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PRIMARY DECOMPOSITION OF TORSION $R[X]$-MODULES

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ABSTRACT. This paper is concerned with studying hereditary properties of primary decompositions of torsion $R[X]$-modules M which are torsion free as R-modules. Specifically, if an $R[X]$-submodule of M is pure as an R-submodule, then the primary decomposition of M determines a primary decomposition of the submodule. This is a generalization of the classical fact from linear algebra that a diagonalizable linear transformation on a vector space restricts to a diagonalizable linear transformation of any invariant subspace. Additionally, primary decompositions are considered under direct sums and tensor product.

KEYWORDS. Primary decomposition of modules and endomorphisms, torsion submodule, pure submodule, diagonalizable endomorphism.
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If R is a principal ideal domain (PID) and M is a torsion R-module, then M is a direct sum of its primary submodules (see Hungerford [3], page 222). The two most important cases of this result (known as the primary decomposition theorem) are when $R = \mathbb{Z}$ (abelian groups) or when $R = F[X]$ where F is a field. In the latter case, to give an $F[X]$-module structure on an F-vector space M is equivalent to giving an F-linear transformation $T : M \to M$. Note that M will be a torsion $F[X]$-module for all choices of $T \in \mathrm{End}_F(M)$ if $\dim_F(M) < \infty$. If $\dim_F(M) = \infty$ then M need not be a torsion $F[X]$-module. If the ring R is not a PID, then a natural generalization of the primary decomposition theorem fails. (See Example 2.)

The purpose of the present note is to study when a primary decomposition of an $R[X]$-module M determines a primary decomposition of an $R[X]$-submodule. Equivalently, if an R-module M possesses a primary decomposition determined by an endomorphism $T \in \mathrm{End}_R(M)$, and N is a T-invariant R-submodule of M, under what conditions does N posses a primary decomposition as an $R[X]$-module? This question has been considered for diagonalizable endomorphisms of M by Weintraub [4]. We will begin by establishing notation.

Let R be an integral domain, M a torsion $R[X]$-module which is torsion free as an R-module, and let K denote the quotient field of R. By extending the scalars to K we obtain a torsion $K[X]$-module $M^K = K \otimes_R M$. If $p(X) \in K[X]$ is an irreducible polynomial then let $M^K(p(X))$ denote the set of elements of M^K which are annihilated by a power of $p(X)$. The primary decomposition theorem then gives

$$M^K = \oplus M^K(p(X))$$

where the sum is over all distinct primes $p(X) \in K[X]$. Let $\iota : M \to M^K$ be the canonical map, which is injective since M is assumed to be torsion free over R. Thus we may identify M with its image in M^K, i.e., we identify $x \in M$ with $1 \otimes x \in M^K$. With these notations, we make the following definition.

DEFINITION 1. We say that M has a *primary decomposition over* $R[X]$ if

$$M = \oplus(M \cap M^K(p(X)))$$

where the sum is over all distinct primes in $K[X]$. If $T \in \mathrm{End}_R(M)$ is determined by multiplication by $X \in R[X]$, then we will say that T has a *primary decomposition* if M has a primary decomposition over $R[X]$. The submodule $M(p(X)) = M \cap M^K(p(X))$ is called the $p(X)$*-primary submodule* of M. A *primary submodule* of M is a submodule of $M(p(X))$ for some prime $p(X)$ of $K[X]$.

The following is an example of a torsion $\mathbf{Z}[X]$-module which does not have a primary decomposition.

EXAMPLE 2. Let $R = \mathbf{Z}[X]$, $M = \mathbf{Z}^2$, and consider M as an R-module via the $\mathbf{Z}$-module endomorphism $T(x, y) = (y, x)$. Then M is a torsion $\mathbf{Z}[X]$-module since $T^2 = 1_M$. But the maximal primary $\mathbf{Z}[X]$-submodules of M are $\langle(1, 1)\rangle$ and $\langle(1, -1)\rangle$, which do not generate M.

DEFINITION 3. If R is an integral domain and M is a torsion free R-module, then a submodule $N \subseteq M$ is *pure* if M/N is torsion free, i.e., if $\alpha y \in N$ and $\alpha \neq 0$, then $y \in N$.

