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ON THE RELATIONSHIP OF INTERIOR-POINT METHODS

RUEY-LIN SHEU

AT&T Bell Laboratories, Holmdel

and

SHU-CHERNG FANG

Operations Research & Industrial Engineering
North Carolina State University
Box 7913, Raleigh, NC 27695-7913, USA

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ABSTRACT. In this paper, we show that the moving directions of the primal-affine scaling method (with logarithmic barrier function), the dual-affine scaling method (with logarithmic barrier function), and the primal-dual interior point method are merely the Newton directions along three different algebraic "paths" that lead to a solution of the Karush-Kuhn-Tucker conditions of a given linear programming problem. We also derive the missing dual information in the primal-affine scaling method and the missing primal information in the dual-affine scaling method. Basically, the missing information has the same form as the solutions generated by the primal-dual method but with different scaling matrices.

KEY WORDS AND PHRASES. Linear programming, interior-point method, Newton method, duality theory.
AMS SUBJECT CLASSIFICATION CODE. 90C05.

1. INTRODUCTION.

Since Karmarkar [7] proposed his polynomial-time projective scaling algorithm for solving linear programming problems in 1984, the interest of studying interior-point methods has been arising to a peak in recent years. In particular, Vanderbei, Meketon, and Freeman [15], and independently, Barnes [2] extended Karmarkar's algorithm to the "pure affine scaling" method for a linear program in its standard form: $\{Minimize\ c^tx \mid Ax = b,\ x \geq 0\}$ where A is an mxn matrix; $x,\ c \in R^n$ and $b \in R^m$. Adler et al. [1] applied the same affine scaling technique to its dual problem: $\{Maximize\ b^ty \mid A^ty + s = c,\ s \geq 0\}$ where $y \in R^m$ and $s \in R^n$. Both extensions have been effective in practice, but neither was proven to be of polynomial-time bound.

Gill et al. [5] discovered that Karmarkar's algorithm is equivalent to a "projected barrier method" that comes from adding a "logarithmic barrier function" to the linear objective function. Moreover, Gonzaga's combination [6] of such a barrier function with the "pure affine scaling" results in a "primal-affine scaling method" (with centering force) which exhibits a polynomial complexity of $O(n^3L)$. The same type of combination applied to the dual problem produces a "dual-affine scaling method" (with centering force) with the same complexity.

Monteiro and Adler [11] and, independently, Kojima, Mizuno, and Yoshise [8] focused their attention on solving the "K-K-T optimality conditions" consisting of the "primal and dual feasibility" along with the "complementary slackness". Enforcing primal and dual interiority by replacing each complementary slackness condition $x_i s_i = 0$ with a relaxation of $x_i s_i = \mu$ for $\mu > 0$, they showed a "primal-dual method" that exhibits the same complexity of $O(n^3 L)$.

According to Shanno and Bagchi [12], the moving directions of the primal-affine scaling, the dual-affine scaling, and the primal-dual algorithms can all be represented as a combination of a "steepest descent direction" and a centering vector obtained from the logarithmic barrier function method. In particular, if $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^t$ is a current primal interior feasible point, $e = (1, 1, \cdots, 1)^t$, and $D = diag(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ is a diagonal matrix of $\bar{x}$, then the moving direction Δx of the primal-affine scaling method in literature becomes

$$\Delta x = \frac{-1}{\mu} D[I - DA^t(AD^2A^t)^{-1}AD]Dc + D[I - DA^t(AD^2A^t)^{-1}AD]e \tag{1.1}$$

A new interior feasible point is given by $x = \bar{x} + \theta \Delta x$ with $0 < \theta < 1$.

Similarly, if $\bar{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m)^t$ is a current dual feasible solution with $\bar{s} = c - A^t\bar{y}$, $\bar{s} > 0$, and $Z = diag(\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n)$ is a diagonal matrix, then the moving direction in the dual-affine scaling method is

$$\Delta y = \frac{1}{\mu}(AZ^{-2}A^t)^{-1}b - (AZ^{-2}A^t)^{-1}AZ^{-1}e. \tag{1.2}$$

A new dual feasible interior solution is then given by $y = \bar{y} + \theta \Delta y$ and $s = c - A^t y$ where $0 < \theta < 1$.

