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SOME SPECTRAL INCLUSIONS ON D-COMMUTING SYSTEMS

R.U. VERMA

Duke University
Department of Mathematics
Durham, NC 27706, U.S.A.

and

S.S. JOU

Winston - Salem State University
Department of Mathematics and Computer Science
Winston-Salem, NC 27110, U.S.A.

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ABSTRACT. Some spectral inclusions for the Taylor joint Browder spectra of D-commuting systems are established. The obtained results are applied in generalizing the spectral mapping theorems.

KEY WORDS AND PHRASES. Taylor joint Browder spectrum, D-Commuting systems, Fredholm complex, densely - defined operators.

1. INTRODUCTION.

For two complex Hilbert spaces X and Y, let L(X,Y) denote the set of all closed linear operators, defined on linear subspaces of X, with values in Y. Let C(X,Y) denote the subset of those operators from L(X,Y) which are defined everywhere, and hence continuous. For L(X,X) and C(X,X), we shall simply write L(X) and C(X), respectively. For any operator $A \in L(X,Y)$, D(A), N(A), and R(A) denote, respectively, the domain, null space, and range of A.

Let $\{X^p\}_{p \in Z}$ (where Z is the ring of integers) be a family of Hilbert spaces, and let $a^{(p)} \in L(X^p, Y^p)$ be a family of operators such that $R(a^{(p)}) \subset N(a^{(p+1)})$, for all $p \in Z$. Let the following sequence represent the complex of Hilbert spaces:

$$\cdots \xrightarrow{a^{(p-1)}} X^p \xrightarrow{a^{(p)}} X^{p+1} \xrightarrow{a^{(p+1)}} \cdots .$$

Let $\{H^p(X,a)\}_{p \in Z}$ denote the cohomology of the complex $(X,a)=(X^p,a^{(p)})$, for all $p \in Z$, that is, $H^p(X,a)=N(a^{(p)})/R(a^{(p-1)})$.

A complex $(X,a) = (X^p, a^{(p)})$ is said to be Fredholm if $\inf\{r(a^{(p)})\} > 0$ (where $r(a^{(p)})$ is the reduced minimum modulus of $a^{(p)}$), $\dim(H^p(X,a)) < \infty$ for each $p \in Z$, and $H^p(X,a) \neq 0$ only for a finite number of indices.

Note that for a Fredholm complex (X,a), the quotient space $H^p(X,a)$ is isomorphic to the subspace $N(a^{(p)}) \ominus R(a^{(p-1)})$, for all $p \in Z$, and hence $H^p(X,a)$ will carry this meaning in the sequel. We also assume without loss of generality that $D(a^{(p)})$ is dense in X^p for each $p \in Z$ so that this emphasis is attached to the complex $(X,a) = (X^p, a^{(p)})$.

We recall some definitions [1]. Let $s=(s_1,\ldots,s_n)$ be a system of n indeterminates, and let $E(s)$ be the exterior algebra over the complex field C, generated by the indeterminates $s_1,\ldots,s_n$. Then, for any integer p, $0 \leq p \leq n$, $E^p(s)$ is the vector subspace of $E(s)$ containing all homogeneous exterior forms of degree p $(p=0,1,\ldots,n)$ in $s_1,\ldots,s_n$. For any Hilbert space X, $E(X,s)$ (resp. $E^p(X,s)$) shall denote the tensor product $X \otimes E(s)$ (resp. $X \otimes E^p(s)$). An element $\xi \in E^p(X,s)$ will be written

$$\xi = \sum_{1 \leq j_1 < \ldots < j_p \leq n} x_{j_1 \ldots j_p} \, \delta_{j_1} \wedge \ldots \wedge \delta_{j_p} ,$$

where $x_{j_1 \ldots j_p} \in X$, and $x_{j_1 \ldots j_p} \, \delta_{j_1} \ldots \delta_{j_p}$ stands for $x_{j_1 \ldots j_p} \otimes \delta_{j_1} \wedge \ldots \wedge \delta_{j_p}$. The space $E^p(X,s)$ is identified with a direct sum of $\binom{n}{p}$ copies of X, therefore, $E^p(X,s)$ can be endowed with a natural structure of Hilbert space. For two Hilbert spaces X and Y, there is a natural identification of the space $E(X,s) \otimes E(y,t)$ with space $E(X \bar{\otimes} Y,(s,t))$, where $(t_1,\ldots,t_n)$ is another system of indeterminates.

Let $H_j : E(s) \longrightarrow E(s)$ be defined by $H_j \eta = s_j \, \eta$, $\eta \in E(s)$.