If N is a direct summand of M, then it is a pure submodule. If R is a PID and M is finitely generated, then $N \subseteq M$ is pure if and only if it is a direct summand of M, while if M is not finitely generated, then N may be pure without being a direct summand. (See [1], page 172.) Thus the concept of pure submodule is somewhat more general than that of direct summand. In terms of the extension of scalars, we have that N is a pure submodule of M if and only if $KN \cap M = N \subseteq M^K$. The concept of pure submodule we are using is more restrictive than the definition used in the theory of infinite abelian groups. See Fuchs [2], page 76, for the more general concept.

THEOREM 4. *Let R be an integral domain and M an $R[X]$-module which is torsion free as an R-module and has a primary decomposition over $R[X]$. If N is an $R[X]$-submodule which is pure as an R-submodule, then N has a primary decomposition.*

PROOF. If $N = \{0\}$ the result is obvious, so assume that $N \neq \{0\}$. If $p(X) \in K[X]$ is a prime, let $M(p(X)) = M \cap M^K(p(X))$. By hypothesis,

$$M = \oplus M(p(X)) \tag{1}$$

where the direct sum is over all distinct primes of $K[X]$. Let v be a nonzero element of N. By Equation (1) we may write

$$v = v_1 + \cdots + v_r \tag{2}$$

where $v_i \in M(p_i(X))$ and $p_1(X), \ldots, p_r(X)$ are distinct primes of $K[X]$. Thus there is n_i $(1 \leq i \leq r)$ such that $p_i(X)^{n_i} v_i = 0$. Let

$$h(X) = \prod_{j=2}^{r} p_j(X)^{n_j} \tag{3}$$

Then $p_1(X)^{n_1}$ and $h(X)$ are relatively prime in $K[X]$ so that

$$p_1(X)^{n_1} g_1(X) + h(X) g_2(X) = 1. \tag{4}$$

By clearing denominators in all polynomials in Equation (4), we obtain an equation in $R[X]$:

$$\overline{p}_1(X)^{n_1} \overline{g}_1(X) + \overline{h}(X) \overline{g}_2(X) = \alpha \in R. \tag{5}$$

Multiplying by v_1 gives

$$\alpha v_1 = \overline{h}(X) \overline{g}_2(X) v_1. \tag{6}$$

But

$$\overline{h}(X) v = \overline{h}(X)(v_1 + \cdots + v_r) = \overline{h}(X) v_1. \tag{7}$$

Equations (6) and (7) give

$$\overline{g}_2(X) \overline{h}(X) v = \overline{g}_2(X) \overline{h}(X) v_1 = \alpha v_1. \tag{8}$$

Since N is an $R[X]$-submodule of M we conclude that $\alpha v_1 \in N$ and since N is a pure R-submodule of M, it follows that $v_1 \in N$.

A similar calculation shows that $v_j \in N \cap M(p_j(X))$ for $2 \leq j \leq r$. Since v was an arbitrary element of N, it follows that

$$N = \oplus(N \cap M(p(X))),$$

i.e., N has an $R[X]$-primary decomposition.

DEFINITION 5. We say that $T \in \mathrm{End}_R(M)$ is *block diagonalizable* if $M = \oplus_{j \in J} N_j$ where N_j is an R-submodule of M such that $T|_{N_j} = \lambda_j 1_{N_j}$ where $\lambda_j \in R$.

COROLLARY 6. (Weintraub [4]) *Suppose that R is an integral domain, M is a torsion free R-module, and $T \in \mathrm{End}_R(M)$ is a block diagonalizable endomorphism. If N is a T-invariant pure submodule of M, then $T|_N$ is block diagonalizable.*

PROOF. If T is block diagonalizable, then the primary components of T are $\mathrm{Ker}(T - \lambda_j)$ $(j \in J)$. But by Theorem 4 $T|_N$ has a primary decomposition, and in fact the primary components are just $N \cap \mathrm{Ker}(T - \lambda_j)$.

EXAMPLE 7. Theorem 4 is false without the assumption that the $R[X]$-submodule N be pure as an R-submodule. As an example, let $M = \mathbf{Z}^2$ have the $\mathbf{Z}[X]$-module structure determined by the endomorphism $T(x, y) = (x, -y)$, and let $N = \{(x, y) \in M : x + y \text{ is even}\}$. If $(x, y) \in N$, then $x - y = (x + y) - 2y$ is even so $T(x, y) \in N$, i.e., N is T-invariant so that it is a $\mathbf{Z}[X]$-submodule of M. M has a primary decomposition as a $\mathbf{Z}[X]$-module, but the submodule N does not. Of course, N is not a pure $\mathbf{Z}$-submodule of M.