As to the primal-dual method, let $(\bar{x}, \bar{y}, \bar{s})$ be a current interior feasible solution that satisfies $A\bar{x} = b$, $A^t\bar{y} + \bar{s} = c$ and $\bar{x} > 0$, $\bar{s} > 0$, then the moving directions are given by

$$\Delta x = -[Z^{-1} - Z^{-1}DA^t(AZ^{-1}DA^t)^{-1}AZ^{-1}]v(\mu), \tag{1.3a}$$

$$\Delta y = (AZ^{-1}DA^t)^{-1}AZ^{-1}v(\mu), \tag{1.3b}$$

$$\Delta s = -A^t(AZ^{-1}DA^t)^{-1}AZ^{-1}v(\mu), \tag{1.3c}$$

where $v(\mu) = DZe - \mu e$ and $\Delta s = s - \bar{s}$.

In this paper, we show that the moving directions of the primal-affine scaling method (1.1), the dual-affine scaling method (1.2), and the primal-dual interior point method (1.3a), (1.3b) and (1.3c) are merely the Newton directions along three different "algebraic paths" that lead to the solution of the Karush-Kuhn-Tucker conditions of a given linear programming problem. We also derive the dual information in the primal-affine scaling method and the primal information in the dual-affine scaling method. Basically, they have the same form as in (1.3) but with different scaling matrices.

2. MOVING ALONG THREE PATHS.

Consider a linear programming problem in its standard form and its dual problem. For a positive scalar μ, we can incorporate a logarithmic barrier function into either the primal and consider the problem (P_μ): $\{Minimize\ c^t x - \mu \sum_{i=1}^{n} \ln x_i \mid Ax = b,\ x > 0\}$, or into the dual and consider (D_μ): $\{Maximize\ b^t y + \mu \sum_{i=1}^{n} \ln s_i \mid A^t y + s = c,\ s > 0\}$. A straight-forward derivation [14] shows that the K-K-T conditions of both (P_μ) and (D_μ) lead to the same system of equations:

$$\begin{cases} A^t y + s - c = 0, \\ Ax - b = 0, \\ \mu e - Xs = 0,. \\ x > 0, s > 0 \end{cases} \tag{2.1}$$

where $X = diag(x_1, x_2, \cdots, x_n)$ and $e^t = (1, 1, \cdots, 1)$.

To assure the existence of a unique optimal solution to (P_μ) and (D_μ), or equivalently the existence of a unique solution to system (2.1), we assume that (1) there exists a primal interior feasible solution, i.e., the set $W = \{x \in R^n : Ax = b, x > 0\}$ is nonvoid; (2) there exists a dual interior feasible solution, i.e., the set $T = \{y \in R^m, s \in R^n : A^t y + s = c, s > 0\}$ is nonvoid; and (3) matrix A has full rank. Note that these three assumptions are commonly accepted in most, if not all, related papers.

Now focus on system (2.1). We know that, under the above assumptions, as μ approaches 0, the unique solution of (2.1) solves the given linear programming problem. However, for any $\mu > 0$, we can actually approach the solution of $\mu e - Xs = 0$ from many different but equivalent "algebraic paths". Here a "path" means the contour of an algebraic function. More specifically, for $x_i > 0, s_i > 0$, consider the following three functions

$$f(x_i, s_i) = \mu - x_i s_i,$$

$$g(x_i, s_i) = \frac{\mu}{x_i} - s_i,$$

and

$$h(x_i, s_i) = \frac{\mu}{s_i} - x_i (i = 1, \cdots, n).$$

Note that although the above three functions look different, they are algebraically equivalent to the complementary slackness condition in (2.1), since $\{(x, s) \mid f(x_i, s_i) = 0, x_i > 0, s_i > 0,$ for $i = 1, \cdots, n\} = \{(x, s) \mid g(x_i, s_i) = 0, x_i > 0, s_i > 0,$ for $i = 1, \cdots, n\} = \{(x, s) \mid h(x_i, s_i) = 0, x_i > 0, s_i > 0,$ for $i = 1, \cdots, n\} = \{(x, s) \mid \mu e - Xs = 0, x > 0, s > 0\}$.

