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A COMMUTATIVITY THEOREM FOR LEFT s-UNITAL RINGS

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ABSTRACT. In this paper we generalize some well-known commutativity theorems for associative rings as follows: Let R be a left s-unital ring. If there exist non-negative integers $m > 1$, $k \geq 0$, and $n \geq 0$ such that for any x,y in R, $[x^k y - x^n y^m, x] = 0$, then R is commutative.

KEY WORDS AND PHRASES. Associative ring, s-unital ring, ring with unity, commutativity of rings.
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1. INTRODUCTION.

Throughout this paper, R denotes an associative ring (may be without unity), Z(R) represents the center of R, N the set of all nilpotent elements of R, N' the set of all zero divisors of R, and C(R) the commutator ideal of R. For any $x,y \in R$, we write $[x, y] = xy - yx$.

As stated in Hirano and Kobayashi [1] and Quadri and Khan [2], a ring R is called left (resp. right) s-unital if $x \in Rx$(resp. $x \in xR$) for each $x \in R$. Further, R is called s-unital if it is both left as well as right s-unital, that is $x \in Rx \cap xR$, for every $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R, there exists an element $e \in R$ such that $ex = e = xe$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a pseudo-identity (resp. pseudo left identity or pseudo right identity) of F in R.

The famous Jacobson theorem stated that any ring R in which for every $x \in R$ there exists a positive integer $n = n(x) > 1$ such that $x^n = x$ is commutative, has been generalized as follows: if for each pair $x,y \in R$ there exists a positive integer $n = n(x,y) > 1$ such that $(xy)^n = xy$, then R is commutative. Recently, Ashraf and Quadri [3] investigated the commutativity of the rings satisfying the following condition: For all $x,y \in R$ there is a fixed integer $n > 1$ such that $x^n y^n = xy$. In fact, Ashraf and Quadri [3] have generalized the above results as follows: Let R be a ring with unity 1 in which $[xy - x^n y^m, x] = 0$, for all x,y in R and fixed integers $m > 1$, $n \geq 1$. Then R is commutative.

The objective of this paper is to generalize the above mentioned results. Indeed, we prove the following:

THEOREM 1.1. Let R be a left s-unital ring with the property that

(P) "there exist positive integers $m > 1$, $k \geq 0$, and $n \geq 0$ such that $[x^k y - x^n y^m, x] = 0$ for all $x, y \in R$".

Then R is commutative.

We notice that the property (P) of the above theorem can be rewritten as follows:

$$x^k[x,y] = x^n[x,y^m]. \quad (1.1)$$

Thus for any integer $t \geq 1$, we have

$$\begin{aligned} x^{tk}[x,y] &= x^{(t-1)k}\,(x^k[x,y]) \\ &= x^{(t-1)k}\,(x^n[x,y^m]) \\ &= x^{(t-2)k}\,(x^n x^k[x,y^m]) \\ &= x^{(t-2)k}(x^{2n}[x,y^{m^2}]) \\ &= \ldots \end{aligned}$$

By repeating the above process and using (1.1), we get

$$x^{tk}[x,y] = x^{tn}\,[x,y^{m^t}]. \quad (1.2)$$

2. PRELIMINARY LEMMS.

In preparation for the proof of the above theorem we start by stating without proof the following well-known Lemmas.

LEMMA 2.1 (Bell [4, Lemma]). Suppose x and y are elements of a ring R with unity 1, satisying $x^m\, y = 0$ and $(1+x)^m\, y = 0$ for some positive integer m. Then $y = 0$.

LEMMA 2.2. (Bell [5, Lemma 3]). Let x and y be in R. If $[x,y]$ commutes with x, then $[x^k, y] = k\, x^{k-1}[x,y]$ for all positive integers k.

LEMMA 2.3 ([2, Lemma 3]). Let R be a ring with unity 1. If $(1 - y^k)x = 0$, then $(1 - y^{km})\; x = 0$, for any positive integers m and k.

LEMMA 2.4 ([1, Proposition 2]). Let f be a polynomial in non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:

1) For any ring R satisfying $f = 0$, C(R) is a nil ideal.
2) Every semiprime ring satisfying $f = 0$ is commutative.
3) For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.

3. MAIN RESULTS.

The following lemmas will be used in the proof our main theorem.

LEMMA 3.1. Let R be a left s-unital ring satisfying $[x^k y - x^n y^m, x] = 0$, for each $x, y \in R$ and any non-negative integers k,n and $m > 1$. Then R is s-unital.

PROOF. Let $u \in N$. Then for any $x \in R$, and $t \geq 1$, we have $x^{tk}[x,u] = x^{tn}[x,u^{m^t}]$.

For sufficiently large t, we have $x^{tk}[x,u] = x^{tn}[x,u^{m^t}] = 0$, since u is nilpotent and $u^{m^t} = 0$.

