Brasil

(Received December 30, 1987)

ABSTRACT. In continuing from previous papers, where we studied the existence and uniqueness of the global solution and its asymptotic behavior as time t goes to infinity, we now search for a time-periodic weak solution u(t) for the equation whose weak formulation in a Hilbert space H is

$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha\, b(u,v) + \beta a(u,v) + (G(u),v) = (h,v)$$

where: ' = d/dt;(,) is the inner product in H; b(u,v), a(u,v) are given forms on subspaces $U \subset W$, respectively, of H; $\delta > 0$, $\alpha \geq 0$, $\beta \geq 0$ are constants and $\alpha + \beta > 0$; G is the Gateaux derivative of a convex functional $J: V \subset H \to [0,\infty)$ for V = U, when $\alpha > 0$ and V = W when $\alpha = 0$, hence $\beta > 0$; v is a test function in V; h is a given function of t with values in H.

Application is given to nonlinear initial-boundary value problems in a bounded domain of R^n.

KEYWORDS AND PHRASES. Periodic weak solution, Gateaux derivative.
1980 AMS SUBJECT CLASSIFICATION CODE. 35L20, 35B40.

1. INTRODUCTION.

In continuation of Brito [1], [2], where we studied existence and uniqueness of the global solution and its asymptotic behavior as time t goes to infinity, we now search for a time-periodic weak solution u(t), i.e., such that

$$u(0) = u(T);\ u'(0) = u'(T)$$

for the equation whose weak formulation in a Hilbert space H is

$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v) \qquad (1.1)$$

where

= d/dt; (,) is the inner product in H; b(u,v), a(u,v) are given forms on subspaces $U \subset W$, respectively, of H; $\delta > 0$, $\alpha \geq 0$, $\beta \geq 0$ are constants and $\alpha + \beta > 0$; G is the Gateaux derivative of a convex functional $J: V \subset H \to [0,\infty)$, for V = U, when $\alpha > 0$, and V = W, when $\alpha = 0$, hence $\beta > 0$; v is a test function in V; h is a given function of t with values in H.

Application is given to initial-boundary value problems in a bounded domain Ω of R^n for the following equations, in which $p > 2$ depends on n, $\alpha > 0$, $\beta \geq 0$, $k > 0$:

$$u'' + \delta u' - \Delta u + |u|^{p-2}u = h \tag{1.2}$$

$$u'' + \delta u' + \alpha\Delta^2 u - \beta\Delta u + u + |u|^{p-2}u = h \tag{1.3}$$

$$u'' + \delta u' + \alpha\Delta^2 u - \{\beta + k \int_\Omega (\nabla u)^2 d\Omega\}\Delta u = h \tag{1.4}$$

and the generalization of (1.4) in a Hilbert space H

$$u'' + \delta u' + \alpha A^2 u + \beta Au + M(|A^{1/2}|^2)Au = h \tag{1.5}$$

for A a linear operator in H, M a real function.

For related problems, we refer to Biroli [3], Lovicar [4]; see also references in Brito [1].

2. PRELIMINARIES.

We consider three Hilbert spaces $U \subset W \subset H$ each continuously embedded and dense in the following.

We assume the injection $W \subset H$ compact.

Let (,) denote the inner product in H and $|\ \ |$ its norm.

Let a(u,v) and b(u,v) be two continuous, symmetric, bilinear forms in W and U, respectively. We shall write a(v) for a(v,v), b(v) for b(v,v). We shall assume that $(a(v))^{1/2}$ defines in W a norm equivalent to the norm of W and, similarly, that $(b(v))^{1/2}$ defines in U a norm equivalent to the norm of U.

Let $c > 0$ be such that

$$c|v|^2 \leq a(v) \text{ for } v \text{ in } W. \tag{2.1}$$

Let A be a linear operator in H, with domain D(A) such that $U \subset D(A) \subset W$ and

$a(u,v) = (Au,v)$ for u in U, v in W

$b(u,v) = (Au, Av)$ for u,v in U

$\delta > 0$, $\alpha \geq 0$, $\beta \geq 0$ are constants and $\alpha + \beta > 0$.

Assume

$V = U$ if $\alpha > 0$, $\beta \geq 0$;

$V = W$ if $\alpha = 0$, $\beta > 0$.

Consider a convex functional

$J: V \to [0,\infty)$ such that $J(0) = 0$.

Let $G: V \to H$ be the Gateaux derivative of J. We assume G is Gateaux differentiable, locally Lipschitz and $G(0) = 0$.

