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ON MAXIMAL MEASURES WITH RESPECT TO A LATTICE

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ABSTRACT. Outer measures are used to obtain measures that are maximal with respect to a normal lattice. Alternate proofs are then given extending the measure theoretic characterizations of a normal lattice to an arbitrary, non-negative finitely additive measure on the algebra generated by the lattice. Finally these general results are used to consider σ-smooth measures with respect to the lattice when further conditions on the lattice hold.

KEY WORDS AND PHRASES. Lattice regular measures, normal lattices, σ-smooth measures.
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1. INTRODUCTION AND BACKGROUND.

A measure theoretic (equivalently filter) characterization of normal lattices is well known (see e.g. Frolik [1]) and we will give here an alternate proof of this. M. Szeto has considered (see [2]) the relationship between measures that are maximal with respect to a lattice and lattice regular measures in the case of normal and arbitrary lattices of subsets. We consider here the case of a normal lattice, and first give an alternate presentation to the one given by M. Szeto. We then apply these results to extend the characteristic result of normal lattices from zero-one valued measures to arbitrary non-negative, non-trivial finitely additive measures on the algebra generated by the lattice (see Theorem 2.2). Finally in the third and last section we extend the results of Szeto [2] by considering a measure which is σ-smooth with respect to a lattice, and give results about the associated maximal measure when the lattice is normal (see e.g. Theorem 3.4), and also countably paracompact (see e.g. Theorem 3.2).

We adhere to standard lattice and measure theoretic terminology consistent with Frolik [1], Szeto [3] and Wallman [4], and we give the main definitions and notations that will be used throughout this paper before considering normal lattices.

Let X be an abstract set, and $\mathcal{L}$ denote the lattice of subsets of X. We assume that ϕ, $X \in \mathcal{L}$ for most of our results. First:

Lattice Terminology:

$A(\mathbf{L})$ is the algebra generated by $\mathbf{L}$.

$\sigma(\mathbf{L})$ is the σ-algebra generated by $\mathbf{L}$.

$\delta(\mathbf{L})$ is the lattice of all countable intersections of sets from $\mathbf{L}$. We have a delta lattice (δ-lattice) if $\delta(\mathbf{L}) = \mathbf{L}$.

$\tau(\mathbf{L})$ is the lattice of arbitrary intersections of sets of $\mathbf{L}$.

$\mathbf{L}$ is complemented if $L \in \mathbf{L} \Rightarrow L' \in \mathbf{L}$ ($\mathbf{L}$ is an algebra).

$\mathbf{L}$ is normal if for all $L_1, L_2 \in \mathbf{L}$ such that $L_1 \cap L_2 = \phi$ there exists $\hat{L}_1, \hat{L}_2 \in \mathbf{L}$ such that $L_1 \subset \hat{L}_1'$, $L_2 \subset \hat{L}_2'$ and $\hat{L}_1 \cap \hat{L}_2' = \phi$.

$\mathbf{L}$ coallocates iteself if $L \subset L_1' \cup L_2'$ where $L, L_1, L_2 \in \mathbf{L}$ then $L = L_3 \cup L_4$ where $L_3 \subset L_1'$ and $L_4 \subset L_2'$ and $L_3, L_4 \in \mathbf{L}$. Note $\mathbf{L}$ coallocates itself if and only if $\mathbf{L}$ is normal.

$\mathbf{L}$ is compact if every covering of X by elements of $\mathbf{L}'$ has a finite subcovering.

$\mathbf{L}$ is countably compact if every countable covering of X by elements of $\mathbf{L}'$ has a finite subcovering.

$\mathbf{L}$ is countably paracompact if, whenever $A_n \downarrow \phi$, $A_n \in \mathbf{L}$ there exists $B_n \in \mathbf{L}$ such that $A_n \subset B_n'$ and $B_n' \downarrow \phi$.

Measure Terminology

We denote by $M(\mathbf{L})$ the finitely additive bounded measures on $A(\mathbf{L})$ (we may and do assume all elements of $M(\mathbf{L})$ are ≥ 0).

