Internat. J. Math. & Math. Sci.
VOL. 13 NO. 1 (1990) 139-144



# EIGENVALUES OF THE TIME — DEPENDENT FLUID FLOW PROBLEM I.

**EL-SAYED M. ZAYED and EL-SAYED F. ELSHEHAWEY**

Department of Mathematics and Computer Sciences
United Arab Emirates University
Al-Ain, P.O. BOX 15551
United Arab Emirates

(Received April 19, 1988 and in revised form November 19, 1988)

ABSTRACT. The direct and inverse boundary value problems for the linear unsteady viscous fluid flow through a closed conduit of a circular annular cross-section $\Omega$ with arbitrary time-dependent pressure gradient under the third boundary conditions have been investigated.

KEY WORDS AND PHRASES. Inverse boundary value problems, unsteady viscous flow, closed conduits, annular cross-section.
1980 AMS SUBJECT CLASSIFICATION CODE. 76D05, 31A25.

## 1. INTRODUCTION.

The problems of unsteady fluid flow through a closed conduit of annular cross-section $\Omega$ have been solved by many authors (see, for example Bansal [1], Muller [2], Ojalvo [3], Sanyal [4], Szymanski [5], and Wadhwa [6]). In this paper we shall combine the ideas of Ojalvo [3] and Zayed [7] to solve the third boundary value problem (2.1) - (2.3) mentioned in Section 2, where the direct and inverse problems are considered. For inverse problems, we shall follow Zayed's work [7], which requires the determination of the geometrical properties of the circular annular cross-section $\Omega$ from the complete knowledge of its spectrum, Spec $(\Omega)$, that is

$$\text{Spec }(\Omega) = \{0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots \to \infty \qquad (1.1)$$

using the asymptotic expansion of the spectral function

$$\theta(t) = \operatorname{tr}(e^{-t\nabla^2}) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \qquad (1.2)$$

for small positive t, where $\nabla^2$ is the Laplace operator in $R^2$. Note that the spectrum (1.1) is discrete and consists of all $\lambda$'s such that there exist non-zero solutions $\phi_k$ of $\nabla^2\phi_k = \lambda_k\phi_k$; each $\lambda$ is written in the spectrum of $\Omega$ as a number of times equal to its multiplicity, that is dim $\{\phi_k : \nabla^2\phi_k = \lambda_k\phi_k\}$. Note also that the spectral function (1.2) for the vibrating membranes has been studied by many authors using the heat equation approach (see, for example Zayed [7], Gottlieb [8,9], Kac [10], Sleeman and Zayed [11,12], and Stewartson [13]).

In this paper, the authors have arrived at the fact that the same analysis of the spectral function (1.2) for the vibrating annular membrane case, holds for the case of unsteady viscous flow through conduits, even though the governing partial differential equation is different. This fact enables us to determine the spectral function (1.2) for unsteady viscous flow through conduits of annular cross-section $\Omega$.

## 2. FORMULATION OF THE MATHEMATICAL PROBLEM.

In this section, we discuss the following initial-boundary value problem of a circular annular cross-section $\Omega$ with radii a and b; $b > a$, for unsteady viscous fluid flow when the pressure gradient $F(t)$ is an arbitrary function of time t:

$$\frac{\partial u}{\partial t} = -F(t) + \nu\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})\} \text{ in } \Omega,\ t > 0, \tag{2.1}$$

together with the initial condition

$$u(r,0) = 0,\ a < r < b, \tag{2.2}$$

and the artificial third boundary conditions

$$(\frac{\partial u}{\partial r} + \gamma_1 u)_{r=a} = (\frac{\partial u}{\partial r} + \gamma_2 u)_{r=b} = 0, \tag{2.3}$$

where $\nu$, is the kinematic coefficient of viscosity, and $\gamma_1$ and $\gamma_2$ are positive constants.

