Internat. J. Math. & Math. Sci.
VOL. 13 NO. 4 (1990) 783-798

INDUCED MEASURES ON WALLMAN SPACES

EL-BACHIR YALLAOUI

University of Setif, Algeria

(Received April 14,1989 and in revised form October 3, 1989)

ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. To each lattice-regular measure μ, we associate two induced measures $\hat{\mu}$ and $\tilde{\mu}$ on suitable lattices of the Wallman space $I_R(\mathcal{L})$ and another measure μ' on the space $I_R^\sigma(\mathcal{L})$. We will investigate the reflection of smoothness properties of μ onto $\hat{\mu}, \tilde{\mu}$ and μ' and try to set some new criterion for repleteness and measure repleteness.

KEY WORDS AND PHRASES. Lattice regular measure, Wallman space and remainder, replete and measure replete lattices, σ-smooth, τ-smooth and tight measures.

1980 AMS SUBJECT CLASSIFICATION CODE. 28C15

1. INTRODUCTION.

Let X be an abstract set and $\mathcal{L}$ a lattice subsets of X. To each lattice regular measure μ, we associate following Bachman and Szeto [1] two induced measures $\hat{\mu}$ and $\tilde{\mu}$ on suitable lattices of subsets of the Wallman space $I_R(\mathcal{L})$ of $(X,\mathcal{L})$; we also associate to μ a measure μ' on the space $I_R^\sigma(\mathcal{L})$ (see below for definitions).

We extend the results of [1], by further investigation of the reflection of smoothness properties of μ onto $\hat{\mu}, \tilde{\mu}$ and μ' and investigate more closely the regularity properties of $\hat{\mu}$ and $\tilde{\mu}$ (see in particular theorems 4.7, 4.8, 4.9, 5.4, and 5.6). We are then in a position to get new criterion for repleteness and measure repleteness etc. These general results are then applied to specific lattices in a topological space to obtain some new and some old results pertaining to measure compactness, real compactness, α-real compactness, ets...in an entirely different manner.

We give in section 2, a brief review of the lattice notation and terminology relevant to the paper. We will be consistent with the standard terminology as used, for example, in Alexandroff [2], Frolik [3], Grassi [4], Nöbeling [5], and Wallman [6].

We also give a brief review of the principal Theorems of [1] that we need in order to make the paper reasonably self-contained.

2. DEFINITIONS AND NOTATIONS.

Let X be an abstract set, then $\mathcal{L}$ is a lattice of subsets of X; if $A,B \subset X$ then $A \cup B \in \mathcal{L}$ and $A \cap B \in \mathcal{L}$. Throughout this work we will always assume that $\varnothing$ and X are in $\mathcal{L}$. If $A \subset X$ then we will denote the complement of A by A' i.e. $A' = X - A$. If $\mathcal{L}$ is a Lattice of subsets of X then $\mathcal{L}'$ is defined $\mathcal{L}' = \{L' \mid L \in \mathcal{L}\}$.

Lattice Terminology

DEFINITIONS 2.1. Let $\mathcal{L}$ be a Lattice of subsets of X. We say that:

1- $\mathcal{L}$ is a δ-Lattice if it is closed under countable intersections.

2- $\mathcal{L}$ is separating or T_1 if $x, y \in X$; $x \neq y$ then $\exists L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$.

3- $\mathcal{L}$ is Hausdorff or T_2 if $x, y \in X$; $x \neq y$ then $\exists A, B \in \mathcal{L}$ such that $x \in A', y \in B'$ and $A' \cap B' = \varnothing$.

4- $\mathcal{L}$ is disjunctive if for $x \in X$ and $L \in \mathcal{L}$ where $x \notin L; \exists A, B \in \mathcal{L}$ such that $x \in A, L \subset B$ and $A \cap B = \varnothing$.

5- $\mathcal{L}$ is regular if for $x \in X, L \in \mathcal{L}$ and $x \notin L; \exists A, B \in \mathcal{L}$ such that $x \in A', L \subset B'$ and $A' \cap B' = \varnothing$.

6- $\mathcal{L}$ is normal if for $A, B \in \mathcal{L}$ where $A \cap B = \varnothing \exists \tilde{A}, \tilde{B} \in \mathcal{L}$ such that $A \subset \tilde{A}', B \subset \tilde{B}'$ and $\tilde{A}' \cap \tilde{B}' = \varnothing$.

7- $\mathcal{L}$ is compact if $X = \bigcup_{\alpha} L'_{\alpha}$ where $L_{\alpha} \in \mathcal{L}$ then there exists a finite number of L_{α} that cover X i.e. $X = \bigcup_{i=1}^{n} L'_{\alpha i}$ where $\in \mathcal{L}$.

8- $\mathcal{L}$ is countably compact if for $X = \bigcup_{i=1}^{\infty} L'_i$ then $X = \bigcup_{i=1}^{n} L'_i$.

9- $\mathcal{L}$ is Lindelöf if $X = \bigcup_{\alpha} L_{\alpha}, a \in \wedge$ then $X = \bigcup_{i=1}^{\infty} L'_{\alpha i}$ where $L_{\alpha i} \in \mathcal{L}$.

10- $\mathcal{L}$ is countably paracompact if for every sequence $\{L_n\}$ in $\mathcal{L}$ such that $L_n \downarrow \varnothing$ there exists a sequence $\{\tilde{L}_n\}$ in $\mathcal{L}$ such that $L_n \subset \tilde{L}'_n$ and $\tilde{L}'_n \downarrow \varnothing$.

11- $\mathcal{L}$ is complemented if $L \in \mathcal{L}$ then $L' \in \mathcal{L}$.

12- $\mathcal{L}$ is complement generated if $L \in \mathcal{L}$ then $L = \bigcap_{i=1}^{\infty} \tilde{L}'_i$ where $L_i \in \mathcal{L}$.

13- $\mathcal{L}$ is T_4 if it isnormal and T_1.

14- $\mathcal{L}$ is $T_{3\frac{1}{2}}$ if it is completely regular and T_2.

$A(\mathcal{L})$ = the algebra generated by $\mathcal{L}$.
$\sigma(\mathcal{L})$ = the σ-algebra generated by the $\mathcal{L}$.
$\delta(\mathcal{L})$ = the Lattice of countable intersections of sets of $\mathcal{L}$.
$\tau(\mathcal{L})$ = the Lattice of arbitrary intersection of sets of $\mathcal{L}$.
$\rho(\mathcal{L})$ = the smallest class containing $\mathcal{L}$ and closed under countable unions and intersections.

If $A \in \mathcal{A}(\mathcal{L})$ then $A = \bigcup_{i=1}^{n} (L_i - \tilde{L}'_i)$ where the union is disjoint and $L_i, \tilde{L}_i \in \mathcal{L}$. If X is a topological space we denote:

$\mathcal{O}$ = Lattice of open sets

$\mathcal{F}$ = Lattice of closed sets

$\mathcal{Z}$ = Lattice of zero sets of continuous functions

$\mathcal{K}$ = Lattice of compacts sets, with X adjoined

$\mathcal{C}$ = Lattice of clopen sets

Measure Terminology

Let $\mathcal{L}$ be a lattice of subsets of X. $M(\mathcal{L})$ will denote the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$. Clearly since any measure in $M(\mathcal{L})$ can be written as a difference of two non-negative measures there is no loss of generality in assuming that the measures are non-negative, and we will assume so throughout this paper.

DEFINITIONS 2.2.

1- A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{L}$ if for $L_n \in \mathcal{L}$ and $L_n \downarrow \varnothing$ then $\mu(L_n) \to 0$.

2- A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{A}(\mathcal{L})$ if for $A_n \in \mathcal{A}(\mathcal{L}), A_n \downarrow \varnothing$ then $\mu(A_n) \to 0$.

3- A measure $\mu \in M(\mathcal{L})$ is said to be τ-smooth on $\mathcal{L}$ if for $L_\alpha \in \mathcal{L}\alpha \in \Lambda, L_\alpha \downarrow \varnothing$ then $\mu(L_\alpha) \to 0$.

4- A measure $\mu \in M(\mathcal{L})$ is said to be $\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$,

$$\mu(A) = \sup_{\substack{L \subset A \\ L \in \mathcal{L}}} \mu(L).$$

If $\mathcal{L}$ is a lattice of subsets of X, then we will denote by:

$M_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$

$M_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$

$M^\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$

$M_R^\sigma(\mathcal{L})$ = the set of regular measures of $M^\sigma(\mathcal{L})$

$M_R^\tau(\mathcal{L})$ = the set of τ-smooth measures on $\mathcal{L}$ of $M_R(\mathcal{L})$

$M_\tau(\mathcal{L})$ = the set of τ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$.

Clearly

$$M_R^T(\mathcal{L}) \subset M_R^\sigma(\mathcal{L}) \subset M_R(\mathcal{L}).$$

DEFINITION 2.3. If $A \in \mathcal{A}(\mathcal{L})$ then μ_x is the measure concentrated at $x \in X$.

$$\mu_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

$I(\mathcal{L})$ is the subset of $M(\mathcal{L})$ which consists of non-trivial zero-one measures which are finitely additive on $\mathcal{A}(\mathcal{L})$.

$I_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$

$I_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$

$I^\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $I(\mathcal{L})$

$I_\tau(\mathcal{L})$ = the set of τ-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$

$I_R^\sigma(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I^\sigma(\mathcal{L})$

$I_R^\tau(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I_\tau(\mathcal{L})$

DEFINITION 2.4. If $\mu \in M(\mathcal{L})$ then we define the support of μ to be:

$$S(\mu) = \cap\{L \in \mathcal{L}/\mu(L) = \mu(X)\}.$$

Consequently if $\mu \in I(\mathcal{L})$,

$$S(\mu) = \cap\{L \in \mathcal{L}/\mu(L) = 1\}.$$

DEFINITION 2.5. If $\mathcal{L}$ is a Lattice of subsets of X we say that $\mathcal{L}$ is replete if for any $\mu \in I_R^\sigma(\mathcal{L})$ then $S(\mu) \neq \varnothing$.