Hence we can consider system (2.1) in terms of these three functions, i.e.,

$$\begin{cases} A^t y + s - c = 0, \\ Ax - b = 0, \\ f(x_i, s_i) = 0, \quad i = 1, 2, \cdots, n, \\ x > 0, \quad s > 0; \end{cases} \tag{2.2}$$

$$\begin{cases} A^t y + s - c = 0, \\ Ax - b = 0, \\ g(x_i, s_i) = 0, \quad i = 1, 2, \cdots, n, \\ x > 0, \quad s > 0; \end{cases} \tag{2.3}$$

and

$$\begin{cases} A^t y + s - c = 0, \\ Ax - b = 0, \\ h(x_i, s_i) = 0, \quad i = 1, 2, \cdots, n, \\ x > 0, \quad s > 0; \end{cases} \tag{2.4}$$

Assume that $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^t > 0, \bar{s} = (\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n)^t > 0$ and $\bar{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n)^t$ are given with $A\bar{x} = b$ and $A^t \bar{y} + \bar{s} = c$. Our objective is to solve (2.2) $\sim$ (2.4) via Newton's method.

Note that functions f, g and h are the only nonlinear expressions in each of these three systems, therefore we only have to linearize them when applying Newton's method.

2.1. The Primal-Affine Scaling Method

Focus on system (2.3) and one Newton step with the linearization of $g(x_i, s_i) = 0$ yields

$$0 - g(\bar{x}_i, \bar{s}_i) = [\nabla g(\bar{x}_i, \bar{s}_i)]^t \begin{pmatrix} x_i - \bar{x}_i \\ \\ s_i - \bar{s}_i \end{pmatrix}.$$

Substituting the formula for g and multiplying it out, we have

$$\bar{s}_i - \frac{\mu}{\bar{x}_i} = \left[-\frac{\mu}{\bar{x}_i^2}, -1 \right] \begin{pmatrix} x_i - \bar{x}_i \\ \\ s_i - \bar{s}_i \end{pmatrix}.$$

Hence
$$s_i = \frac{2\mu}{\bar{x}_i} - \frac{\mu}{\bar{x}_i^2} x_i. \tag{2.5}$$

Remember that $D = diag\,(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, (2.5) becomes

$$s = 2\mu D^{-2}\bar{x} - \mu D^{-2} x.$$

Since we move along the Newton's direction, we know $A^t y + s = c$ and, hence,

$$x = \frac{1}{\mu} D^2 [A^t y + 2\mu D^{-2}\bar{x} - c]. \tag{2.6}$$

Multiplying matrix A on both sides of (2.6), we have

$$b = Ax = \frac{1}{\mu} AD^2 [A^t y + 2\mu D^{-2}\bar{x} - c].$$

Consequently,

$$y = (AD^2 A^t)^{-1}[AD^2 c - \mu b]. \tag{2.7}$$

Plugging (2.7) into (2.6), we obtain that

$$\begin{aligned} \Delta x &= x - \bar{x} \\ &= \frac{-1}{\mu} D[I - DA^t(AD^2A^t)^{-1}AD]Dc \\ &\quad + D[I - DA^t(AD^2A^t)^{-1}AD]D\, D^{-1}e. \end{aligned}$$

This direction is exactly the moving direction (1.1) of the primal-affine scaling method.

2.2 The Dual-Affine Scaling Method.

If we work on system (2.4), the moving direction of the dual-affine scaling method can be obtained. To verify this, note that one Newton step with the linearization of $h(x_i, s_i) = 0$ results in

$$0 - h(\bar{x}_i, \bar{s}_i) = [\nabla h(\bar{x}_i, \bar{s}_i)]^t \begin{pmatrix} x_i - \bar{x}_i \\ \\ s_i - \bar{s}_i \end{pmatrix}.$$

