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ON THE MONOTONICITY OF THE QUOTIENT OF CERTAIN ABELIAN INTEGRALS

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ABSTRACT. We prove that the quotient of abelian integrals associated to an elliptic surface is bounded and strictly increasing by first determining the Picard-Fuchs equation satisfied by the abelian integrals and the Riccati equation satisfied by the quotient of the abelian integrals.

KEY WORDS AND PHRASES. Abelian integrals, elliptic surfaces, Picard-Fuchs equation, Riccati equation, monotonicity.
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1. INTRODUCTION

Complex elliptic surfaces, which are families of elliptic curves parametrized by a Riemann surface, play an important role in algebraic geometry and number theory (see e.g. [8]). Certain aspects of the theory of elliptic surfaces have recently been investigated in connection with the bifurcation theory. In [5] Il'yashenko proved that the quotient of certain abelian integrals associated to a family of real elliptic curves has a bounded range and strictly increasing on a finite interval in order to investigate the limit cycles arising from perturbations of phase curves of certain Hamiltonian systems (see also [6]). In [3] Cushman and Sanders corrected some mistakes made in [5] and proved the monotonicity of the quotient of abelian integrals associated to a real elliptic surface by using the Picard-Fuchs equation and the Riccati equation. They applied this result to consider a global Hopf bifurcation problem treated by Keener [7]. In [2] Cushman and Sanders considered a similar problem for another family of real elliptic curves and proved the uniqueness of the limit cycle for certain two-parameter family of planar vector fields.

It is well known that an elliptic curve can be expressed by an equation of the form

$$y^2 = x(x-1)(x-t)$$

called the Legendre normal form, which can be regarded as a family of elliptic curves parametrized by $t \in \mathbf{C}$. Given such a family the abelian integrals of the differentials $(1/y)dx$ and $(x/y)dx$ satisfy the hypergeometric equation

$$t(t-1)\frac{d^2f}{dt^2} + (2t-1)\frac{df}{dt} + \frac{1}{4}f = 0$$

called the Picard-Fuchs equation of the elliptic surface associated to the given differentials (see e.g. [1]). Picard-Fuchs equations of elliptic surfaces are essentially their Gauss-Manin connections which play an important role in the theory of variation of Hodge structures on complex algebraic manifolds (see e.g. [4]). In this paper we consider the family of real elliptic curves given by the Legendre normal form $y^2 = x(x-1)(x-s)$ for $0 < s < 1$. For each s the corresponding real elliptic curve has two connected components, one of which is compact and the other is noncompact. We consider the abelian integrals of the differentials ydx and $xydx$ over the compact component of the elliptic curve Γ_s corresponding to s. We use the method of Cushman and Sanders [3] to prove that the quotient of these abelian integrals is bounded and strictly increasing on the interval $0 \le s \le 1$ by first determining the Picard-Fuchs equation satisfied by the abelian integrals and the Riccati equation satisfied by the quotient of the abelian integrals.

2. ABELIAN INTEGRALS

Let s be a real number with $0 \le s \le 1$, and let Γ_s be the real elliptic curve in the xy-plane given by

$$y^2 = x^3 - (s+1)x^2 + sx = x(x-s)(x-1). \tag{1}$$

If $s \neq 1$, Γ_s has two connected components, one compact with x-intercepts at $x = 0, s$ and the other noncompact with x-intercept at $x = 1$. We denote by γ_s the compact component of Γ_s. As $s \to 0$, γ_s approaches γ_0 which coincides with the origin $(0,0)$ in the xy-plane. If $s = 1$, the elliptic curve Γ_s has one connected component which has a singularity at $(1,0)$. We define the differentials α, β, a and b by

$$\alpha = y\,dx, \qquad \beta = xy\,dx,$$

and

$$a = \frac{x}{y}\,dx, \qquad b = \frac{x^2}{y}\,dx.$$

Let $\mathcal{A}$, $\mathcal{B}$, A and B be the integrals of these differentials over the closed curve γ_s of α, β, a and b respectively, that is, these are the abelian integrals given by

$$\mathcal{A} = \int_{\gamma_s} \alpha, \qquad \mathcal{B} = \int_{\gamma_s} \beta$$

and

$$A = \int_{\gamma_s} a, \qquad B = \int_{\gamma_s} b.$$

In this section we express $\mathcal{A}$ and $\mathcal{B}$ as linear combinations of A and B with coefficients depending on the parameter s.