$\mu \in M(\mathbf{L})$ is $\mathbf{L}$-regular if for any $A \in A(\mathbf{L})$, $\mu(A) = \sup\{\mu(L) \mid L \subset A, L \in \mathbf{L}\}$;

(equivalently) $= \inf\{\mu(L') \mid A \subset L', L \in \mathbf{L}\}$.

$\mu \in M(\mathbf{L})$ is σ-smooth on $\mathbf{L}$ if $L_n \in \mathbf{L}$, $n = 1, 2, \ldots$ and $L_n \downarrow \phi \Rightarrow \mu(L_n) \to 0$.

$\mu \in M(\mathbf{L})$ is σ-smooth on $A(\mathbf{L})$ if $A_n \in A(\mathbf{L})$, $n = 1, 2, \ldots$ and $A_n \downarrow \phi \Rightarrow \mu(A_n) \to 0$.

Note μ is σ-smooth on $A(\mathbf{L})$ iff μ is countably additive.

We will use the following notations:

$M_R(\mathbf{L})$ = the set of $\mathbf{L}$-regular measures of $M(\mathbf{L})$.

$M_\sigma(\mathbf{L})$ = the set of σ-smooth measures on $\mathbf{L}$ of $M(\mathbf{L})$.

$M^\sigma(\mathbf{L})$ = the set of σ-smooth measures on $A(\mathbf{L})$ of $M(\mathbf{L})$.

$M_R^\sigma(\mathbf{L})$ = the set of $\mathbf{L}$-regular measures of $M^\sigma(\mathbf{L})$. Note that if $\mu \in M_R(\mathbf{L})$ and $\mu \in M_\sigma(\mathbf{L})$ then $\mu \in M_R^\sigma(\mathbf{L})$.

Also we denote by $I(\mathbf{L})$, $I_R(\mathbf{L})$, $I_\sigma(\mathbf{L})$, $I^\sigma(\mathbf{L})$ and $I_R^\sigma(\mathbf{L})$ the subsets of $M(\mathbf{L})$, $M_R(\mathbf{L})$, $M_\sigma(\mathbf{L})$, $M^\sigma(\mathbf{L})$ and $M_R^\sigma(\mathbf{L})$ consisting of the zero-one valued measures.

Now we consider $\mu_1, \mu_2 \in M(\mathbf{L})$: $\mu_1 \leq \mu_2(\mathbf{L})$ means $\mu_1(L) \leq \mu_2(L)$ for $L \in \mathbf{L}$. Note $\mu_1 \leq \mu_2(\mathbf{L})$ and $\mu_1(X) = \mu_2(X) \Rightarrow \mu_2 \leq \mu_1(\mathbf{L}')$. We have the following results:

1). If $\mathbf{L}$ is a normal lattice and if $\mu \in I(\mathbf{L})$ and if $\nu_1, \nu_2 \in I_R(\mathbf{L})$ and $\mu \leq \nu_1(\mathbf{L})$, $\mu \leq \nu_2(\mathbf{L})$. Then $\nu_1 = \nu_2$.

2). Let $\mu_1, \mu_2 \in M_R(\mathbf{L})$, $\mu_1 \leq \mu_2(\mathbf{L})$ and $\mu_1(X) = \mu_2(X)$, then $\mu_1 = \mu_2$.

We shall prove 1): Let X be an arbitrary set and $\mathbf{L}$ a lattice of subsets with ϕ, $X \in \mathbf{L}$, and also let $\mu \in I(\mathbf{L})$. For $E \subset X$, we define $\mu'(E) = \inf\{\mu(L') \mid E \subset L', L \in \mathbf{L}\}$.

It is easy to see that μ is a finitely subadditive outer measure, and $\mu \leq \mu'(\mathbf{L})$. Moreover $\mu = \mu'$ on $\mathbf{L}$ if and only if $\mu \in I_R(\mathbf{L})$. Next let $\mu \in I(\mathbf{L})$ and define $F = \{L \in \mathbf{L} | \mu'(L)=1\}$. It is easy to see that F is an **L**-filter.

