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FOURIER TRANFORMS IN GENERALIZED FOCK SPACES

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ABSTRACT, A classical Fock space consists of functions of the form,

$$\Phi \longleftrightarrow (\phi_o, \phi_1, \ldots, \phi_q, \ldots),$$

where $\phi_o \varepsilon$ C and $\phi_q \varepsilon L^2 (\mathbb{R}^{3q})$, $q \geq 1$. We will replace the ϕ_q, $q \geq 1$ with q-symmetric rapid descent test functions within tempered distribution theory. This space is a natural generalization of a classical Fock space as seen by expanding functionals having generalized Taylor series. The particular coefficients of such series are multilinear functionals having tempered distributions as their domain. The Fourier transform will be introduced into this setting. A theorem will be proven relating the convergence of the tranform to the parameter, s, which sweeps out a scale of genralized Fock spaces.

KEY WORDS AND PHRASES. Generalized Fock Sapces, tempered distibutions, Fourier transforms, and rapid descent test functions.

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1. INTRODUCTION.

Rapid descent test functions, $S(R^q)$, and their dual tempered distributions, $S'(R^q)$, are excellent spaces to do the analysis of the Fourier transform (Bogolubov and Logunov [1], Constantinescu [2], Freidman [3], Gelfand and Shilov [4], and Lighthill [5]). The classical Fourier transform analysis examines spaces having test functions defined on a finite number of independent variables. By this we mean the independent variables of a rapid descent test function, $\phi(t_1,\ldots t_q)$, belonging to a q-dimensional Euclidean space. This paper will indicate a method that will enjoy the property that the number of independent variables becomes infinite, that is in some sense the dimension, $q \to \infty$. The need for this analysis is essential in advanced physics. An infinite number of particles are described by state vectors in a Fock space. The classical results are developed in a Hilbert space. Traditionally the Lebseque integrable functions, $L^p(R^q)$, are implemented in the construction of a direct sum of these spaces. However, when you want to describe a frequency of a particle the Fourier transform must be studied. This presents a significant problem since the kernel, $e^{-2\pi i t w}$, does not belong to any $L^p(R^q)$ space. This kernel problem is solved in tempered distribution theory (Constantinescu [2], Gelfand and Shilvo [4], Lighthill [5], and Zemanian [6]) but the infinite number of variables problem still remains. This paper will implement tempered distributions together with a holomorphic functional theory developed in Schmeelk [7-10] to solve the infinite number of variables problem.

We briefly recall in $S(R^q)$ and $S'(R^q)$ the Fourier transforms are respectively defined as

$$(F\phi)(w_1 \ldots w_q) \triangleq \int_{R^q} \exp - 2\pi[t_1 w_1 + \ldots + t_q w_q \; \phi(t_1 \ldots t_q) dt_1 \ldots dt_q$$

and

$$\langle F\, F(w_1 \ldots w_q),\ \phi(t_1 \ldots t_q) \rangle \triangleq \langle F(w_1 \ldots w_q),\ F(\phi(t_1 \ldots t_q)) \rangle$$

for all $\phi(t_1 \ldots t_q)$ a $\varepsilon\ S(R^q)$ and all $\mathbf{F}(w_1 \ldots w_q)\ \varepsilon\ S'(R^q)$. The advantages of $S(R^q)$ and $S'(R^q)$ are many but the fundamental result is that the Fourier transform exists a homeomorphism and has the appropriate derivative - multiplication property. This paper will not include a survey of the many Fourier transform properties which are contained in Constantinescu [2], Friedman [3], Zemanian [6], Bracewell [11], Gonzalez and Wintz [12], and Papoulis [13].

We will extend the Fourier transform into generalized Fock spaces. The principle result will be the existence of the transform in the scale of Frechet spaces

$\Gamma^{pB} = \bigcup_{s>1} \Gamma^{p,sB}$ and its corresponding dual, $(\Gamma^{pB})'$. A comprehensive examination of these spaces are contained in Schmeelk [7-10]. We will only review these spaces in sections 2 and 3.