DEFINITION 2.6. Let $\mathcal{L}$ be a lattice of subsets of X. We say that $\mathcal{L}$ is measure replete if $S(\mu) \neq \varnothing$ for all $\mu \in M_R^\sigma(\mathcal{L}), \mu \neq 0$.

Separation Terminology

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lattices of subsets of X.

DEFINITION 2.7. We say that $\mathcal{L}_1$ separates $\mathcal{L}_2$ if for $A_1 \in \mathcal{L}_1$ and $A_2 \in \mathcal{L}_2$ and $A_1 \cap A_2 = \varnothing$ then there exists $B_1 \in \mathcal{L}_1$ such that $A_2 \subset B_1$ and $B_1 \cap A_1 = \varnothing$.

DEFINITION 2.8. $\mathcal{L}_1$ separates $\mathcal{L}_2$ if for $A_2, B_2 \in \mathcal{L}_2$ and $A_2 \cap B_2 = \varnothing$ then there exists $A_1, B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1, B_2 \subset B_1$ and $A_1 \cap B_2 = \varnothing$.

DEFINITION 2.9. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. $\mathcal{L}_2$ is $\mathcal{L}_1$-countably paracompact if given $A_n \in \mathcal{L}_2$ with $A_n \downarrow \varnothing$, there exists $B_n \in \mathcal{L}_1$ such that $A_n \subset B'_n$ and $B'_n \downarrow \varnothing$.

DEFINITION 2.10. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. We say that $\mathcal{L}_2$ is $\mathcal{L}_1$-countably bounded ($\mathcal{L}_2$ is $\mathcal{L}_1$ - cb) if for any sequence $\{B_n\}$ of sets of $\mathcal{L}_2$ with $B_n \downarrow \varnothing$ then there exists a sequence $\{A_n\}$ of sets of $\mathcal{L}_1$ such that $B_n \subset A_n$ and $A_n \downarrow \varnothing$. If $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mu \in M(\mathcal{L}_2)$ then the restriction of μ on $\mathcal{A}(\mathcal{L}_1)$ will be denoted by $\nu = \mu|_{\mathcal{L}_1}$.

REMARK 2.1. We now list a few known facts found in [1] which will enable us to characterize some previously defined properties in a measure theoretic fashion.

1. $\mathcal{L}$ is disjunctive if and only if $\mu_x \in I_R(\mathcal{L}), \forall x \in X$.
2. $\mathcal{L}$ is regular if and only if for any $\mu_1, \mu_2 \in I(\mathcal{L})$ such that $\mu_1 \leq \mu_2$ on $\mathcal{L}$ we have $S(\mu_1) = S(\mu_2)$.
3. $\mathcal{L}$ is T_2 if and only if $S(\mu) = \varnothing$ or a singleton for any $\mu \in I(\mathcal{L})$.
4. $\mathcal{L}$ is compact if and only if $S(\mu) \neq \varnothing$ for any $\mu \in I_R(\mathcal{L})$.

3. LATTICE REGULAR MEASURES.

In this section, we shall consider lattice properties which are intimately related to measures on the generated algebra. First we list a few properties that are easy to prove, but which are important and will be used throughout the paper.

PROPOSITION 3.1. If $\mu \in M_R(\mathcal{L})$, then $\mu \in M_\sigma(\mathcal{L})$ implies $\mu \in M^\sigma(\mathcal{L})$.

PROPOSITION 3.2. If $\mu \in M_R^\sigma(\mathcal{L})$, then μ (extended to $\sigma(\mathcal{L})$) is $\delta(\mathcal{L})$-regular on $\sigma(\mathcal{L})$.

LEMMA 3.3. If $\mathcal{L}$ is a complement generated lattice of subsets of X, then $\mathcal{L}$ is c. p.

PROOF. Suppose $L_n \in \mathcal{L}$. Then since $\mathcal{L}$ is complement generated, $L_n = \bigcap_{i=1}^{\infty} L'_{ni}$ where $L'_{ni} \in \mathcal{L}$ (may assume $L_{ni} \downarrow$). Let

$$A'_n = \bigcap_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} L'_{ij} \text{ where } A'_n \in \mathcal{L}'$$

so that

$$L_n \subset A'_n = L'_{11} \cap L'_{12} \ldots \cap L'_{1n} \cap L'_{2n} \cap \ldots \cap L_{nn}' \text{ and clearly } A'_n \downarrow \varnothing.$$

THEOREM 3.4. If $\mathcal{L}$ is complement generated, then $\mu \in M_\sigma(\mathcal{L}')$ implies $\mu \in M_R^\sigma(\mathcal{L})$.

PROOF. If $L \in \mathcal{L}$, then $L = \bigcap_{i=1}^{\infty} L'_i$ where $L_i \in \mathcal{L}$ (may assume $L_i \downarrow$). Clearly, $L \cap L' = \bigcap_{i=1}^{\infty} (L' \cap L'_i) = \varnothing$ and $(L' \cap L'_i) \downarrow \varnothing$. Since $\mu \in M_R(\mathcal{L}')$, then $\mu(L' \cap L'_i) \to 0$ and hence $\mu(L'_i) \to \mu(L)$. Therefore, $\mu(L) = \inf_{L \subset L'_i, L_i \in \mathcal{L}} \mu(L'_i)$. Thence $\mu \in M_R(\mathrm{L})$.

Now, we show that $\mu \in M_\sigma(\mathcal{L})$. Since $\mathcal{L}$ is complement generated we know from lemma 3.3 that $\mathcal{L}$ is countably paracompact. Let $L_n \downarrow \varnothing$. Then, since $\mathcal{L}$ is c. p., there exist $\hat{L}_n \in \mathcal{L}$ such that $L_n \subset \hat{L}'_n$ and $\hat{L}'_n \downarrow \varnothing$. Then, $\mu(L_n) \leq \mu(\hat{L}'_n) \to 0$ because $\mu \in M_\sigma(\mathcal{L}')$. Now, using Proposition 3.1 and the fact that $\mu \in M_R(\mathcal{L})$, we have that $\mu \in M_R^\sigma(\mathcal{L})$.

DEFINITION 3.5. μ is strongly σ-smooth on $\mathcal{L}$ if for $L_n \in \mathcal{L}, L_n \downarrow$ and $\cap L_n \in \mathcal{L}$, $\mu(\cap L_n) = \inf_n \mu(L_n)$.

THEOREM 3.6. Let $\mathcal{L}$ be a complement generated and normal lattice of subsets of X. If μ is strongly σ-smooth on $\mathcal{L}$, then $\mu \in M_R^\sigma(\mathcal{L})$.

REMARK. If $\mathcal{L}$ is a δ-lattice, $\sigma(\mathcal{L}) \subset s(\mathcal{L})$ and $\mu \in M^\sigma(\mathcal{L})$ then $\mu \in M_R^\sigma(\mathcal{L})$. This result follows from Choquet's theorem on capacities [7].

Next, we generalize a result of Gardner [8].

THEOREM 3.7. Let $\mathcal{L}$ be a lattice of subsets of X and suppose that

1) $\mu \in M_\sigma(\mathcal{L})$,

2) $\mathcal{L}$ is regular and

3) if $L_\alpha \in \mathcal{L}$ and $L_\alpha \downarrow$ then, $\mu^*\left(\bigcap_\alpha L_\alpha\right) = \inf_\alpha \mu(L_\alpha)$.

Then, $\mu \in M_R^\tau(\mathcal{L})$.

PROOF. Let $L \in \mathcal{L}$. Then by regularity, $L = \bigcap_\alpha L_\alpha$ where $L \subset L_\alpha \subset \mathcal{L}$ (may assume $L_\alpha \downarrow$). Let $x \in L' = \bigcup_\alpha L'_\alpha, L_\alpha \uparrow$. Then, $x \in L_\alpha \geq \alpha_0$ for some α_0. Clearly, $x \notin L_{\alpha \geq \alpha_0}$ and $L = \bigcap_{\alpha \geq \alpha_0} L_\alpha$. Since $\mathcal{L}$ is regular, there exist $\hat{L}_\alpha, \tilde{L}_\alpha \in \mathcal{L}$ such that $x \in \hat{L}'_\alpha, L_\alpha \subset \tilde{L}'_\alpha$ and $\hat{L}'_\alpha \cap \tilde{L}'_\alpha = \varnothing$. Hence, $L_\alpha \subset \tilde{L}'_\alpha \subset \hat{L}_\alpha = L_\alpha$. Now taking intersections with respect to α, we get,

$$L = \bigcap_\alpha L_\alpha = \bigcap_\alpha \tilde{L}'_\alpha = \bigcap_\alpha \hat{L}_\alpha$$

Therefore $\mu(L) = \mu^*\left(\bigcap_\alpha L_\alpha\right) = \mu^*\left(\bigcap_\alpha \tilde{L}'_\alpha\right) = \mu^*\left(\bigcap_\alpha \hat{L}_\alpha\right) = \inf_\alpha \mu(L_\alpha) = \inf_\alpha(\tilde{L}'_\alpha) = \inf_\alpha(\hat{L}_\alpha)$. By the argument used in Theorem 3.4, we find that $\mu \in M_R(\mathcal{L})$. But, since $\mu \in M_\sigma(\mathcal{L})$ then $\mu \in M_R^\sigma(\mathcal{L})$. Now, let $L_\alpha \downarrow \varnothing$. Then $\mu^*\left(\bigcap_\alpha L_\alpha\right) = \inf_\alpha(L_\alpha) = 0$. Hence, $\mu \in M_R^\tau(\mathcal{L})$.