Using the formula of function h, we have

$$\bar{x}_i - \frac{\mu}{\bar{s}_i} = \left[-1, -\frac{\mu}{\bar{s}_i^2} \right] \begin{pmatrix} x_i - \bar{x}_i \\ s_i - \bar{s}_i \end{pmatrix}.$$

and
$$x_i = \frac{2\mu}{\bar{s}_i} - \frac{\mu}{\bar{s}_i^2} s_i. \tag{2.8}$$

Remember that $Z = diag(\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n)$, (2.8) becomes

$$x = 2\mu Z^{-1} e - \mu Z^{-2} s. \tag{2.9}$$

Since we move along the Newton's direction, therefore $Ax = b$ and $A^t y + s = c$, and (2.9) turns out to be

$$\begin{aligned} b = Ax &= 2\mu A Z^{-1} e - \mu A Z^{-2} s \\ &= 2\mu A Z^{-1} e - \mu A Z^{-2} (c - A^t y). \end{aligned} \tag{2.10}$$

Now, substituting c for $A^t\bar{y} + \bar{s}$ in (2.10), we have

$$b = 2\mu A Z^{-1} e - \mu A Z^{-2} A^t \bar{y} - \mu A Z^{-2} \bar{s} + \mu A Z^{-2} A^t y.$$

Hence,

$$\Delta y = \frac{1}{\mu} (A Z^{-2} A^t)^{-1} b - (A Z^{-2} A^t)^{-1} A Z^{-1} e.$$

This is exactly the moving direction (1.2) of the dual-affine scaling method.

2.3. The Pimal-Dual Method.

Finally, we work on system (2.2) to derive the moving directions of the primal-dual interior method. Simply taking a Newton step with the linearization of $f(x_i, s_i) = 0$, we have

$$0 - f(\bar{x}_i, \bar{s}_i) = [\nabla f(\bar{x}_i, \bar{s}_i)]^t \begin{pmatrix} x_i - \bar{x}_i \\ s_i - \bar{s}_i \end{pmatrix}.$$

Putting in the formula for f results in

$$\bar{x}_i \bar{s}_i - \mu = -(\bar{s}_i, \bar{x}_i) \begin{pmatrix} x_i - \bar{x}_i \\ s_i - \bar{s}_i \end{pmatrix}. \tag{2.11}$$

Equation (2.11) can be represented in terms of Δx and Δs, in this case,

$$D\Delta s + Z\Delta x = -DZe + \mu e. \tag{2.12}$$

Moreover, since we are moving along the Newton's direction,

$$A\Delta x = 0, \tag{2.13}$$

and

$$A^t \Delta y + \Delta s = 0. \tag{2.14}$$

Equations (2.12), (2.13) and (2.14) form a system of linear equations with unknown variables $\Delta x, \Delta y$ and Δs. Using (2.13) and (2.14) to eliminate Δx and Δs in (2.12), we get

$$\Delta y = (A Z^{-1} D A^t)^{-1} A Z^{-1} \nu(\mu),$$

where $\nu(\mu) = DZe - \mu e$.

Plugging Δy in (2.14), we have

$$\Delta s = -A^t(AZ^{-1}DA^t)^{-1}AZ^{-1}\nu(\mu).$$

After Δs is known, Δx immediately follows from (2.12) as

$$\Delta x = -[Z^{-1} - Z^{-1}DA^t(AZ^{-1}DA^t)^{-1}AZ^{-1}]\nu(\mu).$$

This describes the moving directions (1.3a, b, c) of the primal-dual method.

Combining the results we have shown in the previous three subsections, we have our main theorem:

THEOREM 1. The moving directions used in the primal-affine scaling, dual-affine scaling, and primal-dual methods are the Newton's directions along three different, yet equivalent, algebraic paths that lead to the solution of the K-K-T conditions (2.1).

3. MISSING INFORMATION.

Since the moving directions of both the primal-affine scaling and dual-affine scaling methods are closely related to that of the primal-dual method, we can further exploit the dual information in the primal approach and the primal information in the dual approach.

