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ON CERTAIN REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. We classify certain real hypersurfaces ot a quaternionic projective space satisfying the condition $\sigma(R(X,Y)SZ) = 0$.

KEY WORDS AND PHRASES: Quaternionic projective space, real hypersurface.
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1. INTRODUCTION.

Let M be a connected real hypersurface of a quaternionic projective space QP^n, $n \geq 2$, with metric g of constant quaternionic sectional curvature 4. Let ξ be the unit local normal vector field on M and $\{\psi_1, \psi_2, \psi_3\}$ a local basis of the quaternionic structure of QP^n, (See [1]). Then $U_1 = \psi_i\xi$, i=1,2,3 are tangent to M. It is known, [3], that the unique Einstein real hypersurfaces of QP^n are the open subsets of geodesic hyperspheres of QP^n of radius r such that $\cot^2 r = 1/(2n)$. This paper is devoted to study real hypersurfaces M of QP^n satisfying the following condition

$$R(X,Y)SZ + R(Y,Z)SX + R(Z,X)SY = 0 \tag{1.1}$$

for any X,Y,Z tangent to M, where R denotes the curvature tensor and S the Ricci tensor of M. Concretely we prove the following:

THEOREM 1. Let M be a real hypersurface of QP^n, $n \geq 2$, satisfying Condition (1.1) and such that U_i, i=1,2,3, are principal. Then M is an open subset of a geodesic hypersphere of QP^n of radius r, $0 < r < \pi/2$, such that $\cot^2 r = l/(2n)$.

Clearly condition (1.1) is weaker than R.S=0. Thus we also obtain

COROLLARY 2. The unique real hypersurfaces of QP^n, $n \geq 2$, satisfying R.S=0 and such that U_i, i=1,2,3, are principal are open subsets of geodesic hyperspheres of radius r, $0 < r < \pi/2$, such that $\cot^2 r = 1/(2n)$.

COROLLARY 3. A real hypersurface of QP^n, $n \geq 2$, with U_i, i=1,2,3, principal cannot satisfy the condition R.R=0.

Where for any X,Y tangent to M, $R(X,Y).T = \nabla_X\nabla_Y T - \nabla_Y\nabla_X T - \nabla_{[X,Y]}$ for any tensor field T on M, (see, for example, [5]).

2. PRELIMINARIES

Let X be a vector field tangent to M. We write $\psi_i X = \phi_i X + f_i(X)\xi$, i=1,2,3, where $\phi_i X$ denotes the tangential component of $\psi_i X$ and $f_i(X) = g(X,U_i)$. From this, [4], we have

$$g(\phi_i X,Y) + g(X,\phi_i Y) = 0, \qquad \phi_i U_i = 0, \qquad \phi_j U_k = -\phi_k U_j = U_t \tag{2.1}$$

for any X and Y tangent to M, i=1,2,3 and (j.k.t) being a cyclic permutation of (1,2,3).

From the expression of the curvature tensor of QP^n, [4], the equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}\{g(\emptyset_iY,Z)\emptyset_iX - g(\emptyset_iX,Z)\emptyset_iY + $$

$$+ 2g(X,\emptyset_iY)\emptyset_iZ\} + g(AY,Z)AX - g(AX,Z)AY \tag{2.2}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^{3}\{f_i(X)\emptyset_iY - f_i(Y)\emptyset_iX + 2g(X,\emptyset_iY)U_i\} \tag{2.3}$$

for any X,Y,Z tangent to M, where A denotes the Weingarten endomorphism of M. The Ricci tensor of M has the following expression

$$SX = (4n+7)X - 3\sum_{i=1}^{3}\{f_i(X)U_i + hAX - A^2X \tag{2.4}$$

for any X tangent to M, h being the trace of A.
If U_i, i=1,2,3, are principal and have the same principal curvature α, this is constant, [4], and from (2.3) it is easy to see that

$$2A\emptyset_iAX = \alpha(A\emptyset_i + \emptyset_iA)X + 2\emptyset_iX + 2f_k(X)U_j - 2f_j(X)U_k \tag{2.5}$$

for any X tangent to M, where (i,j,k) is a cyclic permutation of (1,2,3).