We make use of the following extension theorem a proof of which can be found in [9].

THEOREM 3.8. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Then any $\mu \in M_R(\mathcal{L}_1)$ can be extended to $\nu \in M_R(\mathcal{L}_2)$ and the extension is unique if $\mathcal{L}_1$ separates $\mathcal{L}_2$. If we further assume that $\mathcal{L}_2$ is $\sigma(\mathcal{L}_1)$-cb and $\mathcal{L}_1$ is a δ-lattice then any $\mu \in M_R^\sigma(\mathcal{L}_1)$ can be extended to $\nu \in M_R^\sigma(\mathcal{L}_2)$.

COROLLARY 3.9. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. If $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. or $\mathcal{L}_1$ c.b., then any $\mu \in M_R^\sigma(\mathcal{L}_1)$ can be extended to $\nu \in M_R^\sigma(\mathcal{L}_2)$.

COROLLARY 3.10. If X a topological c.b. space, then every $\mu \in M_R^\sigma(\mathcal{L}_1)$ can be extended to $\nu \in M_R^\sigma(\mathcal{L}_2)$.

LEMMA 3.11. If $\mathcal{L}_1 \subset \mathcal{L}_2, \mathcal{L}_2$ is c.p. and $\mathcal{L}_1$ separates $\mathcal{L}_2$ then $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p.

COROLLARY 3.12. If X is a coutably paracompact and normal space, then every $\mu \in M_R^\sigma(\mathcal{Z})$ extends to $\nu \in M_R^\sigma(\mathcal{F})$ and the extension is unique.

PROOF. Let $\mathcal{L}_1 = \mathcal{Z}$ and $\mathcal{L}_2 = \mathcal{F}$. Then $\mathcal{L}_2$ is $\mathcal{L}_1$-countably bounded, $\mathcal{L}_1$ separates $\mathcal{L}_2$ and $\mathcal{L}_1$ is a δ-lattice. Now use the previous Theorem 3.8. This result is due to Marik [10].

Next, we have a restriction theorem, which although generally known, we prove for the reader's convenience.

THEOREM 3.13. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Suppose $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$ and $\nu \in M_R(\mathcal{L}_2)$. Then $\mu = \nu\,|_{\mathcal{A}(\mathcal{L}_1)} \in M_R(\mathcal{L}_1)$.

PROOF. The proof of this Theorem is well known and will be omitted.

4. SPACES AND MEASURES ASSOCIATED WITH LATTICE REGULAR MEASURES.

We will briefly review the fundamental properties of this Wallman space associated with a regular lattice measure μ, and then associate with a regular lattice measure μ, two measures $\tilde{\mu}$ and $\hat{\mu}$ on certain algebras in the Wallman space (see [3]). We then investigate how properties of μ reflect to those of $\hat{\mu}$ and $\tilde{\mu}$, and conversely, and then give a variety of applications of these results.

Let X be an abstract set and $\mathcal{L}$ a disjunctive lattice of subsets of X such that $\varnothing$ and X are in $\mathcal{L}$. For any A in $\mathcal{A}(\mathcal{L})$, defined to be $W(A) = \{\mu \in I_R(\mathcal{L}) : \mu(A) = 1\}$. If $A, B \in \mathcal{A}(\mathcal{L})$ then

1) $W(A \cup B) = W(A) \cup W(B)$.

2) $W(A \cap B) = W(A) \cap W(B)$.

3) $W(A') = W(A)'$.

4) $W(A) \subset W(B)$ if and only if $A \subset B$.

5) $W(A) = W(B)$ if and only if $A = B$.

6) $W[\mathcal{A}(\mathcal{L})] = \mathcal{A}[W(\mathcal{L})]$.

Let $W(\mathcal{L}) = \{W(L), L \in \mathcal{L}\}$. Then $W(\mathcal{L})$ is a compact lattice of $I_R(\mathcal{L})$, and $I_R(\mathcal{L})$ with $\tau W(\mathcal{L})$ as the topology of closed sets is a compact T_1 space (the Wallman space) associated with the pair $X, \mathcal{L}$. It is a T_2-space if and only if $\mathcal{L}$ is normal.

For $\mu \in M(\mathcal{L})$ we define $\hat{\mu}$ on $\mathcal{A}(W(\mathcal{L}))$ by: $\hat{\mu}(W(A)) = \mu(A)$ where $A \in \mathcal{A}(\mathcal{L})$. Then $\hat{\mu} \in M(W(\mathcal{L}))$, and $\hat{\mu} \in M_R(W(\mathcal{L}))$ if and only if $\mu \in M_R(\mathcal{L})$.

Finally, since $\tau W(\mathcal{L})$ and $W(\mathcal{L})$ are compact lattices, and $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$, then $\hat{\mu}$ has a unique extension to $\tilde{\mu} \in M_R(\tau W(\mathcal{L}))$ (see Theorem 3.4).

We note that by compactness $\hat{\mu}$ and $\tilde{\mu}$ are in $M_R^{\tau}(W(\mathcal{L}))$ and $M_R^{\tau}(\tau W(\mathcal{L}))$ respectively, where they are certainly τ-smooth and of course σ-smooth. $\hat{\mu}$ can be extended to $\sigma(W(\mathcal{L}))$ where it is $\delta W(\mathcal{L})$-regular; while $\tilde{\mu}$ can be extended to $\sigma(\tau(W(\mathcal{L})))$, the Borel sets of $I_R(\mathcal{L})$, and is $\tau W(\mathcal{L})$-regular on it.

One is now concerned with how further properties of μ reflect over to $\hat{\mu}$ and $\tilde{\mu}$ respectively. The following are known to be true (see [1]) and we list them for the reader's convenience.

THEOREM 4.1. Let $\mathcal{L}$ be a separating and disjunctive lattice. Let $\mu \in M_R(\mathcal{L})$, then the following statements are equivalent.

1. $\mu \in M_R^{\sigma}(\mathcal{L})$.

2. If $\{L_i\} \in \mathcal{L}, L_i \downarrow$ and $\bigcap_1^{\infty} W(L_i) \subset I_R(\mathcal{L}) - X$ then $\hat{\mu}\left[\bigcap_1^{\infty} W(L_i)\right] = 0$.

3. If $\{L_i\} \in \mathcal{L}; L_i \downarrow$ and $\bigcap W(L_i) \subset I_R(\mathcal{L}) - I_R^{\sigma}(\mathcal{L})$ then $\hat{\mu}\left[\bigcap_1^{\infty} W(L_i)\right] = 0$.

4. $\hat{\mu}^*(X) = \hat{\mu}(I_R(\mathcal{L}))$.

5. $\hat{\mu}^*[I_R^{\sigma}(\mathcal{L})] = \hat{\mu}(I_R(\mathcal{L}))$.

THEOREM 4.2. If $\mathcal{L}$ is separating, disjunctive, δ, normal and countably paracompact; and $\mu \in M_R(\mathcal{L})$ then the following statements are equivalent:

1. $\mu \in M_R^{\sigma}(\mathcal{L})$.

2. $\hat{\mu}(K) = 0$ for all $K \subset I_R(\mathcal{L}) - X$ and $K \in \mathcal{Z}(\tau W(\mathcal{L}))$.

Note that $Z \in \mathcal{Z}(\tau W(\mathcal{L})) \Rightarrow Z \in \sigma[W(\mathcal{L})]$.

THEOREM 4.3. Let $\mathcal{L}$ be a separating and disjunctive lattice. If $\mu \in M_R(\mathcal{L})$ then the following statements are equivalent:

1. $\mu \in M_R^{\tau}(\mathcal{L})$.

2. If $\{L_\alpha\} \in \mathcal{L} L_\alpha \downarrow$ and $\bigcap_\alpha W(L_\alpha) \subset I_R(\mathcal{L}) - X$ then $\tilde{\mu}\left(\bigcap_\alpha W(L_\alpha)\right) = 0$.

3. $\tilde{\mu}^*(X) = \tilde{\mu}(I_R(\mathcal{L}))$.

THEOREM 4.4. If $\mathcal{L}$ is a separating and disjunctive lattice of subsets of X then, $\tilde{\mu} \in M_R^{\tau}(\mathcal{L})$ if and only $\tilde{\mu}$ vanishes on every closed subset of $I_R(\mathcal{L})$, contained in $E_R(\mathcal{L}) - X$.

THEOREM 4.5. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X and $\mu \in M_R(\mathcal{L})$, then the two statements are equivalent:

1. $\mu \in M_R^{\tau}(\mathcal{L})$.

2. $\tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}^*(X)$.

THEOREM 4.6. Let $\mathcal{L}$ be a separating, disjunctive and normal lattice of subsets of X. Let $\mu \in M_R(\mathcal{L})$ then the two statements are equivalent:

1. $\mu \in M_R^{\tau}(\mathrm{L})$.

2. X is $\tilde{\mu}^{*}$-measureable and $\tilde{\mu}^{*}(X) = \tilde{\mu}(I_R(\mathcal{L}))$.

We now establish some further properties pertaining to the induced measures $\hat{\mu}$ and $\tilde{\mu}$. First we show

THEOREM 4.7. Let $\mathcal{L}$ be a separating and disjunctive lattice, and $\mu \in M_R(\mathcal{L})$ then $\tilde{\mu}$ is $W(\mathcal{L})$ regular on $(\tau W(\mathcal{L}))'$.