3.1. Dual Information in the Primal-Affine Scaling Method.

From (2.5), we have

$$\begin{aligned} s &= 2\mu D^{-2}\bar{x} - \mu D^{-2}x \\ &= 2\mu D^{-2}\bar{x} - \mu D^{-2}(\bar{x} + \Delta x) \\ &= \mu D^{-2}\bar{x} - \mu D^{-2}D[I - DA^t(AD^2A^t)^{-1}AD]\left(\tfrac{-1}{\mu} Dc + e\right) \\ &= \mu D^{-2}\bar{x} - \mu D^{-2}D\left(\tfrac{-1}{\mu}Dc + e\right) + \mu A^t(AD^2A^t)^{-1}AD\left(\tfrac{-1}{\mu} Dc + e\right) \\ &= c - A^t(AD^2A^t)^{-1}AD(Dc - \mu e)\,. \end{aligned}$$

Since we are moving along the Newton's direction, both the primal and dual feasibility are kept. Hence we can define

$$y = (AD^2A^t)^{-1}AD(Dc - \mu e)$$

and

$$\begin{aligned} \Delta y = y - \bar{y} &= (AD^2A^t)^{-1}AD(Dc - \mu e) - \bar{y} \\ &= (AD^2A^t)^{-1}AD^2(c - A^t\bar{y} - \mu D^{-1}e) \\ &= (AD^2A^t)^{-1}AD(DZe - \mu e)\,. \end{aligned} \tag{3.1}$$

Compare (3.1) with (1.3b), we see the dual moving direction embedded in the primal-affine scaling method has exactly the same form as that of the primal-dual method except the scaling matrix becomes D instead of $Z^{-1/2}D^{1/2}$.

3.2. Primal Information in the Dual-Affine Scaling Method.

Similarly, we can derive the embedded primal moving direction of the dual-affine scaling method. Starting from (2.9), we have

$$\begin{aligned} x &= 2\mu Z^{-1}e - \mu Z^{-2}s \\ &= 2\mu Z^{-1}e - \mu Z^{-2}[\bar{s} - \tfrac{1}{\mu} A^t(AZ^{-2}A^t)^{-1}(b - \mu AZ^{-1}e)] \\ &= \mu Z^{-1}[e + Z^{-1}A^t(AZ^{-2}A^t)^{-1}(\tfrac{1}{\mu} ADe - AZ^{-1}e)] \end{aligned}$$

$$= \mu Z^{-1}[e + Z^{-1}A^t(AZ^{-2}A^t)^{-1}AZ^{-1}(\tfrac{1}{\mu}ZDe - e)]$$

$$= \mu Z^{-1}[I - Z^{-1}A^t(AZ^{-2}A^t)^{-1}AZ^{-1}](\tfrac{-1}{\mu}ZDe + e) + \bar{x}$$

Hence we know

$$\Delta x = -[Z^{-1} - Z^{-2}A^t(AZ^{-2}A^t)^{-1}AZ^{-1}](DZe - \mu e)\ . \tag{3.2}$$

Compare (3.2) with (1.3a), we see, this time, the primal moving direction embedded in the dual-affine scaling method has exactly the same form as that of the primal-dual method except the scaling matrix becomes Z^{-1} instead of $Z^{-1/2}D^{1/2}$.

Summarizing the results in the previous two subsections, we have the following theorem:

THEOREM 2. The dual moving direction ebmedded in the primal-affine scaling method has the same form as that of the primal-dual method but with a different scaling matrix. Similarly, the primal moving direction embedded in the dual-affine method has the same form as that of the primal-dual method but with a different scaling matrix.

4. CONCLUSION AND DISCUSSION.

In this paper, we have shown that the moving directions of the primal-affine scaling method, the dual-affine scaling method, and the primal-dual interior point method are merely the Newton directions along three different "algebraic paths" that lead to the solution of the Karush-Kuhn-Tucker conditions of a given linear programming problem. We have also derived the dual information embedded in the primal-affine scaling method and the primal information embedded in the dual-affine scaling method.

The view of "algebraic paths" not only unifies the existing three major interior-point methods, but also provides us a platform to study new interior-point algorithms. At least in theory there are infinitely many algebraic paths that could lead us to the solution of the K-K-T conditions and each path may generate a new moving direction associated with a potential interior-point algorithm. If a suitable stepsize can be decided at each iteration and convergence can be proved for a potential candidate, this "algebraic paths" approach will provide a fertile source of new algorithms. More detailed information can be referred to Sheu and Fang [14].

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