Since, R is a left s-unital ring, we have u = eu for some $e \in R$. But $e^{tk}[e,u] = 0$ which gives u = ue. For arbitrary $x \in R$, there exists $e' \in R$ such that $e'x = x$. Further, for some $e'' \in R$, we have $e''e' = e'$. Thus $e''x = x$ and $(x - xe'')^2 = 0$, that is $(x - xe'') \in N$. Since $e'(x - xe'') = x - xe''$, we have $x - xe'' = (x - xe'')e' = 0$ which implies $x = xe''$. Hence R is s-unital.

LEMMA 3.2. Let R be a ring with unity 1 which satisfies the property (P). Then every nilpotent element of R is central.

PROOF. Let u be a nilpotent element of R. Then by (1.2) for any $x \in R$ and a positive integer $t > 1$ we have $x^{tk}[x,u] = x^{tn}[x,u^{m^t}]$. But $u \in N$, then $u^{m^t} = 0$, for sufficiently large t, and hence $x^{tk}[x,u] = 0$ for each $x \in R$. By Lemma 2.1 this yields $[x,u] = 0$, which forces $N \subseteq Z(R)$. Thus every nilpotent element of R is central.

LEMMA 3.3. Let R be a ring with unity 1 which satisfies the property (P), then $C(R) \subseteq Z(R)$.

PROOF. Now, R satisfies $[x^ky - x^ny^m, x] = 0$ for all $x,y \in R$, which is a polynomial identity with relatively prime integral coefficients. Let $x = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we find that no ring of 2 x 2 matrices over GF(p), p a prime, satisfies the above polynomial identity. Hence by Lemma 2.4, the commutator ideal C(R) of R is nil. Therefore $C(R) \subseteq Z(R)$.

In view of Lemma 3.3 it is guaranteed that the conclusion of Lemma 2.2 holds for each pair of elements x,y in a ring R with unity 1 which satisfies the property (P).

LEMMA 3.4. Let R be a ring with unity 1, satisfying (P), then R is commutative.

PROOF. Since R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i each of which as a homomorphic image of R satisfies the property (P) placed on R, R itself can be assumed to be a subdirectly irreducible ring. Let S be the intersection of all its non-zero ideals, then $S \neq (0)$.

Let k = n = 0, in (1.1). Then we have $[x,y] = [x,y^m]$ or $[x,y - y^m] = 0$ for all $x,y \in R$. This forces commutativity of R by Herstein [6, Theorem 18]. Next, we assume k = n = 1 in (1.1). Then replacing x by (x + 1), we obtain $[x,y] = [x,y^m]$, for every $x,y \in R$, and again by [6, Theorem 18] R is commutative. If (k,n) = (1,0), then $x[x,y] = [x,y^m]$ and hence by replacing x by (x + 1) we have $[x,y] = 0$, for all $x,y \in R$. Therefore R is commutative. If (k,n) = (0,1), then $[x,y] = x[x,y^m]$, and hence by replacing x by x + 1 we have $[x,y^m] = 0$, for all $x,y \in R$. Thus $[x,y] = x[x,y^m] = 0$ for all $x,y \in R$. Thus R is commutative.

Next, we suppose that k > 1, and n > 1. Let $q = 2^m-2$ be a positive integer. Then by (1.1) we have

$$\begin{aligned} q\, x^k [x,y] &= 2^m x^k[x,y] - 2\, x^k[x,y] \\ &= 2^m x^n[x,y^m] - x^k[x,2y] \\ &= x^n[x, (2y)^m] - x^n[x,(2y)^m] \\ &= 0, \end{aligned}$$

that is, $qx^k\,[x,y] = 0$. By replacing x by (x + 1) and using Lemma 2.1, this yields $q[x,y] = 0$ for all $x,y \in R$. Now combining Lemma 3.3 with Lemma 2.2, we get $[x^q,y] = q\,x^{q-1}[x,y] = 0$ which yields

$$x^q \in Z(R) \text{ for all } x,y \in R. \tag{3.1}$$

Replacing y by y^m in (1.1), we get

$$x^k[x,y^m] = x^n[x,(y^m)^m]. \tag{3.2}$$

By applying Lemma 3.3 and Lemma 2.2, we obtain

$$\begin{aligned} x^k[x,y^m] &= [x,y^m]\,x^k \\ &= my^{m-1}[x,y]x^k \\ &= my^{m-1}x^k[x,y] \\ &= m\,y^{m-1}\,x^n[x,y^m] \\ &= m\,y^{m-1}\,[x,y^m]\,x^n \end{aligned}$$

and, using similar techniques, we get

$$\begin{aligned} x^n[x,\ (y^m)^m] &= [x,\ (y^m)^m]\,x^n \\ &= m(y^m)^{m-1}[x,y^m]x^n \\ &= m\,y^{m^2-m}\,[x,y^m]\,x^n \\ &= m\,y^{m-1}y^{(m-1)^2}\,[x,y^m]\,x^n. \end{aligned}$$