PROOF. We know that $W(\mathcal{L})$ and $\tau W(\mathcal{L})$ are compact lattices and that $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$. Since $\mu \in M_R(\mathcal{L})$ then $\hat{\mu} \in M_R[W(\mathcal{L})]$. Extend $\hat{\mu}$ to $\tau W(\mathcal{L})$ and the extension is

$$\tilde{\mu} \in M_R[\tau W(\mathcal{L})] = M_R^{\sigma}[\tau W(\mathcal{L})] = M_R^{\tau}[\tau W(\mathcal{L})] = M_R^{\tau}[\tau W(\mathcal{L})].$$

Let $0 \in [\tau W(\mathcal{L})]'$ then since $\tilde{\mu} \in M_R[\tau W(\mathcal{L})]$ there exists $F \in \tau W(\mathcal{L}), F \subset 0$ and

$$|\tilde{\mu}(0) - \tilde{\mu}(F)| < \in; \in > 0.$$

Since $F \in \tau W(\mathcal{L}), F = \bigcap_{\alpha \in \Lambda} W(L_\alpha), L_\alpha \in \mathcal{L}$. Also since $F \subset 0$ then $F \cap 0' = \varnothing$ i.e. $\bigcap_{\alpha} W(L_\alpha) \cap 0' = \varnothing$ by compactness there must exist $\alpha_0 \in \Lambda$ such that $W(L_{\alpha 0}) \cap 0' = \varnothing$ thus $\overset{\alpha}{F} \subset W(L_{\alpha 0}) \subset 0'' = 0$ so

$$|\tilde{\mu}(0) - \tilde{\mu}(W(L_{\alpha 0}))| < \in$$

i.e. $\tilde{\mu}$ is $W(\mathcal{L})$ regular on $(\tau W(\mathcal{L}))'$.

THEOREM 4.8. Let $\mu \in M_R(\mathcal{L})$ then $\hat{\mu}^{*} = \tilde{\mu}$ on $\tau W(\mathcal{L})$.

PROOF. Since $\mu \in M_R(\mathcal{L})$ and $W(\mathcal{L})$ is compact then $\hat{\mu} \in M_R[W(\mathcal{L})] = M_R^{\tau}[W(\mathcal{L})]$ and since $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$ and $\tau W(\mathcal{L})$ is compact then $\tilde{\mu} \in M_R[\tau W(\mathcal{L})] = M_R^{\tau}[\tau W(\mathcal{L})]$ furthermore $\tilde{\mu}$ extends $\hat{\mu}$ to $\tau W(\mathcal{L})$ uniquely. Let $F \in \tau W(\mathcal{L})$ then

$$\hat{\mu}^{*}(F) = \inf \sum_{i=1}^{\infty} \hat{\mu}(A_i), F \subset \bigcup_{i=1}^{\infty} A_i \text{ and } A_i \in \mathcal{A}[W(\mathcal{L})]$$

and since $\hat{\mu} \in M_R^{\tau}[W(\mathcal{L})]$ then

$$\hat{\mu}(A_i) = \inf \hat{\mu}[W(L'_i)], A_i \subset W(L'_i), L_i \in \mathcal{L}.$$

Thus $F \subset \bigcup_{i=1}^{\infty} W(L'_i)$ but since $W(\mathcal{L})$ is compact then $F \subset \bigcup_{i=1}^{n} W(L'_i) = W(L')$ where $L \in \mathcal{L}$ and

$$\hat{\mu}^{*}(F) = \inf \hat{\mu}[W(L')]; F \subset W(L') \text{ and } L \in \mathcal{L}.$$

Now $F \subset W(L') \Rightarrow F \cap W(L) = \varnothing$ then since $W(\mathcal{L})$ separates $\tau W(\mathcal{L}) \exists \tilde{L} \in \mathcal{L}$ such that $F \subset W(\tilde{L})$ and $W(\tilde{L}) \cap W(L) = \varnothing$. Therefore $W(L') \subset W(\tilde{L})$ and hence

$$\hat{\mu}^{*}(F) = \inf \hat{\mu}[W(\tilde{L})]: \text{where } F \subset W(\tilde{L}); \tilde{L} \in \mathcal{L}$$

i.e. that $\hat{\mu}^{*}$ is regular on $\tau W(\mathcal{L})$. On the other hand since $\tau W(\mathcal{L})$ is δ then

$$F = \bigcap_{\alpha} W(L_\alpha) \text{ and } \tilde{\mu}\left[\bigcap_{\alpha} W(L_\alpha)\right] = \inf_{\alpha} \tilde{\mu}(W(L_\alpha)) = \inf \hat{\mu}(W(L_\alpha))$$

where $F \subset W(L_\alpha), L_\alpha \in \mathcal{L}$. Therefore $\hat{\mu}^{*} = \tilde{\mu}$ on $\tau W(\mathcal{L})$.

THEOREM 4.9. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$. If $\nu \in M_R^{\sigma}(\mathcal{L})$ then $\nu = \mu^{*}$ on $\mathcal{L}'_2$ and $\nu = \mu_{*}$ on $\mathcal{L}'_2$ where $\mu = \nu|_{\mathcal{L}_1}$.

PROOF. Let $\nu \in M_R^{\sigma}(\mathcal{L}_2)$ then since $\mathcal{L}_1$ separates $\mathcal{L}_2, \mu \in M_R^{\sigma}(\mathcal{L}_1)$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$ then $\sigma(\mathcal{L}_1) \subset \sigma(\mathcal{L}_2)$; Let $E \subset X$ then

$$\nu^{*}(E) = \inf_{E \subset B, B \in \sigma(\mathcal{L}_2)} \nu(B) \leq \inf_{E \subset A, A \in \sigma(\mathcal{L}_1)} \nu(A) = \mu^{*}(E)$$

therefore, $\nu^* \le \mu^*$. Now on $\mathcal{L}_2, \nu^* = \le \mu^*$. Suppose $\exists L_2 \in \mathcal{L}_2$ such that $\nu(L_2) < \mu^*(L_2)$ then since

$$\nu \in M_R^\sigma(\mathcal{L}_2), \nu(L_2) = \inf \nu(\tilde{L}'_2), L_2 \subset \tilde{L}'_2 \text{ and } \tilde{L}_2 \in \mathcal{L}_2$$

then $L_2 \cap \tilde{L}_2 = \varnothing$ and by separation $\exists L_1, \tilde{L}_2 \in \mathcal{L}_1$ such that $L_2 \subset L_1, \subset \tilde{L}'_1 \subset \tilde{L}'_2$ and therefore

$$\begin{aligned}\nu(L_2) &= \inf_\alpha \mu(L_{1\alpha}) \text{ where } L_2 \subset L_{1\alpha}\\ &= \inf_\beta \nu(\tilde{L}'_{2\beta}) \text{ where } L_2 \subset \tilde{L}'_{2\beta}\\ &< \mu^*(L_2).\end{aligned}$$

$\forall \varepsilon > 0 \exists L_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1$ and $\mu(L_1) - \varepsilon < \nu(L_2) < \mu(L_1)$ but since $L_2 \subset L_1$ then $\mu^*(L_2) \le \mu(L_1) < \nu(L_2) + \varepsilon$ which is a contradiction to our assumption. Therefore $\nu = \mu^*$ on $\mathcal{L}_2$ and thus $\nu = \mu_*$ on $\mathcal{L}'_2$.

This theorem is a generalization of the previous one in which we used the compactness of $W(\mathcal{L})$ to have a regular restriction of the measure. Also this theorem enables us to improve corollary 3.12 namely: If X is coutably paracompact and normal then each measure $\mu \in M_R(\mathcal{Z})$ extends to a measure $\nu \in M_R(\mathcal{F})$ which is $\mathcal{Z}$-regular on $\mathcal{O}$.

THEOREM 4.10. Suppose $\mathcal{L}$ is a separating and disjunctive lattice. Let $x \in X$ then $\{x\} = \bigcap_1^\infty L'_n$ if and only if $\{x\} = \bigcap_1^\infty W_\sigma(L'_n)$.

PROOF.

1. Suppose $\bigcap_1^\infty L'_n = \{x\}$ where $L_n \in \mathcal{L}$. Consider $\bigcap_1^\infty W_\sigma(L_n)'$ in $I_R^\sigma(\mathcal{L})$. Let $\mu \in \bigcap_1^\infty W_\sigma(L_n)' \Rightarrow \mu \in W_\sigma(L'_n)$ for all $n \Rightarrow \mu(L'_n) = 1$ for all n and since $x = \bigcap_{i=1}^\infty L'_n$ and one can extend μ to $\sigma(\mathcal{L})$ then $\mu(\{x\}) = 1$ therefore if $A \in \mathcal{A}(\mathcal{L})$ and $x \in A \Rightarrow \mu(A) = 1$ therefore

$$\mu_x \le \mu \text{ on } \mathcal{L}, \mu_x \in I_R(\mathcal{L}) \text{ i.e. } \mu_x = \mu \text{ and hence } \bigcap_1^\infty W_\sigma(L_n)' = \{x\}.$$

2. If $\{\mu\} = \bigcap_1^\infty 0_n$ in $I_R^\sigma(\mathcal{L})$ where 0_n are open then $\mu \in W_\sigma(L'_n) \subset 0_n$ where $L_n \in \mathcal{L}$. Therefore

$$\{\mu\} = \bigcap_1^\infty W_\sigma(L'_n) = W_\sigma\left(\bigcap_1^\infty (L'_n)\right) \text{ and hence } \bigcap_1^\infty L'_n \ne \varnothing$$

thus

$$x \in \bigcap_1^\infty L'_n \Rightarrow \mu = \mu_x \text{ i.e. } \bigcap_1^\infty L'_n = \{x\}.$$

We now give some applications of the previous results.