Following the method of Ojalvo [3], it is easily seen that, the solution of problem (2.1) - (2.3) can be written in the form

$$u(r,t) = \sum_{k=1}^{\infty} \phi_k(r)\ \psi_k(t) + V(r)\ F(t), \tag{2.4}$$

where $\phi_k(r)$ are the eigenfunctions of the problem

$$-(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr})\ \phi_k(r) = \lambda_k\phi_k(r), \qquad a < r < b, \tag{2.5}$$

$$[\frac{d}{dr}\phi_k(r) + \gamma_1\phi_k(r)]_{r=a} = [\frac{d}{dr}\phi_k(r) + \gamma_2\phi_k(r)]_{r=b} = 0, \tag{2.6}$$

in which $\lambda_k$, $k = 1,2,3,\ldots$ are the corresponding eigenvalues, while $V(r)$ satisfies the problem

$$(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr})\ V(r) = \frac{1}{\nu}, \qquad a < r < b, \tag{2.7}$$

$$[\frac{d}{dr}V(r) + \gamma_1 V(r)]_{r=a} = [\frac{d}{dr}V(r) + \gamma_2 V(r)]_{r=b} = 0. \tag{2.8}$$

If, now, V(r) from (2.7) is expressible in the form

$$V(r) = \sum_{k=1}^{\infty} h_k \phi_k(r), \tag{2.9}$$

then $\psi_k(t)$ should be determined from the initial value problem

$$\left(\frac{d}{dt} + \nu\lambda_k\right) \psi_k(t) = - h_k \frac{dF(t)}{dt}, \tag{2.10}$$

$$\psi_k(0) = - h_k F(0),$$

where $h_k$, $k = 1,2,3,\ldots$ are some constants. Let us now discuss the problem when $\gamma_1$, $\gamma_2$ satisfy the following conditions: (i) $\gamma_1 \gamma_2 >> 1$ (ii) $\gamma_1 >> 1$, $0 < \gamma_2 << 1$ (iii) $0 < \gamma_1 << 1$, $\gamma_2 >> 1$.

3. THE SOLUTION OF THE PROBLEM IN THE CASE $\gamma_1$, $\gamma_2 >> 1$.

In this case, it is straightforward to show that

$$\phi_k(r) = \frac{J_o(r \lambda_k^{\frac{1}{2}})}{[\lambda_k^{\frac{1}{2}} \gamma_1^{-1} J_o'(a \lambda_k^{\frac{1}{2}}) + J_o(a \lambda_k^{\frac{1}{2}})]} - \frac{Y_o(r \lambda_k^{\frac{1}{2}})}{[\lambda_k^{\frac{1}{2}} \gamma_1^{-1} Y_o'(a^{\frac{1}{2}} \lambda_k) + Y_o(a \lambda_k^{\frac{1}{2}})]} \tag{3.1}$$

and

$$V(r) = \frac{1}{4\nu} \left\{(r^2 - a^2 - 2a\gamma_1^{-1}) + \left(\frac{\gamma_1^{-1}}{a} - \ln \frac{r}{a}\right) \times \frac{[2(a\gamma_1^{-1} - \gamma_2^{-1} b) + (a^2 - b^2)]}{\left[\frac{\gamma_1^{-1}}{a} - \frac{\gamma_2^{-1}}{b} + \ln \frac{a}{b}\right]}\right\} \tag{3.2}$$

where $\lambda_k^{\frac{1}{2}}$ are roots of the equation

$$\frac{[\lambda_k^{\frac{1}{2}} \gamma_2^{-1} J_o'(b \lambda_k^{\frac{1}{2}}) + J_o(b \lambda_k^{\frac{1}{2}})]}{[\lambda_k^{\frac{1}{2}} \gamma_1^{-1} J_o'(a \lambda_k^{\frac{1}{2}}) + J_o(a \lambda_k^{\frac{1}{2}})]} - \frac{[\lambda_k^{\frac{1}{2}} \gamma_2^{-1} Y_o'(b \lambda_k^{\frac{1}{2}}) + Y_o(b \lambda_k)^{\frac{1}{2}}]}{[\lambda_k^{\frac{1}{2}} \gamma_1^{-1} Y_o'(a \lambda_k^{\frac{1}{2}}) + Y_o(a \lambda_k^{\frac{1}{2}})]} = 0. \tag{3.3}$$