Thus (3.2) gives

$$m\,y^{m-1}\,(1 - y^{(m-1)^2})\,[x,y^m]\,x^n = 0. \tag{3.3}$$

Again the usual argument of replacing x by (x + 1) in (3.3) and applying Lemma 2.1 yields $m\,y^{m-1}(1-y^{(m-1)^2})[x,y\,]^m = 0$. Then by Lemma 3.3 and Lemma 2.3 we have

$$m\,y^{(m-1)}(1 - y^{q(m-1)^2})\,[x,y^m] = 0. \tag{3.4}$$

Next, we claim that $N' \subseteq Z(R)$. Let $a \in N'$, then by (3.1) $a^{q(m-1)^2} \in N' \cap Z(R)$, and $S\,a^{q(m-1)^2} = (0)$. Since by (3.4), $m\,a^{(m-1)}(1 - a^{q(m-1)^2})\,[x,a^m] = 0$, that is, $(1 - a^{q(m-1)^2})\;m\,a^{m-1}[x,a^m] = 0$.

Now, if $m\,a^{m-1}[x,a^m] \neq 0$, then $(1-a^{q(m-1)^2}) \in N'$, and so $S(1-a^{q(m-1)^2}) = 0$ which leads to the contradiction that $S = (0)$. Hence $m\,a^{m-1}[x,a^m] = 0$. From (1.1) and using Lemma 2.2 repeatedly we get

$$\begin{aligned} x^{2k}[x,a] &= x^k(x^k[x,a^m]) \\ &= x^k\,(x^n[x,a^m]) \\ &= x^n(x^k[x,a^m]) \\ &= x^{2n}[x,(a^m)^m] \\ &= x^{2n}\,m(a^m)^{m-1}\,[x,a^m] \\ &= x^{2n}m\,a^{m-1}a^{(m-1)^2}[x,a^m] \\ &= x^{2n}\,a^{(m-1)^2}m\,a^{m-1}[x,a^m] \\ &= 0. \end{aligned}$$

This implies that $x^{2k}[x,a] = 0$, and so the usual argument of replacing x by (x + 1) and using Lemma 2.1 gives $[x,a] = 0$, and hence,

$$N' \subseteq Z(R). \tag{3.5}$$

Now, for any $x \in R$, x^q and x^{qm} are in $Z(R)$. Then by (1.1) for any $y \in R$, we have

$$\begin{aligned} (x^q - x^{qm})\,x^k[x,y] &= x^q(x^k[x,y]) - x^{qm}(x^k[x,y]) \\ &= x^k(x^q[x,y]) - x^{qm}\,x^n[x,y^m] \\ &= x^k[x,x^qy] - x^n[x,(x^qy)^m] \\ &= x^k[x,x^qy] - x^k[x,x^qy]. \end{aligned}$$

Therefore $(x^q - x^{qm})x^k\,[x,y] = 0$, and hence

$$(x - x^{qm-q+1})\,x^{k+q-1}[x,y] = 0. \tag{3.6}$$

If R is not commutative then by [6, Theorem 18], there exists an element $x \in R$ such that $(x - x^t) \notin Z(R)$, where $t = qm - q + 1$. This also reveals $x \notin Z(R)$. Thus neither $(x-x^t)$ nor x is a zero divisor, and so $(x-x^t)\,x^{k+q-1} \notin N'$. Hence (3.6) forces that $[x,y] = 0$, for all $x,y \in R$. Thus $x \in Z(R)$ which is a contradiction. Hence R is commutative.

PROOF OF THE THEOREM. Let R be a left s-unital ring satisfying (P), then by Lemma 3.1, R is s-unital. Therefore, in view of [1, Proposition 1] and Lemma 3.4, R is commutative, if R with 1 satisfying (P) is commutative.

COROLLARY 3.1([3, Theorem]). Let R be a ring with unity 1 in which $[xy - x^ny^m,x] = 0$ for all $x,y \in R$ and fixed integers $m > 1$, $n > 1$. Then R is commutative.

PROOF. Actually, R satisfies the polynomial identity $x[x,y] = x^n[x,y^m]$ for all $x,y \in R$ and fixed integers $m > 1$, $n > 1$. Put $k = 1$ in (1.1), then R is commutative by Lemma 3.4.

COROLLARY 3.2 (Hirano, Kobayashi, and Tominaga [7, Theorem]). Let m,k be fixed non-negative integers. Suppose that R satisfies the polynomial identity

$$x^k [x,y] = [x,y^m].$$

(a) If R is a left s-unital, then R is commutative except when (m,k) = (1,0).

(b) If R is a right s-unital, then R is commutative except when (m,k) = (1,0), and m = 0, k > 0.

REMARK 3.1. ([7]). In case k > 0 and m = 0 in Corollary 3.2(b), R need not be commutative. For, let K be a field. Then the non-commutative ring $R = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} | \ a,b \in K\}$ has a right identity element and satisfies the polynomial identity $x[x,y] = 0$.

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