We also have:

If **L** is normal, then F is an **L**-ultra filter.

PROOF. Suppose $F \subset H$ = **L**-filter, then there exists $L \in H$ and $L \notin F$. Therefore $\mu'(L) = 0$ which means $L \subset \hat{L}'$, $\mu(\hat{L}') = 0$. This implies $\mu(\hat{L}) = 1$, therefore $\hat{L} \in F \subset H$ using $\mu \leq \mu'(\mathbf{L})$. Therefore $\hat{L} \cap L \in H$ and since $L \subset \hat{L}'$ we get $L \cap \hat{L} = \phi \in H$. This contradicts the fact that H is an **L**-filter. Therefore F is an **L**-ultra filter.

As is well known, with F is associated a $\nu \in I_R(\mathbf{L})$, and $\mu \leq \nu(\mathbf{L})$. Uniqueness follows immediately, for if $\mu \leq \rho(\mathbf{L})$ where $\rho \in I_R(\mathbf{L})$, then $\rho \leq \mu = \mu'(\mathbf{L}')$. Suppose $\rho(A) = 1$ and $\nu(A) = \mu'(A) = 0$ where $A \in \mathbf{L}$. Then $A \subset L' \in \mathbf{L}'$, and $\mu(L') = 0$, so $\rho(L') = 0$ and therefore $\rho(A) = 0$ which is a contradiction. Thus we must have $\rho \leq \nu(\mathbf{L})$ so $\rho = \nu$, since $\rho, \nu \in I_R(\mathbf{L})$.

The more general case of $\mu \in M(\mathbf{L})$ will be considered in the next section.

2. ASSOCIATED OUTER MEASURES.

Let $\mu \in M(\mathbf{L})$ and $\mu'(E) = \inf\{\mu(L'): E \subset L', L \in \mathbf{L}\}$ where E is an arbitrary subset of X. Then it is easy to see that $\mu'(\phi) = 0$, μ' is monotone, and finitely subadditive. We shall investigate μ', and other such "outer measures" associated with μ in this section.

First, we note that if $\mu_1, \mu_2 \in M(\mathbf{L})$ and if $\mu_1 \leq \mu_2(\mathbf{L})$, and $\mu_1(X) = \mu_2(X)$, then $\mu_2' \leq \mu_1'$.

Let $\mathbf{L}_\mu = \{L \in \mathbf{L}: \mu(L) = \mu'(L)\}$ then we have

THEOREM 2.1. a) $\mathbf{L}_\mu$ is a lattice

b) $\mathbf{L}_\mu = S'_\mu \cap \mathbf{L}$

where S'_μ is the collection of μ'-measurable sets.

PROOF. Clearly we need just prove b). Since $\mathbf{L}_\mu \subset S'_\mu$ we have $\mathbf{L}_\mu \cap \mathbf{L} \subset S'_\mu \cap \mathbf{L} \Longrightarrow \mathbf{L}_\mu \subset S'_\mu \cap \mathbf{L}$. Now let $E \in S'_\mu \cap \mathbf{L}$ which implies $\mu'(L') \geq \mu'(L' \cap E) + \mu'(L' \cap E')$. From which it follows that $\mu'(X) \geq \mu'(E) + \mu'(E')$. But $\mu(X) = \mu'(X)$, and $\mu = \mu'(\mathbf{L}')$ so we have $\mu'(X) \geq \mu'(E) + \mu'(E')$ and $\mu(X) \geq \mu(E) + \mu(E')$ implying $\mu(E) = \mu'(E)$, $E \in \mathbf{L}$, which implies $E \in \mathbf{L}_\mu$.

We note that if $\mu \in M_R(\mathbf{L})$ then $\mu = \mu'$ on $A(\mathbf{L})$, and $\mathbf{L}_\mu = \mathbf{L}$.