2. THE SPACE, $\Gamma^{pB} = \bigcup_{s>1} \Gamma^{p,sB}$

For each $s > 1$, the space $\Gamma^{p,sB}(p > 1,\ B = \{B_i\}_{i=0}^{\infty},\ B_i > B_j,\ j > i)$, is called an infinite dimensional Fock space. Then p and B_i, $i > 0$ are all real numbers. These spaces are topological spaces of complex valued functionals on $S'(R; \mathbb{C})$, the space of complex valued distributions. The functionals which are members of $\Gamma^{p,sB}$ are all

$C^{\infty}(S'(R);\mathbb{C})$. The complex or real valued functionals enjoy similar properties. The

real valued functionals which are members of Γ^{pB} are developed in Schmeelk [8].

We also require if $\Phi\ \varepsilon\ \Gamma^{p,sB}$, then

$$\Phi(x) = \sum_{q=0}^{\infty} a_q x^q = \sum_{q=0}^{\infty} a_q \; [x,\ldots,x] \tag{2.1}$$

where $a_0\ \varepsilon\ \mathbb{C}$ and a_q, $q > 1$ are q-multilinear symmetric continuous functionals on the space, $S'(R)\times\ldots\times S'(R)$, (q copies, $q > 1$) to $\mathbb{C}$. We identify for each $\Phi\ \varepsilon\ \Gamma^{p,sB}$ the associated Fock state vector,

$$\Phi \leftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_q \\ \vdots \end{bmatrix} \tag{2.2}$$

We equip our infinite dimensional Fock vector space with the following increasing sequence of norms:

$$|||\Phi|||_{sB_m} = \sup_q \frac{||a_q||_m q!^{1/p}}{(sB_m)^q}, \quad m = 0,1,\ldots \tag{2.3}$$

where

$$||a_q||_m = \sup_{||x||_{-m} < 1} |a_q \, x^q|, \quad x \in S'(R), \; m = 0,1,\ldots \tag{2.4}$$

with

$$||x||_{-m} = \sup_{||\phi||_m < 1} |\langle x, \phi\rangle| \;, \quad \phi \in S(R), \; m = 0,1\ldots \tag{2.5}$$

and

$$||\phi||_m = \sup_{\substack{0 < \alpha_i < m \\ 1 < i < q \\ (t_1,\ldots,t_q) \in R^q}} M_m(t_1,\ldots,t_q)|D^\alpha \phi(t_1\ldots t_q)| \tag{2.6}$$

where

$$M_m(t_1,\ldots,t_q) \triangleq [(1+(2\pi t_1)^2) \ldots (1 + (2\pi t_q)^2)]^m \tag{2.7}$$

and

$$D^\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_q}}{\partial t_1^{\alpha_1}\ldots \partial t_q^{\alpha_q}} .$$

The norms defined in expression (2.6) using the functions $M_m(t_1,\ldots,t_q)$ so defined generate a sequence of norms equivalent to the sequence of norms implementing the functions, $M'_m(t_1,\ldots,t_q) = [(1+|t_1|) \ldots (1+|t_q|)^m$, [2,3].

It was proven in reference [10] that each real valued functional, $\Phi \in \Gamma^{p,sB}$, has a kernel representation which remains valid for complex valued functionals. This representation is as follows,

$$\Phi \leftrightarrow \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_q(t_1,\ldots,t_q) \\ \vdots \end{bmatrix} \tag{2.8}$$

where $\phi_0 = a_0$ and $\phi_q(t_1,\ldots,t_q)$ are symmetric complex valued rapid descent test functions, $S_+(R^q)$ satisfying,

$$|||\Phi|||'_{sB_m} = \sup_q \frac{||\phi_q||_m q!^{1/p}}{(s\;B_m)}, \quad m = 0,1,\ldots \tag{2.9}$$

where

$$||\phi_q||_m = \sup_{\substack{0 \leq \alpha_i \leq m \\ 1 \leq i \leq q \\ (t_1,\ldots t_q) \in R^q}} M_m(t_1,\ldots,t_q)|D^\alpha \phi(t_1\ldots t_q)|.$$

The representation for Φ given in expression (2.8) enjoys the standard square summable property often times postulated for Fock functionals as seen by the following theorem.