It is obvious that

$$H_j H_k + H_k H_j = 0, \text{ for } j,k = 1,\ldots,n.$$

DEFINITION 1.1. A n-tuple $a=(a_1,\ldots,a_n) \in L^n(X)$ is said to be a D-commuting system if there exists a dense subspace D of X in $\bigcap_1^n D(a_j)$ with the following properties [2]:

(i) The restriction $\delta_a = (a_1 \otimes H_1 + \ldots + a_n \otimes H_n) | E(D,s)$ is closable.

(ii) If δ_a is the canonical closure of δ_a, then $R(\delta_a) \subset N(\delta_a)$.

DEFINITION 1.2. A D-commuting system $a=(a_1,\ldots,a_n) \in L^n(X)$ is said to be singular (resp. non-singular) if $R(\delta_a) \neq N(\delta_a)$ (resp. $R(\delta_a) = N(\delta_a)$).

REMARK 1.3. Note that to each D-commuting system we can associate a complex of Hilbert spaces $(E^p(X,s), \delta_a^{(p)})_{p=0}^n$, where

$$\delta_a^{(p)} = \delta_a | \, E^p(x,s) \cap D(\delta_a).$$

DEFINITION 1.4. Let $a=(a_1,\dots,a_n) \varepsilon L^n(X)$ be a D-commuting system associated with a complex of Hilbert spaces $(E^p(X,s),\ \delta_a^{(p)})_{p=0}^n$. Then $a=(a_1,\dots,a_n)$ is said to be Fredholm if the corresponding complex is Fredholm.

DEFINITION 1.5. The joint spectrum of a D-commuting system $a = (a_1,\dots,a_n) \varepsilon L^n(X)$ is the set of those points $\lambda \varepsilon C^n$ such that $a-\lambda I$ is singular, denoted $\sigma^T(a,X)$.

DEFINITION 1.6. The Fredholm spectrum of a D-commuting system $a=(a_1,\dots,a_n) \varepsilon L^n(X)$ is the set of those points $\lambda \varepsilon C^n$ such that $a-\lambda I$ is not Fredholm, denoted $\sigma_e^T(a,X)$.

DEFINITION 1.7. The joint Browder spectrum of a D-commuting system $a=(a_1,\dots,a_n) \varepsilon L^n(X)$ is defined by the relation $\sigma_B^i(a) = \sigma_e^i(a)$ $\cup$\{accumulation points of $\sigma^i(a)$ \}, for $i=1,D,P,T$, where $1,D,P$ and T, respectively, denote the commutant spectrum, Dash spectrum, polynomial spectrum, and Taylor spectrum. For detail, see [3], [1] and [4].

The prime aim of this paper is to establish some inclusions for the Taylor joint Browder spectra of D-commuting systems. The obtained results are also applied in generalizing the spectral mapping theorems on tensor products. Unlike the case of the bounded linear operator systems, the situation is quite different in the case of the closed densely-defined operator systems [3].

2. SOME LEMMAS.

We state and prove (in certain case) some results for our main theorems.

LEMMA 2.1. (Grosu and Vasilescu [2]). Let $a=(a_1,\dots,a_n) \varepsilon L^n(X)$ be D_a-commuting system, and $b=(b_1,\dots,b_m) \varepsilon L^m(Y)$ be a D_b-commuting system. Then

$$\sigma^T(a \otimes b, X \bar{\otimes} Y)=\sigma^T(a,X) \times \sigma^T(b,Y). \tag{2.1}$$

LEMMA 2.2. (Grosu and Vasilescu [2]). Let (X,a) and (Y,b) be two complexes of Hilbert spaces. Then their tensor product $(X \bar{\otimes} Y, a \otimes b)$ is Fredholm and exact iff either (X,a) or (Y,b) is Fredholm and exact.

LEMMA 2.3. Let $a=a_1,\dots,a_n) \varepsilon L^n(X)$ be D_a-commuting system and let $b=(b_1,\dots,b_m)$ $\varepsilon L^m(Y)$ be a D_b-commuting system. Then

$$\sigma_e^T(a \otimes b, X \bar{\otimes} Y) \subseteq [\sigma_e^T(a,X) \times \sigma^T(b,Y)] \cup [\sigma^T(a,X) \times \sigma_e^T(b,Y)]. \tag{2.2}$$

PROOF. In order to prove the inclusion, let us take a point

$$(\lambda,\mu) \notin [\sigma_e^T(a,X) \times \sigma^T(b,Y)] \cup [\sigma^T(a,X) \times \sigma_e^T(b,Y)].$$