Taking the derivative of the equation (1) with respect to x and s, we obtain

$$2y\frac{dy}{dx} = 3x^2 - 2(s+1)x + s \tag{2}$$

and

$$2y\frac{dy}{ds} = x - x^2. \tag{3}$$

First, we shall determine the relations among the differentials. Since we are interested in the integrals of the differentials over the closed curve γ_s, it is sufficient to find the relations among the differentials up to exact differentials. Using (2), we have

$$\begin{aligned}
\alpha &= y\,dx = -x\frac{dy}{dx}\,dx \\
&= -\frac{x}{2y}(3x^2 - 2(s+1)x + s)\,dx \\
&= -\frac{1}{2y}(3(y^2 + (s+1)x^2 - sx) - 2(s+1)x^2 + sx)\,dx \\
&= -\frac{3}{2}\alpha - \frac{1}{2}(s+1)b + sa.
\end{aligned}$$

Hence we obtain

$$\alpha = \frac{2}{5}sa - \frac{1}{5}(s+1)b. \tag{4}$$

Similarly, we have

$$\begin{aligned}
\beta &= xy\,dx = -\frac{1}{2}x^2\frac{dy}{dx}\,dx \\
&= -\frac{1}{4y}(3x^4 - 2(s+1)x^3 + sx^2)\,dx \\
&= -\frac{1}{4y}(3x(y^2 + (s+1)x^2 - sx) \\
&\qquad - (s+1)(y^2 + (s+1)x^2 - sx) + sx^2)\,dx \\
&= -\frac{1}{4y}(3xy^2 + (s+1)y^2 + (s+1)^2x^2 - s(s+1)x - 2sx^2)\,dx \\
&= -\frac{3}{4}\beta - \frac{1}{4}(s+1)\alpha + \frac{1}{4}(2s - (s+1)^2)b + \frac{1}{4}s(s+1)a.
\end{aligned}$$

Using (4), we have

$$\begin{aligned}
\frac{7}{4}\beta &= -\frac{1}{4}(s+1)(\frac{2}{5}sa - \frac{1}{5}(s+1)b) \\
&\qquad + \frac{1}{4}(2s - (s+1)^2)b + \frac{1}{4}s(s+1)a \\
&= \frac{3}{20}s(s+1)a + \frac{1}{10}(-2(s+1)^2 + 5s)b.
\end{aligned}$$

Thus we obtain

$$\beta = \frac{3}{35}s(s+1)a + \frac{2}{35}(5s - 2(s+1)^2)b. \tag{5}$$

From (4) and (5) we obtain the following linear relations among the abelian integrals $\mathcal{A}$, $\mathcal{B}$, A and B:

$$\mathcal{A} = \frac{2}{5}sA - \frac{1}{5}(s+1)B, \tag{6}$$

$$\mathcal{B} = \frac{3}{35}s(s+1)A - \frac{2}{35}(2(s+1)^2 - 5s)B \tag{7}$$

3. THE PICARD-FUCHS EQUATION

In this section we determine a system of differential equations for the abelian integrals $\mathcal{A}$ and $\mathcal{B}$ with respect to the parameter s called Picard-Fuchs equation. First, we express $d\mathcal{A}/ds$ and $d\mathcal{B}/ds$ in terms of A and B. Using (3), we obtain

$$\begin{aligned}\frac{d\mathcal{A}}{ds} &= \int_{\gamma_s} \frac{dy}{ds}\,dx = \frac{1}{2}\int_{\gamma_s} \frac{x-x^2}{y}\,dx \\ &= \frac{1}{2}A - \frac{1}{2}B\end{aligned}$$

and

$$\begin{aligned}\frac{d\mathcal{B}}{ds} &= \int_{\gamma_s} x\frac{dy}{ds}\,dx = \frac{1}{2}\int_{\gamma_s} \frac{x^2-x^3}{y}\,dx \\ &= \frac{1}{2}\int_{\gamma_s} \frac{1}{y}\Big(x^2 - y^2 - (s+1)x^2 + sx\Big)\,dx \\ &= -\frac{1}{2}sB - \frac{1}{2}\mathcal{A} + \frac{1}{2}sA \\ &= -\frac{1}{2}sB - \frac{1}{2}\Big(\frac{2}{5}sA - \frac{1}{5}(s+1)B\Big) + \frac{1}{2}sA \\ &= \frac{3}{10}sA + \frac{1}{10}(1-4s)B.\end{aligned}$$

Thus we have

$$\frac{d}{ds}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix} = \begin{pmatrix}1/2 & -1/2\\ 3s/10 & (1-4s)/10\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}. \tag{8}$$