Since every direct summand is a pure submodule, the following result can be viewed as complementary to Theorem 4.

PROPOSITION 8. *Let R be an integral domain and let M be an $R[X]$-module which is torsion free as an R-module. Suppose that*

$$M = \oplus_{j \in J} N_j$$

where each N_j is an $R[X]$-submodule of M. Then M has a primary decomposition over $R[X]$ if and only if each N_j has a primary decomposition over $R[X]$.

PROOF. First note that M is a torsion $R[X]$-module if and only if each N_j is. If M has a primary decomposition over $R[X]$, then so does each N_j by Theorem 4. Conversely, suppose that each N_j has a primary decomposition over $R[X]$. Thus

$$N_j = \oplus_{p(X)} N_j(p(X)) \tag{9}$$

for all $j \in J$ where the sum is over all distinct primes $p(X) \in K[X]$. We claim that

$$M(p(X)) = \oplus_{j \in J} N_j(p(X)). \tag{10}$$

To see this suppose that $v \in M(p(X))$. Then we may write

$$v = v_{j_1} + \cdots + v_{j_r}$$

where $v_{j_i} \in N_{j_i}$ for $1 \le i \le r$. Since $K[X]$ is a PID and $v \in M(p(X))$, we have that

$$p(X)^n = \mathrm{Ann}(v) = \mathrm{lcm}\{\mathrm{Ann}(v_{j_1}), \ldots, \mathrm{Ann}(v_{j_r})\}.$$

Hence $\mathrm{Ann}(v_{j_i}) = p(X)^{n_i}$ for some n_i. Thus $v_{j_i} \in N_{j_i}(p(X))$ and Equation (10) is satisfied. Equations (9) and (10) then give

$$\begin{aligned} \oplus_{p(X)} M(p(X)) &= \oplus_{p(X)} \oplus_{j \in J} N_j(p(X)) \\ &= \oplus_{j \in J} \left(\oplus_{p(X)} N_j(p(X))\right) \\ &= \oplus_{j \in J} N_j \\ &= M. \end{aligned}$$

Hence M has a primary decompositon over $R[X]$.

It is a standard result in linear algebra that two commuting diagonalizable linear transformations have a basis of common eigenvectors. In the context of torsion $R[X]$-modules, this result generalizes to the following fact.

PROPOSITION 9. *Let M be a torsion free R-module over an integral domain R and let T, $S \in \mathrm{End}_R(M)$ be commuting endomorphisms, each of which has a primary decomposition. Then there is a direct sum decomposition $M = \oplus_{i \in I} M_i$ where M_i is a primary $R[X]$-submodule of M for the $R[X]$-module structures determined by both S and T.*

PROOF. By hypothesis, $M = \oplus M(p(X))$ where $M(p(X))$ is the $p(X)$-primary component of M in the $R[X]$-module structure determined by T, and the direct sum is over all distinct primes $p(X) \in K[X]$. Since $M(p(X))$ is a direct summand of M, it is pure as an R-submodule. If $v \in M(p(X))$ then $p(X)^n v = 0$ for some n (we may assume $p(X) \in R[X]$ without loss of generality), i.e., $p(T)^n v = 0$. Now $p(T)^n(Sv) = S(p(T))^n v = 0$ since $TS = ST$. Thus

$Sv \in M(p(X))$, so that $M(p(X))$ is also an $R[X]$-submodule of M with the module structure determined by S. By Theorem 4, it follows that $S|_{M(p(X))}$ has a primary decomposition

$$M(p(X)) = \oplus M(p(X))(q(X)) \tag{11}$$

where the sum is over all distinct primes $q(X) \in K[X]$. Since T and S commute, it follows that $M(p(X))(q(X))$ is an $R[X]$-submodule of M for both $R[X]$-module structures on M, and

$$M = \oplus_{p(X)} \oplus_{q(X)} M(p(X))(q(X))$$

is the required decomposition.