THEOREM 2.10. Given a $\Phi \in \Gamma^{p,sB}$, its kernel representation given in expression (2.8) satisfies

$$||\Phi||_{L_2} = |\phi_0|^2 + \sum_{q=1}^{\infty} \int_{R^q} |\phi_q(t_1,\ldots,t_q)|^2 dt_1,\ldots,dt_q < \infty.$$

PROOF. Clearly the constant, $|\phi_0|^2$, does not contribute to the convergence problem of the result of the theorem. Also since $\Phi \in \Gamma^{p,sB}$, then by the requirement given in expression (2.9) there must exist a sequence of positive constants, $\{C_m\}_{m=0}^{\infty}$, such that

$$\sup_q ||\phi_q||_m \leq \frac{C_m(sB_m)^q}{q!^{1/p}}$$

for all q and $m = 0, 1,\ldots$. We now consider a partial sum,

$$\sum_{q=1}^{q_0} \int_{R^q} |\phi_q(t_1,\ldots,t_q)|^2 \, dt_1\ldots dt_q$$

$$= \frac{\left[\sum_{q=1}^{q_0} \int_{R^q} \frac{M_{2m}(t_1,\ldots,t_q)}{M_{2m}(t_1,\ldots,t_q)} |\phi_q(t_1,\ldots,t_q)|^2 dt_1\ldots dt_q\right] q!^{1/p}(sB_m)^q}{q!^{1/p}(sB_m)^q}$$

$$\leq \frac{\sum_{q=1}^{q_0} ||\phi_q||_m^2 \left[\int_{R^q} \frac{1}{M_{2m}(t_1,\ldots,t_q)} dt_1\ldots dt_q\right] q!^{1/p}(sB_m)^q}{q!^{1/p}(sB_m)^q}$$

$$\leq \sum_{q=1}^{q_0} \frac{||\phi_q||_m^2 \, \pi^q \, q!^{1/p}(sB_m)^q}{q!^{2/p}(sB_m)^q}$$

$$= \sum_{q=1}^{q_0} \frac{||\phi_q||_m q!^{1/p}}{(sB_m)^q} \frac{||\phi_q||_m \, \pi^q(sB_m)^q}{q!^{1/p}}$$

$$\leq |||\Phi|||'_{sB_m} \sum_{q=1}^{q_0} \frac{||\phi_q||_m^2 \, \pi^q(sB_m)^q}{q!^{1/p}}$$

$$\leq |||\Phi|||'_{sB_m} \sum_{q=1}^{q_0} \frac{C_m \pi^q (sB_m)^{2q}}{q!^{2/p}} < \infty . \tag{2.11}$$

Since expression (2.11) converges for any q_0, the result follows.

3. THE FOURIER TRANSFORM IN Γ^{pB}.

DEFINITION 3.1. The Fourier transform $\mathcal{F}$ on $\Phi \in \Gamma^{pB}$ is defined as follows,

$$\mathcal{F} : \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_q \\ \vdots \end{bmatrix} \begin{bmatrix} \phi_0 \\ \int_{R_1} \exp[-2\pi\, i\; t_1 w_1]\phi_1(t_1)dt_1 \\ \vdots \\ \int_{R^q} \exp[-2\pi\, i(t_1w_1+\dots+t_qw_q)]\phi_q(t_1,\dots,t_q)dt_1,\dots,dt_q \\ \vdots \end{bmatrix}.$$

LEMMA 3.2. $\mathcal{F}(\Phi)$ is well defined for every $\Phi \in \Gamma^{p,sB}$ and moreover

$$\phi_0 + \sum_{q=1}^{\infty} \int_{R^q} \exp\,[-2\pi\, i(t_1w_1+\dots+t_qw_q)]\phi_q(t_1,\dots,t_q)dt_1\dots dt_q < \infty.$$