With these hypothesis, we have, from [2] Theorem 3.1 and [1] Lemma 2.1, respectively

THEOREM 2.1. Given u_0 in V, u_1 in H, h in $L^2(0,T;H)$, there is a unique function u such that

a) $u \in L^\infty(0,T;V)$; $u' \in L^\infty(0,T;H)$; $G(u) \in L^\infty(0,T;H)$

b) for all v in V, u satisfies,

$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v) \qquad (2.2)$$

c) u satisfies the initial conditions

$$u(0) = u_0;\ u'(0) = u_1 \qquad (2.3)$$

d) u satisfies the energy equation

$$E(t) + \delta \int_0^t |u'(s)|^2 ds = E(0) + \int_0^t (h(s),u'(s))ds \qquad (2.4)$$

where

$$2E(t) = |u'(t)|^2 + \alpha b(u(t)) + \beta a(u(t)) + 2J(u(t)).$$

THEOREM 2.2. In the conditions of Theorem 2.1, the map S: V×H → V×H given by

$$S(u_0,u_1) = (u(t),u'(t))$$

is, for fixed t, (sequentially) weakly continuous (i.e., if $\phi_n > \phi$ weakly in V × H, then $s(\phi_n) > s(\phi)$ weakly in V × H).

We shall, further, assume that

$$2J(v) - (G(v),v) \leq 0 \text{ for v in V}. \qquad (2.5)$$

3. EXISTENCE OF TIME-PERIODIC WEAK SOLUTIONS.

We shall refer to u(t) in the conditions of Theorem 2.1 as the solution of (2.2) with initial conditions (u_0,u_1) in V × H, given by (2.3).

THEOREM 3.1. If $h \in C([0,T];H)$ there is at least one solution of (2.2) with initial condition in V × H such that

$$u(0) = u(T);\ u'(0) = u'(T). \qquad (3.1)$$

PROOF. Take v = u(t) in (2.2) multiplied by constant $2\gamma > 0$ and add it to the energy equation (2.4) differentiated and multiplied by 2, to obtain, with (2.5),

$$\frac{d}{dt}\{|u'|^2 + \alpha b(u) + \beta a(u) + 2J(u) + 2\gamma(u,u')\} +$$

$$+ 2\gamma\{|u'|^2 + \alpha b(u) + \beta a(u) + 2J(u) + 2\gamma(u,u')\} +$$

$$+ 2(\delta-2\gamma)\ [|u'|^2 + \gamma(u',u)] \leq 2(h,u' + \gamma u).$$

For $0 < \gamma < \delta/2$, let

$$w(t) = |u' + \gamma u|^2 + \alpha b(u) + \beta a(u) + 2J(u). \qquad (3.2)$$

Then we have

$$w'(t) + 2\gamma w(t) \leq 2(h,u' + \gamma u) - 2(\delta-2\gamma)(u',u' + \gamma u) +$$

$$+ \gamma^2 \frac{d}{dt}|u|^2 + 2\gamma^3|u|^2.$$

The right-hand side of the above inequality is equal to

$$2(h,u' + \gamma u) + 2(\gamma^2 u - (\delta - 2\gamma)u', u' + \gamma u) =$$

$$= 2(h,u' + \gamma u) - 2(\delta - 2\gamma)(u' + \gamma u, u' + \gamma u) + 2(\delta - \gamma)(\gamma u, u' + \gamma u).$$

Therefore

$$w'(t) + 2\gamma w(t) \leq 2(h,u' + \gamma u) + \frac{(\delta - \gamma)^2 \gamma^2 |u|^2}{2(\delta - 2\gamma)}. \tag{3.3}$$

Observing (2.1) and (3.2), we obtain

$$w(t) \geq |u' + \gamma u|^2 + \varepsilon\ |u|^2, \text{ with } \varepsilon = ac^2 + \beta c > 0. \tag{3.4}$$

By assumption, $\beta + \alpha > 0$.

We choose $0 < \gamma < \delta/2$ so that

$$\rho = \frac{(\delta-\gamma)^2\ \gamma^2}{2\ \varepsilon\ (\delta-2\gamma)} < 2\gamma. \tag{3.5}$$

This is possible, because it amounts to choosing γ so that

$$B(\gamma) = (\delta - \gamma)^2\gamma - 4\ \varepsilon\ (\delta - 2\gamma) < 0$$

and $\lim_{\gamma\to 0} B(\gamma) = -4\ \varepsilon\ \delta < 0$.

It follows from (3.3), with (3.4), (3.5), that

$$w'(t) + 2\gamma w(t) \leq 2|h(t)|\ \sqrt{w(t)} + \rho w(t).$$

Hence for $0 \leq t \leq T$,

$$w(t) \leq F(t) \tag{3.6}$$

where

$$F(t) = e^{-2\gamma t}\{w(0) + \int_0^t e^{2\gamma s}[2|h|\sqrt{w} + \rho w]ds\}. \tag{3.7}$$

Therefore, because of (3.6), we have

$$F'(t) \leq (\rho - 2\gamma)F(t) + 2|h(t)|\sqrt{F(t)}.$$

Let $F(t) = r$, with

$$r \geq \max_{0 \leq t \leq T} \frac{2|h(t)|}{2\gamma - \rho}.$$

Then $F'(t) \leq 0$. It follows, using (3.6), (3.7), that if $w(0) \leq r$ then $w(t) \leq r$, for $0 \leq t \leq T$.

Consider

$$K = \{(u_0,u_1) \in V \times H;\ |u_1 + \gamma u_0|^2 + \alpha b(u_0) + \beta a(u_0) + 2J(u_0) \leq r\}.$$

We proved that the map $S: V \times H \to V \times H$ given by

$$S(u_0,u_1) = (u(T),\ u'(T))$$

takes K into K.