Now from (6) and (7) we obtain

$$\begin{pmatrix}A\\ B\end{pmatrix} = \frac{-1}{5s(s-1)^2}\begin{pmatrix}10(-2(s+1)^2+5s) & 35(s+1)\\ -15s(s+1) & 70s\end{pmatrix}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix}. \tag{9}$$

Hence from (8) and (9) we obtain the Picard-Fuchs equation

$$\frac{d}{ds}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix} = \frac{-1}{5s(s-1)^2}\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}M_{11} &= 5(-2(s+1)^2+5s) + \frac{15}{2}s(s+1), \\ M_{12} &= \frac{35}{2}(s+1) - 35s, \\ M_{21} &= 3s(-2(s+1)^2+5s) - \frac{3}{2}s(s+1)(1-4s), \\ M_{22} &= \frac{21}{2}s(s+1) + 7s(1-4s).\end{aligned} \tag{11}$$

4. THE RICCATI EQUATION

In this section we determine the Riccati equation satisfied by the quotient $\xi(s) = \mathcal{B}/\mathcal{A}$ of the abelian integrals $\mathcal{A}, \mathcal{B}$ and use this equation to prove that $\xi(s)$ is strictly increasing on the interval $0 \le s \le 1$. Using the Picard-Fuchs equation (10), we have

$$\begin{aligned} -5s(s-1)^2\frac{d\xi}{ds} &= -5s(s-1)^2\Big(\frac{1}{\mathcal{A}}\frac{d\mathcal{B}}{ds} - \frac{\mathcal{B}}{\mathcal{A}}\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{ds}\Big) \\ &= \frac{1}{\mathcal{A}}(M_{21}\mathcal{A} + M_{22}\mathcal{B}) - \frac{\mathcal{B}}{\mathcal{A}}\frac{1}{\mathcal{A}}(M_{11}\mathcal{A} + M_{12}\mathcal{B}) \\ &= M_{21} + (M_{22} - M_{11})\xi - M_{12}\xi^2, \end{aligned}$$

where M_{11}, M_{12}, M_{21} and M_{22} are as in (11). Thus we obtain the Riccati equation

$$2s(1-s)\frac{d\xi}{ds} = 3s - 2(3s+2)\xi + 7\xi^2. \tag{12}$$

Now we shall determine the values $\xi(0)$ and $\xi(1)$. First, for $s = 1$, we have

$$\mathcal{A}(1) = \int_{\gamma_1} y\,dx = 2\int_0^1 (x-1)\sqrt{x}\,dx = -\frac{8}{15},$$

and

$$\mathcal{B}(1) = \int_{\gamma_1} sy\,dx = 2\int_0^1 x(x-1)\sqrt{x}\,dx = -\frac{8}{35}.$$

Thus we obtain

$$\xi(1) = \frac{\mathcal{B}(1)}{\mathcal{A}(1)} = \frac{3}{7}.$$

For $0 < s < 1$ we denote by D_s the region in the xy-plane enclosed by γ_s.. In order to compute $\xi(0)$ we first prove the following lemma:

Lemma 1. For each $\delta > 0$, there exists $s_0 > 0$ such that $D_s \subset \mathcal{D}(\delta)$ whenever $0 < s < s_0$, where

$$\mathcal{D}(\delta) = \{(x,y) \mid x^2 + y^2 < \delta^2\}.$$

Proof. We consider the intersection points of γ_s and $y = \lambda x$. Solving the equation

$$\lambda^2 x^2 = x^3 - (s+1)x^2 + sx,$$

we obtain

$$x = 0, \quad \frac{1}{2}\Big((s+1+\lambda^2) \pm \sqrt{(s+1+\lambda^2) - 4s}\Big).$$

Thus the intersection points of $y = \lambda x$ and γ_s are $(0,0)$ and (x_s, y_s), where

$$\begin{aligned} x_s &= \frac{(s+1+\lambda^2) - \sqrt{(s+1+\lambda^2) - 4s}}{2} \\ &= \frac{2s}{s+1+\lambda^2+\sqrt{(s+1+\lambda^2) - 4s}} \\ &\leq \frac{2s}{s+1+\sqrt{(s+1)^2 - 4s}} = s \end{aligned}$$

and

$$|y_s| = \sqrt{(s-x)(1-x)x} \leq \sqrt{s \cdot 1 \cdot x} \leq s.$$

Thus $(x_s, y_s) \in \mathcal{D}(\delta)$ if s is sufficiently small. Hence the lemma follows. □