PROOF. $\Phi \in \Gamma^{pB}$ implies $\Phi \in \Gamma^{p,sB}$ for some $s > 1$.
We then have

$$\phi_0 + \Big| \sum_{q=1}^{\infty} \int_{R^q} \Big[\exp - 2\pi\, i\, [t_1w_1+\dots+t_qw_q]\Big]\phi_q(t_1,\dots,t_q)dt_1\dots dt_q\Big| < \infty.$$

$$< \phi_0 + \Big|\sum_{q=1}^{\infty} \int_{R^q} \frac{M_1(t_1,\dots,t_q)}{M_1(t_1,\dots,t_q)}\,|\phi_q(t_1,\dots,t_q)|dt_1,\dots,dt_q$$

$$< |\phi_0| + \sup_{(t_1,\overset{q}{\dots},t_q)\,\in\, R^q} M_1(t_1,\dots,t_q)|\phi_q(t_1,\dots,t_q)|\sum_{q=1}^{\infty} \int_{R^q} \frac{1}{M_1(t_1,\dots,t_q)}dt_1\dots dt_q$$

$$< |\phi_0| + \sum_{q=1}^{\infty} \frac{\pi^q ||\phi_q||_1 q!^{1/p}(sB_m)^q}{q!^{1/p}(sB_m)^q}$$

$$< |\phi_0| + |||\Phi|||'_{sB_m} \sum_{q=1}^{\infty} \frac{\pi^q(sB_m)^q}{q!^{1/p}} < \infty.$$

THEOREM 3.4. The Fourier transform is a linear continuous transformation on Γ^{pB} to Γ^{pB}.

PROOF. Since $\Gamma^{pB} = \bigcup_{s>1} \Gamma^{p,sB}$, we consider the Fourier transform on the space, $\Gamma^{p,sB}$, to the space, $\Gamma^{p,s'B}$, where $s' > s\pi$. We have for any norm $|||\cdot|||_{s'B_m}$, the following,

$$|||\mathcal{F}\phi|||_{s'B_m} = \frac{||\int_{R^q} \exp[-2\pi\, i(t_1w_1+\dots+t_qw_q)]\phi(t_1,\dots,t_q)dt_1\dots,dt_q||_m q!^{1/p}}{(s'B_m)^q}$$

$$= \sup_{\substack{q \\ 0 \le \alpha_i \le m \\ 1 \le i \le q \\ (w_1,\dots,w_q) \in R^q}} M_m(w_1,\dots,w_q) \frac{\left|D^\alpha \int_{R^q} \exp([-2\pi i(t_1w_1+\dots+t_qw_q)]\phi(t_1,\dots,t_q)dt_1\dots dt_q\right| q!^{1/p}}{(s'B_m)^q}$$

$$= \sup_{\substack{q \\ 0 \le \alpha_i \le m \\ 1 \le i \le q \\ (w_1,\dots,w_q) \in R^q}} M_m(w_1,\dots,w_q)$$

$$\frac{\left|\int_{R^q} (-2\pi it_1)^{\alpha_1}\dots(-2\pi it_q)^{\alpha_q}\exp[-2\pi i(t_1w_1+\dots+t_qw_q)]\phi(t_1,\dots,t_q)dt_1\dots dt_q\right| q!^{1/p}}{(s'B_m)^q}$$

$$\le \sup_{\substack{q \\ 0 \le \alpha_i \le m \\ 1 \le i \le q \\ (w_1,\dots,w_q) \in R^q}} \frac{M_m(w_1,\dots,w_q) \int_{R^q} M_m(t_1,\dots,t_q)\left|\phi(t_1,\dots,t_q)\right|dt_1\dots dt_q \; q!^{1/p}}{(s'B_m)^q}$$

$$\sup_{\substack{q \\ 0 \le \alpha_i \le m \\ 1 \le i \le q}} \frac{\int_{R^q} M_{2m}(t_1,\dots,t_q)|\phi(t_1,\dots,t_q)|dt_1\dots dt_q}{(s'_mB^q)} q!^{1/p}$$

$$\sup_{\substack{q \\ 0 \le \alpha_i \le m \\ 1 \le i \le q}} \frac{\int_{R^q} \frac{M_{2m+1}(t_1,\dots,t_q)}{M_1(t_1,\dots,t_q)}\left|\phi(t_1,\dots,t_q)\right|dt_1\dots dt_q}{(s'B_m)^q} q!^{1/p}$$

$$= \sup_{\substack{q \\ 0 \le q_i \le m \\ 1 \le i \le q \\ (t_1,\dots,t_q) \in R^q}} \frac{M_{2m+1}(t_1,\dots,t_q)\left|\phi(t_1,\dots,t_q)\right| \; \pi^q q!^{1/p}}{(s'B_m)^q}$$