THEOREM 4.11. Let $\mathcal{L}$ be a lattice of subsets of $X, \mathcal{L}$ separating and disjunctive. Suppose for every $\mu \in I_R(\mathcal{L}) - X$ there exists $Z \in \mathcal{Z}(\tau W(\mathcal{L}))$ such that $\mu \in Z \subset I_R(\mathcal{L}) - X$. Then $\mathcal{L}$ is replete.

PROOF. Suppose $\mathcal{L}$ is not replete i.e. $X \ne I_R^\sigma(\mathcal{L})$. Let $\mu \in I_R^\sigma(\mathcal{L}) - X$ then from the above condition there exists $Z \in \mathcal{Z}(\tau W(\mathcal{L}))$ such that $\mu \in Z \subset I_R^\sigma(\mathcal{L}) - X$ but $Z = \bigcap_1^\infty W(L_n)'$ where $L_n \in \mathcal{L}$. Therefore

$$\mu \in \bigcap_{n=1}^\infty W_\sigma(L_n)' \subset I_R^\sigma(\mathcal{L}) - X$$

$$\mu \in W_\sigma\left[\bigcap_1^\infty L'_n\right] \subset I_R^\sigma(\mathcal{L}) - x$$

because

$$W_\sigma\left(\bigcap_1^\infty L'_n\right) = \bigcap_1^\infty W_\sigma(L'_n).$$

Therefore $\bigcap_{n=1}^\infty L'_n \neq \varnothing$ because $\mu \in w_\sigma(\cap L'_n)$ which is a contradiction for

$$W_\sigma\left(\bigcap_{n=1}^\infty L'_n\right) \subset I_R^\sigma(\mathcal{L}) - X \text{ i.e. } W_\sigma\left(\bigcap_{n=1}^\infty L'_n\right) \cap X = \varnothing = \bigcap_1^\infty L'_n.$$

Therefore $\mathcal{L}$ must be replete.

THEOREM 4.12. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X. If $\mathcal{L}$ is normal, coutably paracompact and replete then for any $\mu \in I_R(\mathcal{L}) - X$; $\exists Z \in \mathcal{Z}(\tau W(\mathcal{L}))$ such that $\mu \in Z \subset I_R(\mathcal{L}) - X$.

PROOF. Since $\mathcal{L}$ is replete then $I_R^\sigma(\mathcal{L}) = I_R^\tau(\mathcal{L}) = X$. Let $\mu \in I_R(\mathcal{L}) - X = I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})$ then $\exists L_n \in \mathcal{L} L_n \downarrow \varnothing$ such that

$$\mu \in \bigcap_1^\infty W(L_n) \subset I_R(\mathcal{L}) - X.$$

Now since $\mathcal{L}$ is normal and countably paracompact then $\exists A_n \in \mathcal{L}$ such that $L_n \subset A'_n$ and $A'_n \downarrow \varnothing$ so
$\bigcap_1^\infty W(L_n) \subset \cap W(A'_n) = Z$ i.e. $Z \in \mathcal{Z}[\tau W(\mathcal{L})]$ and, $\mu \in \cap W(L_n) \subset Z \subset I_R(\mathcal{L}) - X$.

COROLLARY 4.13. Suppose $\mathcal{L}$ is separating, disjunctive, normal and countably paracompact. Then $\mathcal{L}$ is replete if and only if for all $\mu \in I_R(\mathcal{L}) - X$ there exists $Z \in \mathcal{Z}[\tau W(\mathcal{L})]$ such that $\mu \in Z \subset I_R(\mathcal{L}) - X$. The proof is a simple combination of the two previous theorems.

THEOREM 4.14. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X. $\mathcal{L}$ is replete if and only if for each $\mu \in I_R(\mathcal{L}) - X\ \exists B \in \sigma[W(\mathcal{L})]$ such that $\mu \in B \subset I_R(\mathcal{L}) - X$.

PROOF.

1. If $\nu \in I_R^\sigma(\mathcal{L}) - X \subset I_R(\mathcal{L}) - X$ then

$$\exists B \in \sigma[W(\mathcal{L})] \text{ such that } \nu \in B \subset I_R(\mathcal{L}) - X.$$

Then $\hat{\nu}(B) = 0$ since $\nu \in I_R^\sigma(\mathcal{L})$ but $\hat{\nu}^*(\{\nu\}) = 1$ and $\nu \in B$ which is a contradiction, and thus $I_R^\sigma(\mathcal{L}) = X$.

2. Conversely if $\mathcal{L}$ is replete, let $\mu \in I_R(\mathcal{L}) - X = I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})$ then $\mu \notin I_R^\sigma(\mathcal{L}) - X$. Therefore

$$\exists L_n \in \mathcal{L}, L_n \downarrow \text{ such that } \mu \in \bigcap_{n=1}^\infty W(L_n) \subset I_R(\mathcal{L}) - X, B = \bigcap_{n=1}^\infty W(L_n) \in I_R(\mathcal{L}) - X.$$

This theorem is somewhat more general than the previous corollary because we ask less from the lattice $\mathcal{L}$, however we get a set $B \in \sigma[W(\mathcal{L})]$ rather than a zero set $z \in Z(\tau W(\mathcal{L}))$.

EXAMPLES 4.15.

We are going to apply corollary (4.13) to special cases of lattices.

1. Let X be a $T_{3\frac{1}{2}}$ space and $\mathcal{L} = \mathcal{Z}$ then X is $\mathcal{Z}$-replete if and only if $\forall p \in \beta X - X\ \exists Z$ a zero set of βX such that $p \in Z \subset \beta X - X$.
2. Let X be a T_4, countably paracompact space and $\mathcal{L} = \mathcal{F}$ then X is α-real compact if and only if $\forall p \in \omega X - X\ \exists Z$ a zero set of ωX such that $p \in Z \subset \omega X - X$. Where ωX is the Wallman compactification of X.
3. Let X be a T_1 space and $\mathcal{L} = \mathcal{B}$ ($\mathcal{B}$ is normal and countably paracompact and $I_R(\mathcal{B}) = I(\mathcal{B})$) then X is Borel-replete if and only if $\forall p \in I(B) - X = I_R(B) - X\ \exists Z$ a zero set of $I(B)$ such that $p \in Z \subset I(B) - X$.

Let (Cl) be the following condition: If $\bigcap_\alpha W(L_\alpha) \subset I_R(\mathcal{L}) - X$ there exists a countable sequence $\{L_{\alpha i}\}$ such that $\bigcap_\alpha W(L_\alpha) \subset \bigcap_1^\infty W(L_{\alpha i}) \subset I_R(\mathcal{L}) - X$.

THEOREM 4.16. Suppose that $\mathcal{L}$ is separating and disjunctive then $\mathcal{L}$ is Lindelöf if and only if (Cl) holds.

PROOF.

1. Suppose $\mathcal{L}$ is Lindelöf and let $\bigcap_\alpha W(L_\alpha) \subset I_R(\mathcal{L}) - X$ where $L_\alpha \in \mathcal{L}$ then

$$X \subset \bigcup_\alpha W(L_\alpha)' \Rightarrow X \subset \bigcup_\alpha W(L'_\alpha) \cap X = \bigcup_\alpha L'_\alpha$$

but since $\mathcal{L}$ is Lindelöf then

$$X \subset \bigcup_\alpha L'_\alpha \subset \bigcup_1^\infty L'_{\alpha i} \subset \bigcup_1^\infty (L'_{\alpha i})$$

and therefore

$$\bigcap_\alpha W(L_\alpha) \subset \bigcap_1^\infty W(L'_{\alpha i}) \subset I_R(\mathcal{L}) - X, \text{ i.e. C1 holds.}$$

2. Suppose (C1) holds and let $X = \bigcap_\alpha L'_\alpha, L_\alpha \in X$ then

$$\bigcap_\alpha W(L_\alpha) \subset I_R(\mathcal{L}) - X$$

using (C1) we get

$$\bigcap_\alpha W(L_\alpha) \subset \bigcap_1^\infty W(L_{\alpha i}) \subset I_R(\mathcal{L}) - X$$

so

$$X \subset \bigcup_{i=1}^\infty W(L'_{\alpha i}) \Rightarrow x \subset \bigcup_{i=1}^\infty W(L'_{\alpha i}) \cap X = \bigcup_{i=1}^\infty L'_{\alpha i}$$

and since

$$\bigcup_{i=1}^\infty L'_{\alpha i} \subset X$$

then

$$X = \bigcup_1^\infty L'_{\alpha i}$$

i.e. $\mathcal{L}$ is Lindelöf.

THEOREM 4.17. Suppose $\mathcal{L}$ is separating, disjunctive, normal and countably paracompact then $\mathcal{L}$ is Lindelöf if and only if for any compact $K \subset I_r(\mathcal{L}) - X \ \exists Z$ a zero set, $Z \in \mathcal{Z}(\tau W(\mathcal{L}))$ such that $K \subset Z \subset I_R(\mathcal{L}) - X$.

PROOF. Since $\mathcal{L}$ is normal then $I_R(\mathcal{L})$ is T_2 so if K is compact in $I_R(\mathcal{L})$; K is closed and therefore

$$K = \bigcap_\alpha W(L_\alpha), L_\alpha \in \mathcal{L}.$$

Now from the previous theorem we know that $\mathcal{L}$ is Lindelöf if and only if (C1) holds so if

$$K = \bigcap_\alpha W(L_\alpha) \subset I_R(\mathcal{L}) - X$$

there exists a countable set of L_α such that

$$K = \bigcap_\alpha W(L_\alpha) \subset \bigcap_{i=1}^\infty W(L_{\alpha i}) \subset I_R(\mathcal{L}) - X$$

but we know from previous work that if $\mathcal{L}$ is normal and countably paracompact then there exists a zero set Z such that

$$\bigcap_1^\infty W(L_{\alpha i}) \subset Z \subset I_R(\mathcal{L}) - X$$

so

$$K \subset \bigcap_1^\infty W(L_{\alpha i}) \subset Z \subset I_R(\mathcal{L}) - X$$

so $\mathcal{L}$ is Lindelöf if and only if for each compact $K \in M_R(\mathcal{L})$ there exists a zero set $Z \in \mathcal{Z}(\tau(W(\mathcal{L}))$ such that $K \subset Z \subset I_R(\mathcal{L}) - X$.