Now let $\mu \in M(\mathbf{L})$, and define $\lambda(E) = \sup\{\mu(L): L \subset E, L \in \mathbf{L}\}$. Then:

a) $\lambda = \mu$ on $\mathbf{L}$ and $\lambda \leq \mu$ on $\mathbf{L}'$. Define $\hat{\mu}(E) = \inf\{\lambda(L'): E \subset L', L \in \mathbf{L}\}$ then

b) $\hat{\mu} = \lambda \leq \mu = \mu'$ on $\mathbf{L}'$.

c) $\mu = \lambda \leq \hat{\mu} \leq \mu'$ on $\mathbf{L}$.

PROOF. (a) $\lambda = \mu$ on $\mathbf{L}$ follows immediately from the definition of λ. Now let $E = L'$ then $\lambda(L') = \sup\{\mu(\hat{L}): \hat{L} \subset L', \hat{L} \in \mathbf{L}\}$ and $\mu(\tilde{L}) \leq \mu(L')$, so $\mu(L')$ is an upper bound implies $\sup\{\mu(\hat{L}); \hat{L} \subset L', \hat{L} \in \mathbf{L}\} \leq \mu(L')$ implying $\lambda(L') \leq \mu(L')$ giving $\lambda \leq \mu$ on $\mathbf{L}'$.

(b) $\hat{\mu} = \lambda$ on $\mathbf{L}'$ follows immediately from the definition of $\hat{\mu}$ and combining part (a) and $\mu = \mu'$ on $\mathbf{L}'$ we get $\hat{\mu} = \lambda \leq \mu = \mu'$ on $\mathbf{L}'$.

(c) Let $E = L$ then $\hat{\mu}(L) = \inf\{\lambda(L'): L \subset L',\ L \in \mathbf{L}\}$ and $\lambda(L) < \lambda(L')$. But $\lambda < \mu$ on $\mathbf{L}'$ so $\lambda(L) < \lambda(L') < \mu(L')$. So $\lambda(L)$ is a lower bound implying
$\lambda(L) < \inf\{\lambda(L'): L \subset L',\ L \in \mathbf{L}\} < \inf\{\mu(L'): L \subset L',\ L \in \mathbf{L}\}$ which implies
$\lambda(L) < \hat{\mu}(L) < \mu'(L)$. Now $\mu(L) = \lambda(L)$, $L \in \mathbf{L}$ so $\mu = \lambda < \hat{\mu} < \mu'$ on $\mathbf{L}$.

If $\mathbf{L}$ is normal lattice we can prove more, namely

(d) if $\mathbf{L}$ is normal then λ is finitely subadditive on $\mathbf{L}'$, and $\hat{\mu}$ is finitely subadditive.

PROOF. Let $A, B \in \mathbf{L}$ and $L \subset A' \cup B'$, $L \in \mathbf{L}$. Then $L = L_1 \cup L_2$, $L_1, L_2 \in \mathbf{L}$ and $L_1 \subset A'$, $L_2 \subset B'$ since $\mathbf{L}$ coallocates itself if $\mathbf{L}$ is normal. Then $\mu(L) = \mu(L_1 \cup L_2) <$ $\mu(L_1) + \mu(L_2) = \lambda(L_1) + \lambda(L_2)$ since $\mu = \lambda$ on $\mathbf{L}$. Now since $L_1 \subset A'$, $L_2 \subset B'$, and λ is monotone we get $\mu(L) < \lambda(L_1) + \lambda(L_2) < \lambda(A') + \lambda(B')$. Now $\lambda(A' \cup B')$ $= \sup\{\mu(L): L \subset A' \cup B', L \in \mathbf{L}\}$, so $\lambda(A' \cup B') < \lambda(A') + \lambda(B')$. Proceeding by induction we get λ is finitely subadditive on $\mathbf{L}'$. Now take $E_i \subset L_i'$, $i = 1, 2, \ldots, N$ and $L_i \in \mathbf{L}$. We can say $\lambda(L_i') < \hat{\mu}(E_i) + \frac{\varepsilon}{N}$ by definition of $\hat{\mu}$. So $\hat{\mu}(\bigcup_{i=1}^{N} E_i) < \lambda(\bigcup_{i=1}^{N} L_i')$ where $\bigcup_{i=1}^{N} E_i \subset \bigcup_{i=1}^{N} L_i'$, $\bigcap_{i=1}^{N} L_i \in \mathbf{L}$. Now $\hat{\mu}(\bigcup_{i=1}^{N} E_i) < \lambda(\bigcup_{i=1}^{N} L_i') < \sum_{i=1}^{N} \lambda(L_i')$ using λ is finitely subadditive on $\mathbf{L}'$. So $\hat{\mu}(\bigcup_{i=1}^{N} E_i) < \sum_{i=1}^{N} \hat{\mu}(E_i) + \sum_{i=1}^{N} \frac{\varepsilon}{N}$ implying $\hat{\mu}(\bigcup_{i=1}^{N} E_i) < \sum_{i=1}^{N} \hat{\mu}(E_i) + \varepsilon$. Let $\varepsilon \to 0$ and we get $\hat{\mu}$ is finitely subadditive.