The function $\Psi_k(t)$ has the form

$$\Psi_k(t) = - h_k \{F(0) + \int_0^t \frac{dF(\xi)}{d\xi} e^{\nu\lambda_k \xi} d\xi\} e^{-\nu\lambda_k t} \tag{3.4}$$

where

$$h_k = \frac{\int_a^b r\, V(r)\phi_k(r)\, dr}{\int_a^b r\, \phi_k^2(r)\, dr}. \tag{3.5}$$

On inserting (3.1), (3.2) and (3.4) into (2.4) we arrive at the solution of our problem when $\gamma_1, \gamma_2 >> 1$.

Using the same analysis of Zayed [7], we deduce after lengthy calculations that the asymptotic expansion of the spectral function $\theta(t)$ for unsteady viscous flow through conduits of annular cross-section $\Omega$ in the case where $\gamma_1, \gamma_2 >> 1$ can be written in the form:

$$\theta(t) \sim \frac{\pi(b^2 - a^2)}{4\pi t} - \frac{2\pi[(a + \gamma_1^{-1}) + (b + \gamma_2^{-1})]}{8(\pi t)^{1/2}} +$$

$$\{\frac{1}{a} + \frac{1}{b} - \frac{\gamma_1^{-1}}{a^2} - \frac{\gamma_2^{-1}}{b^2}\} \frac{(\pi t)^{1/2}}{128} + O(t) \text{ as } t \to 0. \tag{3.6}$$

This is the same form as (4.7) in Zayed [7] for the vibrating annular membrane. Note that if $\gamma_1 = \gamma_2 \to \infty$, we obtain the results of Dirichlet boundary conditions on $r = a$ and $r = b$ (See Wadjwa and Wineinger [6] and Sleeman and Zayed [12]).

## 4. THE SOLUTION OF THE PROBLEM WHEN $\gamma_1 >> 1, 0 < \gamma_2 << 1$.

In this case, we can show that $\phi_k(r)$ has the same form as (3.1), while $V(r)$ takes the form

$$V(r) = \frac{1}{4\nu} \{(r^2 - a^2 - 2a\, \gamma_1^{-1}) + (\frac{\gamma_1^{-1}}{a} - \ln \frac{r}{a})\} \times$$

$$\frac{[\gamma_2(2a\gamma_1^{-1} + a^2 - b^2) - 2b]}{[\gamma_2 (\frac{\gamma^{-1}}{a} + \ln \frac{a}{b}) - \frac{1}{b}]}\}, \tag{4.1}$$

where $\lambda_k^{\frac{1}{2}}$ are roots of the equation

$$\frac{[\lambda_k^{\frac{1}{2}} J_o' (b\, \lambda_k^{\frac{1}{2}}) + \gamma_2 J_o(b\, \lambda_k^{\frac{1}{2}})]}{[\lambda_k^{\frac{1}{2}} \lambda_1^{-1} J_o' (a\, \lambda_k^{\frac{1}{2}}) + J_o(a\, \lambda_k^{\frac{1}{2}})]} -$$

$$\frac{[\lambda_k^{\frac{1}{2}} Y_o' (b\, \lambda_k^{\frac{1}{2}}) + \gamma_2 Y_o(b\, \lambda_k^{\frac{1}{2}})]}{[\lambda_k^{\frac{1}{2}} \gamma_1^{-1} Y_o'(a\, \lambda_k^{\frac{1}{2}}) + Y_o(a\, \lambda_k^{\frac{1}{2}})]} = 0. \tag{4.2}$$

On inserting (3.1), (4.1) and (3.4) into (2.4) we arrive at the solution of our problem in the case where $\gamma_1 \gg 1$, $0 < \gamma_2 \ll 1$.

