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# COMPACT DIAGONAL LINEAR OPERATORS ON BANACH SPACES WITH UNCONDITIONAL BASES

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ABSTRACT. Let $E$ and $F$ be Banach spaces with equivalent normalized unconditional bases. In this note we show that a bounded diagonal linear operator $T : E \to F$ is compact if and only if its entries tend to 0, using the concept of weak uniform continuity.

KEY WORDS and PHRASES. Weakly uniformly continuous, diagonal operators, compact operators, unconditional bases.
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## 1. Introduction.

Let $E$ and $F$ be two complex Banach spaces and $E^*$ be the dual space of $E$. A function $f : E \to F$ is said to be weakly uniformly continuous on bounded subsets of $E$ if for each bounded set $B \subset E$ and $\epsilon > 0$, there are $\phi_1, \cdots, \phi_k \in E^*$ and $\delta > 0$ such that if $x, y \in B$, $|\phi_i(x-y)| \leq \delta$ $(i = 1, \cdots, k)$, then $\|f(x) - f(y)\| \leq \epsilon$. R.M. Aron and J.B. Prolla [1] showed that a bounded linear operator $T : E \to F$ is compact if and only if $T$ is weakly uniformly continuous on bounded subsets of $E$. Applying this result we generalize the following well-known Hilbert space fact to a Banach space with an unconditional basis: A diagonal bounded linear operator is compact if and only if its entries tend to 0. See, for example, [2, Proposition 4.6].

We recall some relevant definitions and results about a Banach space with an unconditional basis. Let $E$ be a complex Banach space with an unconditional basis $(e_n)$. For every choice of signs $\theta = (\theta_n)$, we have a bounded linear operator $M_\theta$ on $E$ defined by

$$M_\theta(\sum a_n e_n) = \sum a_n \theta_n e_n. \tag{1.1}$$

The uniform bounded principle implies that the number $K = \sup \|M_\theta\|$ is finite, which is called the unconditional constant of $(e_n)$. Then for every choice of a complex sequence $(a_n)$ such that $\sum a_n e_n$ converges and every choice of a bounded complex sequence $(\alpha_n)$, we have

$$\|\sum \alpha_n a_n e_n\| \leq 2K(sup|\alpha_n|)\|\sum a_n e_n\|. \tag{1.2}$$

For details see [3].

## 2. Main Results.

THEOREM 1. Let $(e_n)$ and $(f_n)$ be equivalent normalized unconditional bases of $E$ and $F$, respectively. Given a bounded sequence $(\alpha_n)$, let $T : E \to F$ be the bounded linear operator with $T(e_n) = \alpha_n f_n$ for each $n$. Then $T$ is compact if and only if $\alpha_n \to 0$.

Proof. Suppose $T$ is compact. Let $(P_n)$ be the sequence of the natural projections associated with $(f_n)$. Then $(P_n \circ T)$ converges uniformly to $T$ on the closed unit ball $B_E$, from which it follows that $\alpha_n \to 0$.

Conversely suppose that $\alpha_n \to 0$. We will show that $T$ is weakly uniformly continuous on bounded subsets of $E$. Let $B_r$ be the closed ball of $E$ with the radius $r$ and the center 0 and $C$ be the positive number with $|\alpha_n| \le C$ for all $n$. Given $\epsilon > 0$, $x = \sum a_n e_n$ and $y = \sum b_n e_n$ in $B_r$,

$$\|T(x) - T(y)\| = \|\sum \alpha_n(a_n - b_n) f_n\| \tag{2.1}$$

$$\le C \sum_{n=1}^{N-1} |a_n - b_n| + \|\sum_{n=N}^{\infty} \alpha_n(a_n - b_n) f_n\| \tag{2.2}$$

$$\le C \sum_{n=1}^{N-1} |a_n - b_n| + 2K(\sup_{n \ge N} |\alpha_n|) \|\sum_{n=N}^{\infty} (a_n - b_n) f_n\|, \tag{2.3}$$

where $K$ is the unconditional constant of $(f_n)$. Since $(e_n)$ and $(f_n)$ are equivalent, it is easy to see that

$$\|\sum_{n=N}^{\infty} (a_n - b_n) f_n\| \le 2(1 + K) r \|T\|. \tag{2.4}$$

Let $(f_n^*)$ be the sequence of coefficient functionals associated with $(f_n)$. Since $\alpha_n \to 0$, choosing sufficiently large $N$, we conclude that

$$\|T(x) - T(y)\| \le \epsilon \tag{2.5}$$

if $|f_1^*(x - y)|, \cdots, |f_{N-1}^*(x - y)|$ are sufficiently small. Hence $T$ is weakly uniformly continuous on bounded subset of $E$.

From the above proof it is easy to see that the Banach space $c_0$ of null complex sequences is isomorphic with the Banach space of compact diagonal linear operators $T : E \to F$, where $E$ and $F$ are Banach spaces with equivalent unconditional bases. We would like to remark that if $(e_n)$ and $(f_n)$ are not equivalent, then given a bounded complex sequence $(\alpha_n)$, the map $T(e_n) = \alpha_n f_n$ is not necessarily extended to a bounded linear operator from $E$ into $F$. For example take $E = \ell_2$, $F = \ell_1$ and $\alpha_n = 1$ for all $n$ with respect to the canonical bases of them.

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## REFERENCE

1. ARON, R.M. and PROLLA, J.B. Polynomial approximation of differentiable functions on Banach spaces, J. Reine Angew. Math. 313 (1980), 195-216.

2. CONWAY, J.B. A Course in Functional Analysis, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1985.

3. LINDENSTRAUSS, J. and TZAFRIRI, L. Classical Banach Spaces I, Springer-Verlag, Berlin-Heidelberg-New York, 1977.