EXAMPLES 4.18.

1. Let X be a $T_{3\frac{1}{2}}$ space and $\mathcal{L} = \mathcal{Z}$ then $\mathcal{L}$ is Lindelöf if and only if for each compact $K \subset I_R(\mathcal{L}) - X$ there exists a zero set Z such that
$$K \subset Z \subset \beta X - X, Z \in \mathcal{Z}(\tau(W(\mathcal{Z})).$$
2. Let X be a 0-dim T_2 space and $\mathcal{L} = \mathcal{C}$ then $\mathcal{L}$ is Lindelöf if and only if for each $K \subset \beta_0 X - X$ there exists a zero set Z such that $Z \in \mathcal{Z}[\tau W(\mathcal{L})]$ and $K \subset Z \subset \beta_0 X - X$.
3. X is a T_1 space and $\mathcal{L} = \mathcal{B}$ then $\mathcal{B}$ is Lindelöf if and only if for each compact $K \subset I(\mathcal{B}) - X$ there exists $Z \in \mathcal{Z}[\tau W(\mathcal{B})]$ such that $K \subset Z \subset I(\mathcal{B}) - X$.

Finally we give some further applications to measure-replete lattices.

THEOREM 4.19. Suppose $\mathcal{L}$ is separating and disjunctive. Let $\mu \in M_R(\mathcal{L})$ and suppose for each $F \subset I_R(\mathcal{L}) - X, F$ closed in $I_R(\mathcal{L})$; $\hat{\mu}^*(F) = 0$ then $\mu \in M_R^\tau(\mathcal{L})$.

PROOF. We saw earlier work that $\hat{\mu}^* = \tilde{\mu}$ on $\tau W(\mathcal{L})$. To show that $\mu \in M_R^\tau(\mathcal{L})$ all we have to do is show that $\tilde{\mu}$ vanishes on each closed set $F \subset I_R(\mathcal{L}) - X$. Since $W(\mathcal{L})$ is compact then $F = \bigcap W(L_\alpha)$ where $L_\alpha \in \mathcal{L}$; may assume $L_\alpha \downarrow, F \subset \tau W(\mathcal{L})$ so $\hat{\mu}^*(F) = \tilde{\mu}(F)$ but $\hat{\mu}^*(F) = 0$ by hypothesis. Therefore $\tilde{\mu}(F) = 0$ and hence $\mu \in M_R^\tau(\mathcal{L})$.

THEOREM 4.20. Suppose $\mathcal{L}$ is separating and disjunctive and for each $F \subset I_R(\mathcal{L}) - X, F$ closed in $I_R(\mathcal{L})$ there exists a set $B \in \sigma[W(\mathcal{L})]$ such that $F \subset B \subset I_R(\mathcal{L}) - X$ then $M_R^\sigma(\mathcal{L}) = M_R^\tau(\mathcal{L})$.

PROOF. Let $\mu \in M_R^\sigma(\mathcal{L})$. We have to show that $\mu \in M_R^\tau(\mathcal{L})$ and that can be achieved if we show that $\hat{\mu}^*(F) = 0$. Recall that if $\mu \in M_R(\mathcal{L})$ then $\hat{\mu} \in M_R[W(\mathcal{L})] = M_R^\tau[W(\mathcal{L})]$ and $\hat{\mu}$ can be extended to $\sigma[W(\mathcal{L})]$ where the extension is $\sigma - W(\mathcal{L})$ regular. From the condition we have that if $F \subset I_R(\mathcal{L}) - X$ and F closed in $I_R(\mathcal{L})$; there exists a set $B \in \sigma[W(\mathcal{L})]$ such that $F \subset B \subset I_R(\mathcal{L}) - X$ therefore, $\hat{\mu}^*(F) \le \hat{\mu}^*(B)$, but since $\mu \in M_R^\sigma(\mathcal{L})$ then $\hat{\mu}^*(I_R(\mathcal{L}))$. Hence $\hat{\mu}^*(B) = 0$ and thus $\hat{\mu}^*(F) = 0$ i.e. $M_R^\sigma(\mathcal{L}) = M_R^\tau(\mathcal{L})$.

THEOREM 4.21. Suppose $\mathcal{L}$ is separating and disjunctive, then $M_R^\sigma(\mathcal{L}) = M_R^\tau(\mathcal{L})$ if and only if $\hat{\mu}^*(F) = 0, \mu \in M_R^\sigma(\mathcal{L})$ for all $F \subset I_R(\mathcal{L}) - X, F$ closed in $I_R(\mathcal{L})$.

PROOF.

1. Suppose $M_R^\sigma(\mathcal{L}) = M_R^\tau(\mathcal{L})$ then
$$\tilde{\mu}(F) = 0 \text{ for all } F \subset I_R(\mathcal{L}) - X, F \text{ closed in } I_R(\mathcal{L})$$
but $F = \bigcap_\alpha W(L_\alpha)$ therefore $\tilde{\mu}(F) = \hat{\mu}^*(F) = 0$.

2. Suppose $\mu \in M_R^\sigma(\mathcal{L})$. Let $F \subset I_R(\mathcal{L}) - X, F$ closed in $I_R(\mathcal{L})$ then $\hat{\mu}^*(F) = \tilde{\mu}(F) = 0$ so $\tilde{\mu}$ vanishes on all closed sets of $I_R(\mathcal{L}) - X$ i.e. $\in M_R^\tau(\mathcal{L})$.

THEOREM 4.22. Suppose $\mathcal{L}$ is a separating and disjunctive lattice. Suppose that for each closed set in $I_R(\mathcal{L}), F \subset I_R(\mathcal{L}) - X$ there exists a Baire set B such that $F \subset B \subset I_R(\mathcal{L}) - X$ then $\mathcal{L}$ is measure replete.

PROOF. Let $\mu \in M_R^\sigma(\mathcal{L})$ and $F \subset I_R(\mathcal{L}) - X, F$ closed in $I_R(\mathcal{L})$ then $\exists B \in \sigma[W(\mathcal{L})]$ such that $F \subset B \subset I_R(\mathcal{L}) - X$ then

$$\tilde{\mu}(F) \le \tilde{\mu}(B) = \hat{\mu}_*(I_R(\mathcal{L}) - X) = 0$$

therefore $\tilde{\mu}(F) = 0$ so $\tilde{\mu}$ vanishes on every closed set of $I_R(\mathcal{L}) - X$ i.e. $\mu \in M_R^\tau(\mathcal{L})$.

EXAMPLES 4.23.

1. X is $T_{3\frac{1}{2}}$; $\mathcal{L} = \mathcal{Z}$ then

$$M_R^\sigma(\mathcal{Z}) = M_R^\tau(\mathcal{Z}) \text{ if and only if } \hat{\mu}^*(F) = \tilde{\mu}(F) = 0$$

for every $F \subset \beta X - X$ and F closed in βx and $\mu \in M_R^\sigma(\mathcal{Z})$.

2. If X is T_1; $\mathcal{L} = \mathcal{B}$ then $M_R(\mathcal{B}) = M(\mathcal{B})$ and

$$M_R^\sigma(\mathcal{B}) = M_R^\tau(\mathcal{B}) \text{ if and only if } \hat{\mu}^*(F) = \tilde{\mu}(F) = 0$$

for every $F \subset I(B) - XF$ closed in $I(B)$.

3. If X is a 0-dim T_2 space $\mathcal{L} = \mathcal{C}$ then $M_R(\mathcal{C}) = M(\mathcal{C})$ and

$$M^\sigma(\mathcal{C}) = M_\tau^\tau(\mathcal{C}) \text{ if and only if } \tilde{\mu}(F) = \hat{\mu}^*(F) = 0$$

for $F \subset \beta_0 X - XF$ closed in $\beta_0 X$.

4. If X is a T_1 space and $\mathcal{L} = \mathcal{F}$ then

$$M_R^\sigma(\mathcal{F}) = M_R^\tau(\mathcal{F}) \text{ if and only if } \hat{\mu}^*(F) = 0$$

for all $F \subset wX - X$; F closed in wX.

5. If X is $T_{3\frac{1}{2}}$ and $\mathcal{L} = \mathcal{Z}$ then Z is measure-compact if for each $F \subset \beta X - X$ and F is closed in βX, there exists a Baire set B of βX such that $F \subset B \subset \beta X - X$.

5. THE SPACE $I_R^\sigma(\mathcal{L})$:

DEFINITION 5.1. Let $\mathcal{L}$ be a disjunctive lattice of subsets of X.

1) $W_\sigma(L) = \{\mu \in I_R^\sigma(\mathcal{L}) \mid \mu(L) = 1\}; L \in \mathcal{L}$

2) $W_\sigma(\mathcal{L}) = \{W_\sigma(L), L \in \mathcal{L}\}$

3) $W_\sigma(A) = \{\mu \in I_R^\sigma(\mathcal{L}) \mid \mu(A) = 1\} A \in \mathcal{A}(\mathcal{L})$
$W_\sigma(\mathcal{L}) = W(\mathcal{L}) \cap I_R^\sigma(\mathcal{L})$

The following properties hold:

PROPOSITION 5.2. Let $\mathcal{L}$ be a disjunctive lattice then for $A, B \in \mathcal{A}(\mathcal{L})$

1) $W_\sigma(A \cup B) = W_\sigma(A) \cup W_\sigma(B)$

2) $W_\sigma(A \cap B) = W_\sigma(A) \cap W_\sigma(B)$

3) $W_\sigma(A') = W_\sigma(A)'$

4) $W_\sigma(A) \subset W_\sigma(B)$ if and only if $A \subset B$

5) $\mathcal{A}[W_\sigma(\mathcal{L})] = W_\sigma[\mathcal{A}(\mathcal{L})]$

The proof is the same as for $W(\mathcal{L})$ by simply using the properties of $W(\mathcal{L})$ and the fact that $W_\sigma(A) = W(A) \cap I_R(\mathcal{L})$ and $W_\sigma(B) = W(B) \cap I_R(\mathcal{L})$.