(e) If $\mathbf{L}$ is normal, then $A(\mathbf{L}) \subset S_{\hat{\mu}}$, the $\hat{\mu}$-measurable sets, and $\hat{\mu}$ restricted to $A(\mathbf{L})$ is in $M_R(\mathbf{L})$, and $\mu < \hat{\mu}(\mathbf{L})$, $\mu(X) = \hat{\mu}(X)$.

PROOF. Let $B' \in \mathbf{L}'$. It is not difficult to see that in order for $B \in S_{\hat{\mu}}$ we must show $\hat{\mu}(A') < \hat{\mu}(A' \cap B') + \hat{\mu}(A' \cap B)$ for all $A' \in \mathbf{L}'$. Now let $D \in \mathbf{L}$ such that $D \subset A' \cap B'$ and let $F \in \mathbf{L}$ such that $F \subset A' \cap D'$. It follows that $A' \cap B' \in \mathbf{L}'$, $A' \cap D' \in \mathbf{L}'$, $D \cap F = \phi$, $D \cup F \subset A'$, and $D \cup F \in \mathbf{L}$. Therefore $\hat{\mu}(A') = \lambda(A') > \mu(D \cup F) = \mu(D) + \mu(F)$ using $\hat{\mu} = \lambda$ on $\mathbf{L}'$ and the definition of λ. Therefore $\hat{\mu}(A') > \mu(D) + \sup\{\mu(F): F \subset A' \cap D', F \in \mathbf{L}\}$ which implies $\hat{\mu}(A') > \mu(D) + \lambda(A' \cap D')$. It follows that $\hat{\mu}(A') > \mu(D) + \hat{\mu}(A' \cap D')$ as $A' \cap D' \in \mathbf{L}'$. Also $D \subset A' \cap B' \Longrightarrow D' \supset A \cup B \Longrightarrow D' \supset B$ so $A' \cap B \subset A' \cap D'$. So by monotonicity of $\hat{\mu}$ we get $\hat{\mu}(A') > \mu(D) + \hat{\mu}(A' \cap B)$ which implies $\hat{\mu}(A') > \sup\{\mu(D): D \subset A' \cap B', D \in \mathbf{L}\} + \hat{\mu}(A' \cap B)$. So $\hat{\mu}(A') > \lambda(A' \cap B') + \hat{\mu}(A' \cap B) = \hat{\mu}(A' \cap B') + \hat{\mu}(A' \cap B)$. Therefore $\hat{\mu}(A') > \hat{\mu}(A' \cap B')$ $+ \hat{\mu}(A' \cap B)$ which implies $\mathbf{L}' \subset S_{\hat{\mu}}$. Therefore $A(\mathbf{L}') \subset S_{\hat{\mu}}$, but $A(\mathbf{L}') = A(\mathbf{L})$, so $A(\mathbf{L}) \subset S_{\hat{\mu}}$. Now for $E \in A(\mathbf{L})$ we have, by definition, $\hat{\mu}(E) = \inf\{\lambda(L'): E \subset L', L \in \mathbf{L}\}$ which implies $\hat{\mu}(E) = \inf\{\hat{\mu}(L'): E \subset L', L \in \mathbf{L}\}$. This means we can cover $E \in A(\mathbf{L})$ by $\mathbf{L}'$ on the outside. In addition, since $A(\mathbf{L}) \subset S_{\hat{\mu}}$ then $\hat{\mu}$ is finitely additive. All this implies $\hat{\mu} \in M_R(\mathbf{L})$. Now $\mu < \hat{\mu}(\mathbf{L})$ from part (c). Using $\mu < \hat{\mu}(\mathbf{L})$, $\hat{\mu} < \mu(\mathbf{L}')$, $X \in \mathbf{L}$, and $X \in \mathbf{L}'$ we can say $\mu(X) < \hat{\mu}(X)$ and $\hat{\mu}(X) < \mu(X)$ giving us $\mu(X) = \hat{\mu}(X)$.