We may assume without loss of generality that $\lambda=0$, and $\mu=0$. If $0 \notin \sigma_e^T(a,X)$ and $0 \varepsilon \sigma^T(a,X)$, then from Lemma 2.2, it follows that $a \otimes b$ is Fredholm. The same thing happens when $0 \notin \sigma^T(b,Y)$. Therefore,

$$(0,0) \notin \sigma_e^T(a \otimes b,\ X \otimes Y).$$

LEMMA 2.4. (Fainshtein [5]). Let $a=(a_1,\dots,a_n) \in L^n(X)$ be a D_a-commuting system. Let f be a family of functions, which are holomorphic in a neighbourhood of $\sigma(a_1) \times \dots \times \sigma(a_n)$. Then

$$\sigma_e^T[f(a_1,\dots,a_n)] = f[\ \sigma_e^T(a_1,\dots,a_n)]. \tag{2.3}$$

LEMMA 2.5. Under the assumptions of Lemma 2.4, we have

$$\sigma_B^T[f(a_1,\dots,a_n)] = f[\sigma_B^T(a_1,\dots,a_n)].$$

3. SPECTRAL INCLUSIONS.

We are about to establish the main results.

THEOREM 3.1. Let $a=(a_1,\dots,a_n) \in L^n(X)$ be a D_a-commuting system, let $b=(b_1,\dots,b_m) \in L^m(Y)$ be a D_b-commuting system, and let $a \otimes b = (a_1 \otimes I,\dots,a_n \otimes I, I \otimes b_1,\dots,I \otimes b_m) \in L^{n+m}(X \otimes Y)$ be a $D_a \otimes D_b$-commuting system. Then the Taylor joint Browder spectrum

$$\sigma_B^T(a \otimes b, X \bar{\otimes} Y) \subseteq [\sigma_B^T(a,X) \times \sigma^T(b,Y)] \cup [\sigma^T(a,X) \times \sigma_B^T(b,Y)]. \tag{3.1}$$

COROLLARY 3.2. For $a \otimes b = (a_1 \otimes I,\ I \otimes b_1) \in L^2(x \bar{\otimes} Y)$, we find

$$\sigma_B^T(a \otimes b, X \bar{\otimes} Y) \subseteq [\sigma_B(a_1,X) \times \sigma(b_1,Y)] \cup [\sigma(a_1,X) \times \sigma_B(b_1,Y)]. \tag{3.2}$$

PROOF OF THEOREM 3.1. Let us assume that $(\lambda,\mu) \in \sigma_B^T(a \otimes b, X \bar{\otimes} Y) \supseteq \sigma_e^T(a \otimes b, X \bar{\otimes} Y)$. If $(\lambda,\mu) \in \sigma_e^T(a \otimes b, X \bar{\otimes} Y)$, then

$$\begin{aligned}(\lambda,\mu) &\in [\sigma_e^T(a,X) \times \sigma^T(b,Y)] \cup [\sigma^T(a,X) \times \sigma_e^T(b,Y)] \\ &\subseteq [\sigma_B^T(a,X) \times \sigma^T(b,Y)] \cup [\sigma^T(a,X) \times \sigma_B^T(b,Y)],\end{aligned}$$

by Lemma 2.3. And, if $(\lambda,\mu) \in \sigma^T(a \otimes b, X \bar{\otimes} Y) \setminus \sigma_e^T(a \bar{\otimes} b, X\ \ Y)$, then (λ,μ) is not an isolated point of $\sigma^T(a \otimes b, X \bar{\otimes} Y)$. Thus, in turn, it implies, by Lemma 2.1, that either λ or μ is not an isolated point. This further implies that

$$(\lambda,\mu) \in [\sigma_B^T(a,X) \times \sigma^T(b,Y)] \cup [\sigma^T(a,X) \times \sigma_B^T(b,Y)],$$

and this completes the proof.

REMARK 3.3. The question whether the inclusion (3.1) is true for the cases of the Commutant, Dash, and Polynomial spectra is still open.