It is easy to check that K is a nonempty, closed, bounded, convex subset of $V \times H$.

The fact that S has a fixed point, i.e., that (3.1) holds for some $(u_0,u_1) \in K$, now follows from Theorem 2.2 as a consequence of the well-known fixed point Theorem:

Let B be a separable, reflexive Banach space, K a nonempty closed, bounded convex subset of B, and S a (sequentially) weakly continuous operator of K into K. Then S has at least one fixed point in K.

4. APPLICATIONS.

We devote this Section to applications of Theorem 3.1 involving initial-boundary value problems in a bounded domain Ω with regular boundary in R^n for equations (1.2), (1.3), (1.4).

In what follows

$$H = L^2(\Omega),\ W = H_0^1(\Omega),\ \text{and}$$

$$a(u,v) = (\nabla u,\nabla v) = \int_\Omega \nabla u \nabla v d\Omega.$$

Let

$$A = -\Delta,\quad D(A) = H_0^1(\Omega) \cap H^2(\Omega),$$

$$b(u,v) = (\Delta u,\Delta v),\quad U = H_0^1(\Omega) \cap H^2(\Omega).$$

Note that similar results are obtained if we suppose $U = H_0^2(\Omega)$.

For $\delta > 0$, $h \in C([0,T];\ H)$, we have

EXAMPLE 4.1. Let $\alpha = 0$, $\beta = 1$, $V = W = H_0^1(\Omega)$ and

$$J(u) = \frac{1}{p} |u|^p{}_{L^p(\Omega)} \text{ for } u \text{ in } V$$

where $2 < p \leq 2(n-1)/(n-2)$ if $n > 2$; $p > 2$ if $n \leq 2$. Then $J(u)$ is well-defined in V and

$$G(u) = |u|^{p-2} u \in H.$$

We refer to [2], example 5.1, for the proof. It is clear that (2.5) holds. Therefore Theorem 3.1 ensures existence of a T-periodic weak solution of

$$u'' + \delta u' - \Delta u + |u|^{p-2}u = h.$$

EXAMPLE 4.2. For $\alpha > 0$, $\beta \geq 0$, $V = U = H_0^1(\Omega) \cap H^2(\Omega)$ (or $H_0^2(\Omega)$), let

$$J(u) = \frac{1}{p}|u|^p_{L^p(\Omega)} + \frac{1}{2}|u|^2_{L^2(\Omega)} \text{ for } u \text{ in } V$$

where

$$2 < p \leq 2(n-2)/n-4) \text{ if } n > 4;\ p > 2 \text{ if } n \leq 4.$$

Then $J(u)$ is well-defined in V and

$$G(u) = |u|^{p-2}u + u \in H.$$

We refer to [2], example 5.2, for the proof. It is clear that (2.5) holds. Then Theorem 3.1 ensures existence of a T-periodic weak solution of

$$u'' + \delta u' + \alpha\Delta^2 u - \beta\Delta u + u + |u|^{p-2}u = h.$$

EXAMPLE. 4.3. For $\alpha > 0$, $\beta > 0$, $k > 0$, $V = U = H_0^1(\Omega) \cap H^2(\Omega)$ (or $H_0^2(\Omega)$), let

$$J(u) = \frac{k}{4} (a(u))^2 \text{ for } u \text{ in } V.$$

Then

$$G(u) = -ka(u)\Delta u \in H.$$

We refer to [2], Example 5.3, for the proof. It is clear that (2.5) holds.

Thus Theorem 3.1. ensures existence of a T-periodic weak solution of

$$u'' + \delta u' + \alpha\Delta^2 u - \{\beta + k \int_\Omega (\nabla u)^2 d\Omega\}\Delta u = h.$$

Generalizing, let M be a C^1 function such that, for $s > 0$,

$$M(s) > k_s \text{ and } M'(s) > 0.$$

Take V = U and A as in Section 1. Let

$$J(u) = \frac{1}{2} \int_0^{a(u)} M(s)ds \text{ for } u \text{ in } V.$$

Then

$$G(u) = M(a(u)), Au \in H.$$

We refer to [2], Example 5.3, for the proof. It is clear that (2.5) holds. Therefore Theorem 3.1 ensures existence of a T-periodic weak solution of

$$u'' + \delta u' + \alpha A^2 u + [\beta + M(|A^{1/2}u|^2)] Au = h.$$

REFERENCES

1. BRITO, E.H., Nonlienar Equations, Applicable Analysis, Vol. 24, 13-26, (1987).
2. BRITO, E.H., Nonlinear Initial-Boundary Value Problems, Nonlinear Analysis, Vol. II, No. 1, (1987), 125-137.
3. BIROLI, M., Sur l'equation des ondes avec un Terme Non linéaire, Monotone dans la Frontiere Inconnue, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. Serie VIII, Vol. LIII (1972), 359-361., 508-515.
4. LOVICAR, V., Periodic Solutions of Nonlinear Abstract, Second Order Equations with Dissipative Terms, Casopis propestovani matematiky, roc. 102, Praha, (1977), 364-369.