$$= \sup_{\substack{q \\ 0 < \alpha_i < m \\ 1 < i < q \\ (t_1,\dots,t_q) \in R^q}} \frac{M_{2m+1}(t_1,\dots,t_q)\left|\phi(t_1,\dots,t_1)\right| q!^{1/p}}{(s\ B_{2m+1})^q} \frac{(sB_{2m+1})_\pi^q q}{(s'B_m)^q}$$

$$< \sup_q \frac{||\phi||_{2m+1} q!^{1/p}}{(sB_{2m+1})^q} \left(\frac{(sB_{2m+1})\pi}{s'B_m}\right)^q . \tag{3.5}$$

Noting that $B_{2m+1} < B_m$ and $s' > s\pi$ implies expression (3.5) is finite.

4. THE FOURIER TRANSFORM ON $(\Gamma^{pB})'$

In a previous paper [9], it was shown that the dual of $\Gamma^{p,sB}$ denoted $(\Gamma^{p,sB})'$ is the union of sets of the form,

$$(\Gamma^{p,sB}_{-m}) = \{(F_0, F_1,\dots,F_q,\dots): F_0 \in C , \tag{4.1}$$

$$F_q \in S'_R + (R^q), \sum_{q=0}^{\infty} ||F_q||_{-m}(sB_m)^q q!^{-1/p} < \infty\}.$$

The generalized Fock dual functionals described in expression (4.1) can also be considered as sequences where the F_q are symmetric tempered distributions all having rank $< m$. We also note if $\Phi \in \Gamma^{p,sB}$ and $F \in (\Gamma^{p,sB})$, then the evaluation of F at Φ is denoted as

$$< < F, \Phi > > \triangleq \sum_{q=0}^{\infty} F_q, \phi_q >. \tag{4.2}$$

EXAMPLE. 4.3 All the sets, $(\Gamma^{p,sB}_{-m})$, contain the generalized Fock Dirac functional,

$$\delta \iff \begin{bmatrix} 1 \\ \delta \\ \delta \otimes \delta \\ \vdots \\ \delta \otimes \delta \otimes \dots \otimes \delta \end{bmatrix} \tag{4.3}$$

where $\delta \otimes \delta \otimes \dots \otimes \delta$ is the tensor product of q copies of the Dirac delta functional [3]. We immediately verify that

$$||\delta||_{-(sB_m)} \triangleq \sum_{q=0}^{\infty} ||F_q||_{-m}(sB_m)^q\ q!^{-1/p}$$

$$= \sum_{q=0}^{\infty} ||\delta \otimes \dots \otimes \delta||_{-m}(sB_m)^q\ q!^{-1/p}$$

$$< \sum_{q=0}^{\infty} 1 \cdot (sB_m)^q\ q!^{-1/p} < \infty.$$

DEFINITION 4.4. The Fourier transform on the space $(\Gamma^{pB})'$ is defined as

$$< < \mathfrak{F} F, \Phi > > \triangleq << F, \mathfrak{F}\Phi > >.$$

EXAMPLE 4.4. We compute the Fourier transform of

$$\delta^{(k)}(t - \tau) \quad <==> \quad \begin{bmatrix} 1 \\ \delta^{(k)}(t_1 - \tau_1) \\ \vdots \\ \delta^{(k)}(t_1 - \tau_1) \vdots \delta^{(k)}(t_2 - \tau_2) \\ \\ \delta^{(k)}(t_1 - \tau_1) \ldots \delta^{(k)}(t_q - \tau_q) \\ \vdots \end{bmatrix} \tag{4.4}$$