3. PROOF OF THEOREM 1.

Let Z be a tangent field to M, orthogonal to U_i, i=1,2,3, and principal with principal curvature λ. Then, from Condition (1.1) and (2.4) we have

$$(4n+7+h\lambda-\lambda^2)R(U_1,U_2)Z + (4n+4+h\alpha_1-\alpha_1^2)R(U_2,Z)U_1 + (4n+4+h\alpha_2-\alpha_2^2)R(Z,U_1)U_2 = 0 \tag{3.1}$$

where α_1 is the principal curvature of U_i, i=1,2,3.
From (3.1) and the identity of Bianchi we obtain

$$(3+h\lambda-\lambda^2)R(U_1,U_2)Z + (h\alpha_1-\alpha_1^2)R(U_2,Z)U_1 + (h\alpha_2-\alpha_2^2)R(Z,U_1)U_2 = 0 \tag{3.2}$$

that is,

$$(3+h\lambda-\lambda^2-h\alpha_1+\alpha_1^2)R(U_1,U_2)Z + h\alpha_2-\alpha_2^2-h\alpha_1+\alpha_1^2)R(Z,U_1)U_2 = 0 \tag{3.3}$$

From (2.2), (3.3) gives $h\alpha_2-\alpha_2^2-h\alpha_1+\alpha_1^2 = 2(3+h\lambda-\lambda^2-h\alpha_1+\alpha_1^2)$. Changing (U_1,U_2) in (3.1) by (U_2,U_3) or (U_3,U_1), respectively, we obtain

$$h\alpha_i-\alpha_i^2+h\alpha_j-\alpha_j^2 = 6+2h\lambda-2\lambda^2, i\neq, i,j=1,2,3 \tag{3.4}$$

From (3.4) we get

$$h(\alpha_i-\alpha_j) = \alpha_1^2-\alpha_j^2 \tag{3.5}$$

thus either $\alpha_i=\alpha_j$ or $\alpha_i+\alpha_j=h$.
Let us suppose that $\alpha_1\neq\alpha_2=\alpha_3$. Then $\alpha_1+\alpha_2=h$. Thus α_i, i=1,2,3, must satisfy the equation $\alpha^2-h\alpha+\alpha_1\alpha_2=0$. Then we have $(hA-A^2)U_i=\alpha_1\alpha_2U_i$, i=1,2,3, and from (2.4)

$$SU_i = (4n+4+\alpha_1\alpha_2)U_i \tag{3.6}$$

From (3.4) we also have $h(\alpha_1+\alpha_2)-\alpha_1^2-\alpha_2^2=6+2h\lambda-2\lambda^2$, but $h=\alpha_1+\alpha_2$. Thus $\alpha_1\alpha_2=3+h\lambda-\lambda^2$. This means that for any Z orthogonal to U_1, i=1,2,3, $(hA-A^2)Z=(\alpha_1\alpha_2-3)Z$, and from (2.4),

$$SZ=(4n+4+\alpha_1\alpha_2)Z \tag{3.7}$$

From (3.6) and (3.7), M must be Einstein. But this is a contradiction (see [3]). Thus $\alpha_i=\alpha_j=\alpha$, $i\neq j$. Then α is constant and from (3.4) we have

$$3+h(\lambda-\alpha)-\lambda^2+\alpha^2=0 \tag{3.8}$$

But from (2.5), $\phi_i Z$ is also principal and its principal curvature is $\mu=(\lambda\alpha+2)/(2\lambda-\alpha)$. Thus we also get

$$3+h(\mu-\alpha)-\mu^2+\alpha^2=0 \tag{3.9}$$

Then from (3.8) and (3.9) we obtain that either $\lambda=\mu$ or $\lambda+\mu=h$. If $\lambda=\mu$, λ must satisfy the equation $\lambda^2-\lambda\alpha-1=0$. If $\lambda+\mu=h$, λ must satisfy the equation $\alpha\lambda^2-2(\alpha^2+4)\lambda+\alpha^3+5\alpha=0$. In both cases all the principal curvatures are constant. Thus, [3], M must be an open subset of either a geodesic hypersphere or of a tube of radius r, $0<r<\pi/2$ over QP^k, $0<k<n-1$. It is easy now to see that the only ones satisfying (3.8) are open subsets of geodesic hyperspheres of radius r, $0<r<\pi/2$, such that $\cot^2 r=1/(2n)$, (see [3]). This concludes the proof.

It is also easy to see that these real hypersurfaces cannot satisfy the condition R.R=0, and then Corollary 3 is proved because R.R=0 implies R.S=0.

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