As an immediate application we have:

THEOREM 2.2. If $\mathbf{L}$ is a normal lattice and if $\mu \in M(\mathbf{L})$ and if $\nu_1, \nu_2 \in M_R(\mathbf{L})$, and $\mu < \nu_1(\mathbf{L})$, $\mu < \nu_2(\mathbf{L})$ with $\mu(X) = \nu_1(X) = \nu_2(X)$, then $\nu_1 = \nu_2$.

PROOF. From $\mu < \nu_1(\mathbf{L})$, $\mu < \nu_2(\mathbf{L})$ we get $\hat{\mu} < \hat{\nu}_1$, and $\hat{\mu} < \hat{\nu}_2$. Now $\mu = \hat{\mu}$ if $\mu \in M_R(\mathbf{L})$ so $\hat{\mu} < \hat{\nu}_1 = \nu_1 \in M_R(\mathbf{L})$ and $\hat{\mu} < \hat{\nu}_2 = \nu_2 \in M_R(\mathbf{L})$. Therefore $\hat{\mu} < \nu_1 \in M_R(\mathbf{L})$; and $\hat{\mu} < \nu_2 \in M_R(\mathbf{L})$. Now $\hat{\mu}(X) = \mu(X)$ therefore $\hat{\mu}(X) = \nu_1(X) = \nu_2(X)$. Recall $\hat{\mu} \in M_R(\mathbf{L})$; therefore $\hat{\mu} = \nu_1$, $\hat{\mu} = \nu_2$ implying $\nu_1 = \nu_2$.

This extends the result of section 1 to $M_R(\mathbf{L})$ from $I_R(\mathbf{L})$.

3. SMOOTHNESS CONSIDERATIONS.

If one assumes certain added smoothness conditions on μ as well as further demands on the lattice, then it is possible to improve some of the results of section 2.

First let $\mu \in M(L)$ and define

$$\tilde{\mu}(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(L_i') : E \subset \bigcup_{i=1}^{\infty} L_i',\ L_i \in \mathbf{L} \right\} .$$

Then $\tilde{\mu}$ is an outer measure in the usual sense. Also we have: $\mu \in M_\sigma(\mathbf{L}) \Longrightarrow \mu \le \tilde{\mu}(\mathbf{L})$.

PROOF. 1). First we consider the following: Can we have $X = \bigcup_{i=1}^{\infty} L_i'$, $L_i \in \mathbf{L}$ and $\sum_{i=1}^{\infty} \mu(L_i') < \mu(X)$? We claim no, this cannot occur and that $\tilde{\mu}(X) = \mu(X)$. The proof of this follows: $\sum_{i=1}^{\infty} \mu(L_i') = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(L_i') \ge \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} L_i') = \mu(X)$ using

$\mu(\bigcup_{i=1}^{n} L_i') \le \sum_{i=1}^{n} \mu(L_i')$, $\bigcup_{i=1}^{n} L_i' \in \mathbf{L}'$ and $\lim_{n\to\infty} \bigcup_{i=1}^{n} L_i' = X$, since $\mu \in M_\sigma(\mathbf{L})$. So

$\sum_{i=1}^{\infty} \mu(L_i') \ge \mu(X)$. Now $X \subset \mathbf{L}'$ and $\tilde{\mu} \le \mu' = \mu$ on $\mathbf{L}'$ so $\tilde{\mu}(X) \le \mu(X)$.