It suffices to consider the q^{th} component,

$$< \mathfrak{F}\, \delta^{(k)}(t_1 - \tau_1) \otimes \ldots \otimes \delta^{(k)}(t_q - \tau_q), \ \phi(w_1 \ldots w_q) >$$

$$\triangleq < \delta^{(k)}(t_1 - \tau_1) \otimes \ldots \otimes \delta^{(k)}(t_q - \tau_q), \mathfrak{F}\phi(w_1 \ldots w_q) >$$

$$= < \delta^{(k)}(t_1 - \tau_1) \otimes \ldots \otimes \delta^{(k)}(t_q - \tau_q), \ \int_{R^q} \exp[-2\pi i(w_1 t_1 + \ldots + w_q t_q)]\, \phi(w_1 \ldots w_q)\, dw_1 \ldots dw_q >$$

$$= (-1)^{qk} < \delta(t_1 - \tau_1) \otimes \ldots \otimes \delta(t_q - \tau_q), \ \frac{d^{k+k+\ldots k}}{dt_1^k \ldots dt_q^k} \int_{R^q} \exp[-2\pi i(w_1 t_1 + \ldots + w_q t_q)]\, \phi(w_1 \ldots w_q)\, dw_1 \ldots dw_q > \tag{4.5}$$

$$= < \delta(t_1 - \tau_1) \otimes \ldots \otimes \delta(t_q - \tau_q), \ \int_{R^q} (2\pi i w_1)^k \ldots (2\pi i w_q)^k \exp[-2\pi i(w_1 t_1 + \ldots + w_q t_q)]\, \phi(w_1 \ldots w_q)\, dw_1 \ldots dw_q >$$

$$= \int_{R^q} (2\pi i w_1)^k \ldots (2\pi i w_q)^k \exp[-2\pi i(w_1 \tau_1 + \ldots + w_q \tau_q)]\, \phi(w_1 \ldots w_q)\, dw_1 \ldots dw_q$$

$$= <(2\pi i w_1)^k\, e^{-2\pi i w_1 \tau_1} \otimes (2\pi i w_2)^k\, e^{-2\pi i w_2 \tau_2} \ldots \otimes (2\pi i w_q)^k\, e^{-2\pi i w_q \tau_q}, \ \phi(w_1 \ldots w_q)>$$

where $(2\pi i w_n)^k\, e^{-2\pi i w_n \tau_n}$, $(1 < n < q)$ is being considered as a regular tempered distribution. In summary we have

$$(\delta^{(k)}(t - \tau)) \leftrightarrow \begin{bmatrix} 1 \\ (2\pi i w_1)^k\, e^{-2\pi i w_1 \tau_1} \\ \vdots \\ \\ (2\pi i w_1)^k (2\pi i w_2)^k \ldots (2\pi i w_q)^k\, e^{-2\pi i [w_1 \tau_1 + \ldots + w_q \tau_q]} \end{bmatrix}. \tag{4.6}$$

It is clear that any q'th entry in expression (4.6) does not belong to $L^2(R^q)$ since clearly

$$\left|(2\pi i w_1)^k \ldots (2\pi i w_q)^k \, e^{-2\pi i[w_1\tau_1+\ldots+w_q\tau_q]}\right| =$$

$$[(2\pi w_1) \ldots (2\pi w_q)]^k \text{ is not integrable over } R^q.$$

However, the expression given in line (4.5) does belong to the $(\Gamma^{pB})'$ space since

$$\begin{aligned}
&|||\quad (\delta^{(k)}(t - \tau))|||_{-sB_m} \\
&= \sum_{q=0}^{\infty} ||F_q||_{-m}(sB_m)^q \, q!^{-1/p} \\
&= \sum_{q=0}^{\infty} ||(2\pi i w_1)^k \ldots (2\pi i w_q)^k \, e^{-2\pi i[w_1\tau_1\ldots w_q\tau_q]}||_{-m}(sB_m)^q q!^{-1/p} \\
&< \sum_{q=0}^{\infty} 1(sB_m)^q \, q!^{-1/p} < \infty.
\end{aligned}$$

EXAMPLE 4.7. In a similar computation it can be shown that

$$\mathfrak{F}: \begin{bmatrix} 1 \\ \delta \\ \vdots \\ \delta \otimes \delta \otimes \ldots \otimes \delta \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$$

and again the Fourier transform is a member of every set, $(\Gamma_{-m}^{p,sB})$. It should be noted that other spaces such as distributions of exponential growth [3] offer some technical achievements that increase the space of Fourier transformable functions. However, we wanted to relate our results to our specialized scales of Frechet spaces developed in Schmeelk [7-10] and Schwartz [15].

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