Now we consider the function $f(x,y) = x$. Since f is continuous at $(0,0)$, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(\mathcal{D}(\delta))| < \epsilon$. By Lemma 1 there is $s_0 > 0$ such that $D_s \subset \mathcal{D}(\delta)$ whenever $0 < s < s_0$. Thus, if $0 < s < s_0$, we have

$$\left|\int_{D_s} f(x,y)\, dx\, dy\right| \leq \epsilon \int_{D_s} dx\, dy;$$

hence we have

$$\frac{\int_{D_s} x\, dx\, dy}{\int_{D_s} dx\, dy} \longrightarrow 0.$$

as $s \to 0$. Using the Stokes' theorem, we have

$$\int_{D_s} x\, dx\, dy = \int_{\gamma_s} xy\, dx$$

and

$$\int_{D_s} dx\, dy = \int_{\gamma_s} y\, dx.$$

Hence it follows that

$$\xi(0) = \lim_{s\to 0^+} \frac{\int_{\gamma_s} xy\, dx}{\int_{\gamma_s} y\, dx} = \lim_{s\to 0^+} \frac{\int_{D_s} x\, dxdy}{\int_{D_s} dxdy} = 0.$$

5. THE MONOTONICITY

In this section we show that $\xi(s)$ is monotonic increasing on the interval $0 \leq s \leq 1$ by using the Riccati equation (12).

Lemma 2. $\xi(s)$ satisfies the inequality

$$0 \le \xi(s) \le \frac{3}{7} \quad \text{for} \quad 0 \le s \le 1.$$

Proof. From (12) it follows that $d\xi/ds = 0$ if the point (s, ξ) lies on the curves in the $s\xi$-plane determined by the equation

$$7\xi^2 - 2(3s+2)\xi + 3s = 0,$$

which is equivalent to

$$\xi = \frac{1}{7}\left(3s + 2 \pm \sqrt{(3s+2)^2 - 21s}\right). \tag{13}$$

The derivative of the function on the right hand side of (13) with respect to s is

$$\frac{3}{7}\left(1 \pm \frac{6s-3}{2\sqrt{9s^2 - 9s + 4}}\right),$$

which is positive because

$$\left(\frac{6s-3}{2\sqrt{9s^2-9s+4}}\right)^2 = \frac{36s^2 - 36s + 9}{36s^2 - 36s + 16} < 1.$$

If we set

$$\mu(s) = \frac{1}{7}\left(3s + 2 + \sqrt{(3s+2)^2 - 21s}\right)$$

and

$$\nu(s) = \frac{1}{7}\left(3s + 2 - \sqrt{(3s+2)^2 - 21s}\right),$$

then $d\xi/ds = 0$ along the curves $\xi = \mu(s)$ and $\xi = \nu(s)$, and these curves are strictly increasing on the interval $0 \le s \le 1$ with

$$\mu(0) = \frac{4}{7}, \qquad \mu(1) = 1$$

and

$$\nu(0) = 0, \qquad \nu(1) = \frac{3}{7}.$$

Hence for $0 \le s \le 1$ we have

$$\frac{d\xi}{ds} = 0 \quad \text{if} \quad \frac{3}{7} < \xi < \frac{4}{7}$$

and

$$\frac{d\xi}{ds} > 0 \quad \text{if} \quad \xi < 0.$$

Thus it follows that $0 \le \xi(s) \le 3/7$ for $0 \le s \le 1$. □

Now we prove our main theorem.

Theorem 3. $\xi(s)$ is strictly monotonic increasing on the interval $0 \le s \le 1$.

Proof. Suppose that $\xi(s)$ has an extremum at s_0 with $0 < s_0 < 1$. Differentiating the both sides of the equation (12) with respect to s, we obtain

$$2(2s-1)\frac{d\xi}{ds} + 2s(s-1)\frac{d^2\xi}{ds^2} = 3 - 6\xi - 6s\frac{d\xi}{ds} + 14\xi\frac{d\xi}{ds}. \tag{14}$$

Evaluating the equation (14) at $s = s_0$, we get

$$2s_0(s_0 - 1)\frac{d^2\xi}{ds^2}(s_0) = 3 - 6\xi(s_0). \tag{15}$$

Since $0 \leq \xi(s_0) \leq 3/7$, from (15) we obtain

$$\frac{d^2\xi}{ds^2}(s_0) > 0.$$

Thus it follows that every relative extremum is a relative minimum; hence $\xi(s_0)$ is a relative minimum. Since $\xi(0) = 0 \leq \xi(s_0)$, there exists t with $0 < t < s_0$ such that $\xi(t)$ is a relative maximum, which is impossible. Hence the theorem follows. □

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