So $\tilde{\mu}(X) \le \mu(X) \le \sum_{i=1}^{\infty} \mu(L_i')$. This implies $\tilde{\mu}(X) \le \mu(X) \le \inf\{ \sum_{i=1}^{\infty} \mu(L_i') : X \subset \bigcup_{i=1}^{\infty} L_i',\ L_i \in \mathbf{L}\}$

implying $\mu(X) = \tilde{\mu}(X)$.

2). Now back to the proof. Suppose $\mu(L) > \tilde{\mu}(L)$, $L \in \mathbf{L}$. This implies

$\sum_{i=1}^{\infty} \mu(L_i') < \mu(L)$ by definition of $\tilde{\mu}$. Now $X = L \cup L'$ so $\tilde{\mu}(X) \le \tilde{\mu}(L) + \tilde{\mu}(L')$ by

countable subadditivity of outer measure $\tilde{\mu}$. So $\tilde{\mu}(X) \le \tilde{\mu}(L)+\tilde{\mu}(L') \le \tilde{\mu}(L)+\mu(L')$ since $\tilde{\mu} \le \mu$ on $\mathbf{L}'$. Now $\tilde{\mu}(X) \le \tilde{\mu}(L)+\mu(L') < \mu(L)+\mu(L')$ since $\tilde{\mu}(L) < \mu(L)$, but $\mu(L)+\mu(L') = \mu(X)$, so $\tilde{\mu}(X) < \mu(X)$. This is a contradiction therefore $\mu \le \tilde{\mu}$ on $\mathbf{L}$.

Now if $\mathbf{L}$ is a normal lattice and countably paracompact, we have some further results:

THEOREM 3.1. If $\mathbf{L}$ is normal and countably paracompact, $\mu(X) = \nu(X)$ and $\mu \le \nu$ on $\mathbf{L}$. Then $\mu \in M_\sigma(\mathbf{L})$ implies $\nu \in M_\sigma(\mathbf{L})$.

PROOF. Let $A_n \downarrow \phi$, $A_n \in \mathbf{L}$ then there exists $B_n \in \mathbf{L}$ such that $A_n \subset B_n'$, and $B_n' \downarrow \phi$ since $\mathbf{L}$ is countably paracompact. Now it follows that $A_n \cap B_n = \phi$, so using $\mathbf{L}$ is normal there exists $C_n, D_n \in \mathbf{L}$ such that $A_n \subset C_n'$, $B_n \subset D_n'$ and $C_n' \cap D_n' = \phi$. Then $C_n' \le D_n$ and one sees $A_n \subset C_n' \le D_n \le B_n'$. Now $\mu(A_n) \le \nu(A_n)$ and $\nu(A_n) \le \hat{\nu}(A_n)$ for $A_n \in \mathbf{L}$. In addition $\hat{\nu}(A) \le \hat{\nu}(C_n')$ by monotonicity of $\hat{\nu}$ and it is easy to show that $\mu \le \nu$ on $\mathbf{L}$ implies $\nu \le \mu$ on $\mathbf{L}'$. So $\mu(A_n) \le \nu(A_n) \le \hat{\nu}(A_n) \le \hat{\nu}(C_n') \le \nu(C_n') \le \mu(C_n')$. Using monotonicity of

μ we get: $\mu(A_n) < \nu(A_n) < \hat{\nu}(A_n) < \hat{\nu}(C_n') < \nu(C_n') < \mu(C_n') < \mu(D_n)$. But $B_n' \downarrow \phi \Rightarrow D_n \downarrow \phi$ and $\mu \in M_\sigma(\mathbf{L}) \Longrightarrow \mu(D_n) \to 0$. So $\mu(A_n)$, $\mu(D_n) \to 0$ as $A_n \downarrow \phi$. Therefore $\lim \nu(A_n) \to 0$ as $A_n \downarrow \phi$ by the last inequality. Therefore $\nu \in M_\sigma(\mathbf{L})$.