In this case, one can show after some reduction that the asymptotic expansion of $\theta(t)$ for unsteady viscous flow through conduits can be written as:

$$\theta(t) \sim \frac{\pi(b^2 - a^2)}{4\pi t} + \frac{2\pi[b - (a + \gamma_1^{-1})]}{8(\pi t)^{\frac{1}{2}}} - \gamma_2 b +$$

$$\{\frac{1}{a} + \frac{7}{b} - 32\,\gamma_2 + 64\,\gamma_2^2\, b - \frac{\gamma_1^{-1}}{a^2}\}\,\frac{(\pi t)^{\frac{1}{2}}}{128} + O(t) \quad \text{as } t \to 0, \qquad (4.3)$$

which has the same form as (4.6) in Zayed [7] for the vibrating annular membrane. If $\gamma_1 \to \infty$ and $\gamma_2 = 0$, we obtain the results of Dirichlet boundary condition on $r = a$, and Neumann boundary condition on $r = b$. (See Sleeman and Zayed [12]).

REMARK 4.1. The solution of our problem in the case where $0 < \gamma_1 \ll 1$, $\gamma_2 \gg 1$ can be deduced from the previous case with the interchanges $\gamma_1 \longleftrightarrow \gamma_2$ and $a \longleftrightarrow b$.

Finally, we close this paper with the remark that the expansions of $\theta(t)$ determine the geometry of $\Omega$ (area, length of the boundary, number of holes, curvatures of the boundary, ...) for unsteady viscous flow through conduits, which are very similar to those obtained in Zayed [7] for the vibrating annular membrane case.

ACKNOWLEDGEMENT. The authors are grateful to the referee for his suggestions and comments. They also express their sincere thanks to the Editor Dr. Lokenath Debnath for his cooperation.

## REFERENCES

1. BANSAL, J.L, Viscous Fluid Dynamics, Oxford & IBH. Publishing Co., 1977.
2. MÜLLER, V.W., Zum Problem des Anlaufstromung Einger Flussigkeit in Geraden Rohr mit Kreisingund Kreisquerschmitt, Z.A.M.M. 16, (1935), 227-238.
3. OJALVO, I.U., An Extension of Separation of Variables for Time Dependent Excitations, Quart. Appl. Math. 20, (1962), 390-394.
4. SANYAL, L., The Flow of a Viscous Fluid in a Circular Tube Under Pressure Gradient Varying Exponentially with Time, Indian. J. Phys. 30, (1956), 57-61.
5. SZYMANSKI, P., Sur L'ecoulement non Permanent du Fluide Vis - quex dans le Tuyau Applied Mech. Congress 3rd., Paris, 1(1930), 249-254.
6. WADHWA, Y.D. and WINEINGER, T.W., Linear Time Dependent Fluid flow Problems, Quart. Appl. Math. XXVI, (1968), 1-9.
7. ZAYED, E.M.E., Eigenvalues of the Laplacian for the Third Boundary Value Problem J. Austral. Math. Soc. Ser.B 29, (1987), 79-87.

8. GOTTLIEB, H.P.W., Hearing the Shape of Annular Drum, J. Austral. Math. Soc. B 24(1983), 435-438.

9. GOTTLIEB, H.P.W., Eigenvalues of the Laplacian with Neumann Boundary Conditions, J. Austral. Math. Soc. Ser. B 26, (1985), 293-309.

10. KAC, M., Can one Hear the Shape of a Drum?, Amer. Math. Monthly 73(4) Part II, (1966), 1-23.

11. SLEEMAN, B.D. and ZAYED, E.M.E., An Inverse Eigenvalue Problem for a General Convex Domain, J. Math. Anal. App. 94(1), (1983), 78-95.

12. SLEEMAN, B.D. and ZAYED, E.M.E., Trace Formulae for the Eigenvalues of the Laplacian, J. Applied. Math. phys (ZAMP), 35, (1984), 106-115.

13. STEWARTSON, K. and WAECHTER, R.T., On Hearing the Shape of a Drum: Further Results, Proc. Camb. Phil. Soc. 69, (1971), 353-363.