REMARK. It is not difficult to show that $\sigma[W_\sigma(\mathcal{L})] = W_\sigma[\sigma(\mathcal{L})]$. Also, for each $\mu \in M(\mathcal{L})$ we define μ' on $\mathcal{A}[W_\sigma(\mathcal{L})]$ as follows:

$$\mu'[W_\sigma(A)] = \mu(A) \text{ where } A \in \mathcal{A}(\mathcal{L})$$

μ' is defined and the map $\mu \to \mu'$ from $M(\mathcal{L})$ to $M(W_\sigma(\mathcal{L}))$ is onto. In addition, it can readily be checked that,

THEOREM 5.3. Let $\mathcal{L}$ be disjunctive then

1) $\mu \in M(\mathcal{L})$ if and only if $\mu' \in M[W_\sigma(\mathcal{L})]$

2) $\mu \in M_R(\mathcal{L})$ if and only if $\mu' \in M_R[W_\sigma(\mathcal{L})]$

3) $\mu \in M^{\sigma}(\mathcal{L})$ if and only if $\mu' \in M_{\sigma}[W_{\sigma}(\mathcal{L})]$

4) $\mu \in M_{\sigma}(\mathcal{L})$ if and only if $\mu' \in M^{\sigma}[W_{\sigma}(\mathcal{L})]$

5) $\mu \in M_R^{\sigma}(\mathcal{L})$ if and only if $\mu' \in M_R^{\sigma}[W_{\sigma}(\mathcal{L})]$

We next consider properties of the lattice $W_{\sigma}(\mathcal{L})$.

PROPOSITION 5.4. Let $\mathcal{L}$ be a disjunctive lattice of subsets of X then:

1) $W_{\sigma}(\mathcal{L})$ is disjunctive.

2) $W_{\sigma}(\mathcal{L})$ is T_1.

3) $W_{\sigma}(\mathcal{L})$ is replete.

PROOF. The proof of this Theorem is known. Let (C2) be the following condition: For each $\mu \in I_R(\mathcal{L})$ there exists at most one $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$.

THEOREM 5.5. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X. Then $(I_R^{\sigma}(\mathcal{L}), \tau W_{\sigma}(\mathcal{L}))$ is T_2 if and only if (C2) holds.

PROOF.

1) Suppose $(I_R^{\sigma}(\mathcal{L}), \tau W(\mathcal{L}))$ is T_2; then $W_{\sigma}(\mathcal{L})$ is T_2; if $\mu' \in I[W_{\sigma}(\mathcal{L})]$ then $S(\mu') = \varnothing$ or $\{\nu\}$, where $\nu \in I_R^{\sigma}(\mathcal{L})$. Since $S(\mu') = \{\nu \in I_R^{\sigma}(\mathcal{L}) \mid \mu \leq \gamma \text{ on } \mathcal{L}\} = \varnothing$ or a singleton then (C2) holds.

2) Suppose (C2) holds and let $\mu' \in I[W_{\sigma}(\mathcal{L})]$ if $S(\mu') \neq \varnothing$ and $\nu_1, \nu_2 \in S(\mu'); \nu_1 \neq \nu_2$ then $\mu \leq \nu_1$ and $\mu \leq \nu_2$ on $\mathcal{L}$ which is a contradiction to (C2) therefore $S(\mu') = \varnothing$ or $\{\nu\}$. i.e. $\tau W_{\sigma}(\mathcal{L})$ is T_2. Let $\mu \in M_R(\mathcal{L})$, then $\mu' \in M_R(W_{\sigma}(\mathcal{L}))$ by theorem 5.1. We wish to investigate conditions under which μ' has further smoothness properties. Recalling the notations of section 4 we have,

THEOREM 5.6. Let $\mathcal{L}$ be a disjunctive lattice of subsets of X. If $\mu \in M_R^{\sigma}(\mathcal{L})$ then the following statements are equivalent:

1. $\mu' \in M_R^{\tau}[W_{\sigma}(\mathcal{L})]$

2. If $\{L_{\alpha}\}$ is a net in $\mathcal{L}$ such that $L_{\alpha} \downarrow, \bigcap_{\alpha} W(L_{\alpha}) \subset I_R(\mathcal{L}) - I_R^{\sigma}(\mathcal{L})$ then $\bar{\mu}\left[\bigcap_{\alpha} W(L_{\alpha})\right] = 0$

3. $\bar{\mu}^{*}(I_R^{\sigma}(\mathcal{L})) = \bar{\mu}(I_R(\mathcal{L}))$

PROOF.

$1 \Rightarrow 2$. Suppose $\mu' \in M_R^{\tau}[W_{\sigma}(\mathcal{L})]$ and let $\{L_{\alpha}\}$ be a net in $\mathcal{L}$ such that $L_{\alpha} \downarrow$ then $W(L_{\alpha}) \downarrow$ and $W_{\sigma}(L_{\alpha}) \downarrow$ then

$$\bar{\mu}\left[\bigcap_{\alpha} W(L_{\alpha})\right] = \inf_{\alpha} \bar{\mu}(W(L_{\alpha})) = \lim_{\alpha} \hat{\mu}(W(L_{\alpha})) = \lim_{\alpha} \mu(L_{\alpha}) = \lim_{\alpha} \mu'[W_{\sigma}(L_{\alpha})]$$

but since $W_{\sigma}(L_{\alpha}) \downarrow$ and $\mu' \in M_R^{\tau}[W_{\sigma}(\mathcal{L})]$ then

$$0 = \lim_{\alpha} \mu'[W_{\sigma}(L_{\alpha})] = \bar{\mu}\left(\bigcap_{\alpha} W(L_{\alpha})\right).$$

$2 \Rightarrow 1$. Let $W_{\sigma}(L_{\alpha}) \downarrow \varnothing, L_{\alpha} \in \mathcal{L}$ then

$$\bigcap_{\alpha} W_{\sigma}(L_{\alpha}) = \varnothing \text{ or } \bigcap_{\alpha}[W(L_{\alpha}) \cap I_R^{\sigma}(\mathcal{L})] = \varnothing.$$

Therefore $\bigcap W(L_{\alpha}) \subset I_R(\mathcal{L}) - I_R^{\sigma}(\mathcal{L})$ and using 2 we get,

$$0 = \tilde{\mu}\left(\bigcap_{\alpha} W(L_\alpha)\right) = \mu'\left[\bigcap_{\alpha} W_\sigma(L_\alpha)\right].$$

$2 \Rightarrow 3$. Assume 2 is true then

$$\tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}_*[I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})] + \tilde{\mu}^\sigma(I_R^\sigma(\mathcal{L}))$$

so

$$\tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) \text{ if and only if } \tilde{\mu}_*[I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})] = 0.$$

Now

$$\tilde{\mu}_*[I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})] = \{\tilde{\mu}(K), K \in \tau W(\mathcal{L}) \text{ and } K \subset I_R - I_R^\sigma(\mathcal{L})\} K \in \tau W(\mathcal{L}) \text{ then}$$

$$K = \bigcap_{\alpha} W(L_\alpha) \subset I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})$$

where we may assume $W(L_\alpha) \downarrow$ then

$$\tilde{\mu}(K) = \tilde{\mu}\left(\bigcap_{\alpha} W(L_\alpha)\right) = 0$$

and therefore

$$\tilde{\mu}_*(I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})) = 0.$$

$3 \Rightarrow 2$. Assume 3 is true and let

$$L_\alpha \in \mathcal{L}, L_\alpha \downarrow \text{ and } \bigcap_{\alpha} W(L_\alpha) \subset I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})$$

then

$$0 \le \tilde{\mu}\left(\bigcap_{\alpha} W(L_\alpha)\right) \le \tilde{\mu}_*(I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})) = 0.$$

COROLLARY 5.7. If $\mathcal{L}$ is a separating, disjunctive and replete lattice of subsets of X then $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$ implies $\mu \in M_R^\tau(\mathcal{L})$.

PROOF. Since $\mathcal{L}$ is replete then $X = I_R^\sigma(\mathcal{L})$ then from the previous theorem we have

$$\tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}^*(X)$$

i.e. $\mu \in M_R^\tau(\mathcal{L})$ from theorem (4.5).

COROLLARY 5.8. Let $\mathcal{L}$ be separating and disjunctive. Suppose $\mu' \in M_R^\tau(W_\sigma(\mathcal{L})) \Rightarrow \mu \in M_R^\tau(\mathcal{L})$ then $\mathcal{L}$ is replete.

PROOF. Let $\mu \in I_R^\sigma(\mathcal{L})$ then since $W_\sigma(\mathcal{L})$ is replete $\mu' \in I_R^\tau[W_\sigma(\mathcal{L})]$ then by hypothesis $\mu \in I_R^\tau(\mathcal{L})$ therefore $I_R^\sigma(\mathcal{L}) = I_R^\tau(\mathcal{L})$ or $\mathcal{L}$ is replete.