THEOREM 3.2. If **L** is normal and countably paracompact, then $\mu \in M_\sigma(\mathbf{L})$ implies $\hat{\mu} \in M_R^\sigma(\mathbf{L})$ (where $\hat{\mu}$ is as defined in Section 2).

PROOF. Since $\mu(X) = \hat{\mu}(X)$ and $\mu < \hat{\mu}$ on **L** from Section 2 this implies $\hat{\mu} \in M_\sigma(\mathbf{L})$ by Theorem 3.1. Now since $\hat{\mu} \in M_R(\mathbf{L})$, therefore $\hat{\mu} \in M_R^\sigma(\mathbf{L})$.

THEOREM 3.3. If **L** is normal, countably paracompact and $\mu \in M_\sigma(\mathbf{L})$, then $\hat{\mu} < \tilde{\mu}$ on **L**.

PROOF. From Theorem 3.2 we get $\mu \in M_R^\sigma(\mathbf{L})$. Since this is true, we assume $\hat{\mu}$ has been extended to $\sigma(\mathbf{L})$ and call it $\hat{\mu}$ still. Now $\tilde{\mu}(L) = \inf\{\sum_{i=1}^{\infty} \mu(L_i'): L \subset \bigcup_{i=1}^{\infty} L_i', L_i \in \mathbf{L}\}$ for $L \in \mathbf{L}$. Now $\hat{\mu}(L) < \hat{\mu}(\bigcup_{i=1}^{\infty} L_i')$ by monotonicity of $\hat{\mu}$ and this expression is valid since $\bigcup_{i=1}^{\infty} L_i' \in \sigma(\mathbf{L})$. Also $\hat{\mu}(\bigcup_{i=1}^{\infty} L_i) < \sum_{i=1}^{\infty} \hat{\mu}(L_i')$ since $\hat{\mu} \in M_R^\sigma(\mathbf{L})$ and $\hat{\mu} < \mu$ on **L'** from Section 2. So $\hat{\mu}(L) < \hat{\mu}(\bigcup_{i=1}^{\infty} L_i') < \sum_{i=1}^{\infty} \mu(L_i')$. Therefore

$$\hat{\mu}(L) < \inf\{\sum_{i=1}^{\infty} \mu(L_i'): L \subset \bigcup_{i=1}^{\infty} L_i', L_i \in \mathbf{L}\} \text{ which implies } \hat{\mu}(L) < \tilde{\mu}(L),\ L \in \mathbf{L}.$$

Therefore $\hat{\mu} < \tilde{\mu}$ on **L**.

Finally we note:

THEOREM 3.4. If **L** is a normal lattice, and if $\mu \in M_\sigma(\mathbf{L})$ then $\hat{\mu}$ restricted to $A(\mathbf{L})$ is in $M_\sigma(\mathbf{L}') \cap M_R(\mathbf{L})$.

PROOF. Using $\hat{\mu} = \lambda$ on **L'** and the definition of λ we have $\hat{\mu}(B_n') = \lambda(B_n') = \sup\{\mu(A_n): A_n \subset B_n', A_n \in \mathbf{L}\}$. Therefore there exists A_n such that $A_n \subset B_n'$ and $\hat{\mu}(B_n') < \mu(A_n) + \varepsilon$. Let $B_n' \downarrow \phi$ we may assume $A_n \downarrow \phi$. Now $\hat{\mu} \in M_R(\mathbf{L})$ since **L** is normal, $\mu \in M_\sigma(\mathbf{L})$ and let $\varepsilon \to 0$ we get $\hat{\mu}(B_n') \to 0$. Therefore $\mu \in M_\sigma(\mathbf{L}') \cap M_R(\mathbf{L})$.

REMARK. If $M_\sigma(\mathbf{L}') \subset M_\sigma(\mathbf{L})$, then we can improve Theorem 3.4, and state that $\hat{\mu}$ restricted to $A(\mathbf{L})$ is in $M_R^\sigma(\mathbf{L})$. This condition is clearly satisfied if **L** is countably paracompact.

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