THEOREM 3.4. Let $X_1,\ldots,X_n$ be Hilbert spaces and let $S_j \in L(X_j)$, for $j=1,\ldots,n$, be densely-defined operators. Let $\tilde{X}=X_1 \bar{\otimes} \ldots \bar{\otimes} X_n$ be the completion of the algebraic tensor product $X_1 \otimes \ldots \otimes X_n$ with respect to the canonical Hilbert norm, and let $\tilde{S}_j \in L(\tilde{X}_j)$ be the canonical closure of the operator

$$I_1 \otimes I_2 \ldots \otimes I_{j-1} \otimes S_j \otimes I_{j+1} \otimes \ldots \otimes I_n .$$

Then $\tilde{S} = (\tilde{S}_1,\ldots,\tilde{S}_n)$ is a D-commuting system, for $D=D(S_1) \otimes \ldots \otimes D(S_n)$, and

$$\begin{aligned}\sigma_B^T(\tilde{S},\tilde{X}) \subseteq\; & [\sigma_B(S_1,X_1) \times \sigma(S_2,X_2)\times\ldots\times \sigma(S_n,X_n)] \\ & \cup [\sigma(S_1,X_1)\times \sigma_B(S_2,X_2)\times \sigma(S_3,X_3)\times\ldots\times^{\sigma}(S_n,X_n)] \cup\ldots\cup \\ & \cup [\sigma(S_1,X_1)\times\ldots\times \sigma(S_{n-1},X_{n-1}) \times \sigma_B(S_n,X_n)].\end{aligned}$$

PROOF. We prove the Theorem by induction. For $n=2$, it reduces to a special case of Corollary 3.2.

Let us assume that the theorem holds for any $n-1$ operators, for $n\geq 3$. Then, if B denotes the canonical closure of the operator

$$I_1 \otimes\ldots\otimes S_j \otimes\ldots\otimes I_{n-1}, \quad (j=1,\ldots,n-1),$$

we obtain

$$\tilde{S} = (B_1 \bar{\otimes} I_n,\ldots,B_{n-1} \bar{\otimes} I_n,\; I_1 \bar{\otimes}\ldots\bar{\otimes} I_{n-1} \bar{\otimes} S_n) .$$

If $B = (B_1,\ldots,B_{n-1})$, $D(B) = D(S_1) \otimes\ldots\otimes D(S_{n-1})$ and $\tilde{X}_{n-1}=X_1 \otimes\ldots\otimes X_{n-1}$, then Theorem 3.1, Lemma 2.1 and the induction hypothesis imply that

$$\begin{aligned}\sigma_B^T(B \otimes S_n,\tilde{X}) & \subseteq [\sigma_B^T(B,\tilde{X}_{n-1}) \times \sigma(S_n,X_n)] \\ & \quad \cup [\sigma^T(B,\tilde{X}_{n-1}) \times \sigma_B(S_n,X_n)] \\ & \subseteq [\{\sigma_B(S_1,X_1) \times \sigma(S_2,X_2)\times\ldots\times \sigma(S_{n-1}, X_{n-1})\} \cup \\ & \quad \{\sigma(S_1,X_1)\times \sigma_B(S_2,X_2)\times \sigma(S_3,X_3)\times\ldots\times\sigma(S_{n-1},X_{n-1})\} \cup\ldots\cup \\ & \quad \cup\{\sigma(S_1,X_1)\times\ldots\times \sigma(S_{n-2},X_{n-2})\times \sigma_B(S_{n-1},X_{n-1})\}]\times \sigma(S_n,X_n)] \\ & \quad \cup[\{\sigma(S_1,X_1)\times\ldots\times \sigma(S_{n-1},X_{n-1})\} \times \sigma_B(S_n,X_n)].\end{aligned}$$

This completes the proof.

THEOREM 3.5. Let $f=(f_1,\ldots,f_r)$ be a family of functions which are holomorphic in a neighbourhood of the Taylor spectrum of the operator family
$a \otimes b = (a_1 \otimes I,\ldots,a_n \otimes I,\ I \otimes b_1,\ldots,I \otimes b_m)$. Then

$$\sigma_e^T[f(a \otimes b, X \bar{\otimes} Y)] \subseteq \{f[\sigma_e^T(a,X),\ \sigma^T(b,Y)]\} \cup \{f[\sigma^T(a,X),\ \sigma_e^T(b,Y)]\}. \tag{3.4}$$

PROOF. Apply Lemma 2.4 to Lemma 2.3.

THEOREM 3.6. Under the assumptions of the Theorem 3.5., we find the following spectral result:

$$\sigma_B^T[f(a \otimes b,\ X \bar{\otimes} Y)] \subseteq \{f[\sigma_B^T(a,X),\ \sigma^T(b,Y)]\} \cup \{f[\sigma^T(a,X),\ \sigma_B^T(b,Y)]\}.$$

PROOF. The proof follows from an application of the Lemma 2.5 to the Theorem 3.1.

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