If we combine the two corollaries we get the following:

THEOREM 5.9. Let $\mathcal{L}$ be separating and disjunctive. Then $\mathcal{L}$ is replete if and only if $\mu' \in M_R^\tau(W_\sigma(\mathcal{L})) \Rightarrow \mu \in M_R^\tau(\mathcal{L})$.

REMARK. Let $\mu \in M_R(\mathcal{L})$. We say that there is a one to one correspondence between $M_R(\mathcal{L})$ and $M_R[W(\mathcal{L})]$, and we defined $\hat{\mu}$ on $\mathcal{A}[W(\mathcal{L})]$ such that for all $A \in \mathcal{A}(\mathcal{L})$, $\hat{\mu}[W(A)] = \mu(A)$. Since $W_\sigma(\mathcal{L}) = W(\mathcal{L}) \cap I_R^\sigma(\mathcal{L})$ we can restrict $\hat{\mu}$ on $\mathcal{A}[W_\sigma(\mathcal{L})]$ and we call the restriciton μ'_0 defined as

$$\mu'_0[W_\sigma(A)] = \mu'_0[W(A) \cap I_R^\sigma(\mathcal{L})] = \hat{\mu}[W(A)].$$

μ'_0 is well defined and the restriction is a 1-1 correspondence since $\hat{\mu}^*(I_R^\sigma(\mathcal{L})) = \hat{\mu}(I_R(\mathcal{L}))$ i.e. by thickness. Hence μ'_0 in $M_R[W_\sigma(\mathcal{L})]$ and $\mu'_0 = \mu'$.

PROPOSITION 5.10. Let $\mathcal{L}$ be a separating, disjunctive and normal lattice. Let $\lambda \in M_R[\tau W(\mathcal{L})]$ and $\lambda^*[I_R^\sigma(\mathcal{L})] = \lambda[I_R(\mathcal{L})]$ then $\lambda = \tilde{\mu}, \mu \in M_R(\mathcal{L})$ and $\mu' \in M_R^\tau(W_\sigma(\mathcal{L}))$.

PROOF. Suppose

$$\lambda \in M_R(\tau W(\mathcal{L})) = M_R^\sigma(\tau W(\mathcal{L})) = M_R^\tau(\tau W(\mathcal{L})) \text{ and } \lambda^*(I_R^\sigma(\mathcal{L})) = \lambda(I_R(\mathcal{L})).$$

Restrict λ to $\hat{\mu} \in M_R(W(\mathcal{L}))$. The restriction is unique because $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$ and since $\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$ then $\lambda = \tilde{\mu}$. $\lambda = \tilde{\mu}$ projects onto $I_R^\sigma(\mathcal{L})$ and is denoted by ν. $\mu' \in M_R^\tau(W_\sigma(\mathcal{L}))$ and has a unique extension to $M_R^\tau(\tau W_\sigma(\mathcal{L}))$ and of course ν is that extension.

$$\nu\left(\bigcap_\alpha W_\sigma(L_\alpha)\right) = \tilde{\nu}\left(\bigcap_\alpha W(L_\alpha)\right) = \inf \tilde{\mu}(W(L_\alpha)) = \inf \mu'(W_\sigma(L_\alpha)).$$

THEOREM 5.11. Suppose $\mathcal{L}$ is a separating, disjunctive and normal lattice of subsets of X, then the following statements are equivalent:

1. $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$
2. $I_R^\sigma(\mathcal{L})$ is $\tilde{\mu}^*$-measurable and $\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$.

PROOF.

$1 \Rightarrow 2$. Suppose 1 holds then $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$ and then using theorem 5.4 we get $\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$. We saw in earlier work that $\tilde{\mu}$ projects on $I_R^\sigma(\mathcal{L})$ where the projection is $\nu \in M_R^\tau[\tau W_\sigma(\mathcal{L})]$ and is the unique extension of $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$. Now since $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$ there exists a compact set $K \in W_\sigma(\mathcal{L})$ such that $\mu'_*(I_R^\sigma(\mathcal{L}) - K) < \varepsilon$ for any $\varepsilon > 0$ so

$$\mu'_*(I_R^\sigma(\mathcal{L}) - K) + \mu'^*(K) = \mu'(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$$

$$\begin{aligned} \mu'^*(K) &= \inf \mu'(A), K \subset A \text{ and } A \in \sigma[W_\sigma(\mathcal{L})] \\ &= \inf \nu(A), K \subset A \text{ and } A \in \sigma[W_\sigma(\mathcal{L})] \\ &\geq \nu(K). \end{aligned}$$

Therefore $\mu'^*(K) \geq \nu(K)$. $K \in \tau W_\sigma(\mathcal{L})$, since $I_R^\sigma(\mathcal{L})$ is T_2 because $\mathcal{L}$ is normal; then $K = \bigcap_\alpha W_\sigma(L_\alpha), L_\alpha \in \mathcal{L}$ and may assume $L_\alpha \downarrow$ so

$$\nu(K) = \inf \nu[W_\sigma(L_\alpha)] \geq \inf_{\substack{A \subset K \\ A \in \sigma[W_\sigma(\mathcal{L})]}} \nu(A) = \mu'^*(K).$$

Therefore $\nu(K) = \mu^{*\prime}(K)$ and

$$\nu[I_R^\sigma(\mathcal{L}) - K] = \mu'_*[I_R^\sigma(\mathcal{L}) - K] = \tilde{\nu}[I_R(\mathcal{L}) - K] < \varepsilon$$

where K is compact in $I_R^\sigma(\mathcal{L})$ and $I_R(\mathcal{L})$ because it is a closed subset of a T_2 space. So $I_R(\mathcal{L}) - K$ is open, $I_R(\mathcal{L}) - K \subset I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})$ and $\tilde{\mu}(I_R(\mathcal{L}) - K) < \varepsilon$. Therefore $\tilde{\mu}^*(I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})) = 0$. So $I_R^\sigma(\mathcal{L})$ is $\tilde{\mu}^*$-measurable and

$$\tilde{\mu}(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L})).$$

$2 \Rightarrow 1$. Suppose 2 holds. Since $\mu' \in M_R[W_\sigma(\mathcal{L})]$ then

$$\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \sup\{\tilde{\mu}(K); K \in \tau W_\sigma(\mathcal{L}) \text{ and } K \subset I_R^\sigma(\mathcal{L})\}$$

then there exists a compact set $K \in \tau W_\sigma(\mathcal{L}), K \subset I_R^\sigma(\mathcal{L})$ and $K - W_\sigma(\mathcal{L})$ such that $\tilde{\mu}(K) > \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) - \in \forall \in > 0$. Let $K' = I_R^\sigma(\mathcal{L}) - K$ then

$$\nu(K') = \mu'_*(K') \Rightarrow \mu'_*(K) = \nu(K) \text{ but}$$

$$\nu(K) = \nu(I_R^\sigma(\mathcal{L}) \cap K) = \tilde{\mu}(K) > \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) - \varepsilon > \tilde{\mu}(I_R(\mathcal{L})) - \varepsilon$$

so

$$\mu'_*(K') = \mu'_*(I_R^\sigma(\mathcal{L}) - K) < \varepsilon$$

i.e. $\mu' \in M_R^t(\mathcal{L})$.

THEOREM 5.12. Let $\mathcal{L}$ be a separating, disjunctive, normal and replete lattice then

$$\mu' \in M_R^t[W_\sigma(\mathcal{L})] \text{ if and only if } \mu \in M_R^t(\mathcal{L}).$$

PROOF.

1. Let $\mu' \in M_R^t[W_\sigma(\mathcal{L})]$ then since $\mathcal{L}$ is replete we have that $X = I_R^\sigma(\mathcal{L})$ and X is $\tilde{\mu}^*$-measurable and

$$\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}(X)$$

then by theorem 4.6 we get that $\mu \in M_R^t(\mathcal{L})$.

2. Conversely suppose $\mu \in M_R^t(\mathcal{L})$ then from theorem 4.6 we get that

$$\tilde{\mu}^*(X) = \tilde{\mu}(I_R(\mathcal{L}))$$

and X is $\tilde{\mu}^*$-measureable but $X \subset I_R^\sigma(\mathcal{L}) \subset I_R(\mathcal{L})$ therefore $\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$, then since $\mathcal{L}$ is replete $X = I_R^\sigma(\mathcal{L})$ so $\tilde{\mu}^*(X) = \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$ then from theorem 5.11 $\mu' \in M_R^t[W_\sigma(\mathcal{L})]$.

REFERENCES

1. Bachman, G. and Szeto, M., On Strongly Measure Replete Lattices, Support of a Measure and the Wallman Remainder, Per. Math. Hung, 15(2) (1984), 127-155.
2. Alexandroff, A. D., Additive Set Functions in Abstract Spaces, Mat. Sb, (N.S.) 8, 50 (1940), 307-348.
3. Frolik, Z., Prime Filter with C.I.P., Comment. Math. Univ. Carolinae, 13, No. 3 (1972), 553-573.
4. Grassi, P., On Subspaces of Replete and Measure Replete Spaces, Canad. Math. Bull., 27(1), (1984), 58-64.
5. Nöbeling, G., Grundlagender Analytischen Topologie, Springer-Verlag, Berlin, 1954.
6. Wallman, H., Lattices and Topological Spaces, Ann. of Math. 39, 1938), 112-126.
7. Choquet, G., Theory of Capacities, Ann. Inst. Fourier, 5 (1953-1954), 131-295.
8. Gardner, R. J., The Regularity of Borel Measure and Borel Measure-Compactness, Proc. London Math. Soc., Ser. 3, 30 (1975), 95-113.
9. Gachman, G. and Sultan, A., On Regular Extensions of Measures, Pacific J. Math., Vol. 86 (1980), 389-395.
10. Marik, J., The Baire and Borel Meausres, Czech. J. Math., 7 (1957), 248-253.