COROLLARY 10. *If $S, T \in \mathrm{End}_R(M)$ are block diagonalizable and $ST = TS$, then S and T are jointly block diagonalizable, i.e.,*

$$M = \oplus_{i \in I} M_i \tag{12}$$

where $T|_{M_i} = \lambda_i 1_{M_i}$ and $S|_{M_i} = \mu_i 1_{M_i}$ with $\lambda_i, \mu_i \in R$.

COROLLARY 11. *If $S, T \in \mathrm{End}_R(M)$ are block diagonalizable and $ST = TS$, then $P(S, T)$ is block diagonalizable for all $P(X, Y) \in R[X, Y]$.*

PROOF. Write $M = \oplus_{i \in I} M_i$ as in Equation (12). Then $S|_{M_i} = \lambda_i 1_{M_i}$ and $T|_{M_i} = \mu_i 1_{M_i}$ so that $P(S, T)|_{M_i} = P(\lambda_i, \mu_i) 1_{M_i}$.

The following result is similar in spirit to Corollary 11.

PROPOSITION 12. *Let $T \in \mathrm{End}_R(M)$ have a primary decomposition over $R[X]$. Then every element of the algebra $R[T]$ has a primary decomposition over $R[X]$.*

PROOF. By hypothesis $M = \oplus M(p(X))$ where the sum is over all distinct primes $p(X) \in K[X]$. Since the property of having a primary decomposition is preserved under direct sums (Proposition 8), it is sufficient to assume that $M = M(p(X))$ where $p(X) \in K[X]$ is irreducible. Let $f(X) \in R[X]$ be arbitrary. We wish to show that $f(T)$ has a primary decomposition.

Let $F = K[X]/\langle p(X) \rangle$. Then F is a finite algebraic extension of K. Let $\pi : K[X] \to F$ be the projection. If $\alpha = \pi(f(X))$ then α is algebraic over K, so let $m_\alpha(X) \in K[X]$ be the minimal polynomial of α over K. By clearing denominators we can assume that $m_\alpha(X) \in R[X]$. Then $m_\alpha(\pi(f(X)) = 0 \in F$. In $K[X]$ this means that

$$m_\alpha(f(X)) = h(X)p(X). \tag{13}$$

By clearing denominators we may assume the polynomials are in $R[X]$, i.e.,

$$cm_\alpha(f(X)) = ch(X)p(X) \tag{14}$$

where $c \in R$. The evaluation at T, ev_T is an R-algebra homomorphism. Thus, if $p(T)^n v = 0$ it follows from Equation (13) that

$$m_\alpha(f(T))^n v = 0.$$

Thus $M = M(p(X)) = M(m_\alpha(X))$ for the $R[X]$-module structure determined by $f(T)$. That is, if M is primary for T, then M is also primary for $f(T)$, and the result is proved.

We conclude with the following example which shows that primary decomposition need not be preserved under tensor product.

EXAMPLE 12. Let $R = \mathbf{Z}$ and let $M = \mathbf{Z}^2$. Give M the $\mathbf{Z}[X]$-module structure determined by the endomorphism $T(x, y) = (-y, x)$ with matrix (with respect to the standard basis) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The minimal polynomial of T is $X^2 + 1$ so that M is primary. Let $N = M \otimes_{\mathbf{Z}} M$ and give N the $\mathbf{Z}[X]$-module structure determined by $T \otimes T$. The matrix of $T \otimes T$ is

$$A \otimes A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which has characteristic polynomial $(X - 1)^2(X + 1)^2$. The eigenspace of 1 has a basis

$$\{(1, 0, 0, 1),\ (0, 1, -1, 0)\}$$

while the eigenspace of -1 has a basis

$$\{(1, 0, 0, -1),\ (0, 1, 1, 0)\}.$$

Thus the sum of the eigenspaces does not generate all of N so that N does not have a primary decomposition following $T \otimes T$.

REFERENCES

1. W. Adkins and S. Weintraub, *Algebra: An Approach via Module Theory*, GTM 136, Springer-Verlag, New York (1992).
2. L. Fuchs, *Abelian Groups*, Third Edition, Akademiai Kidao, Budapest (1966).
3. T. Hungerford, *Algebra*, GTM 73, Springer-Verlag, New York (1980).
4. S. Weintraub, A note on linear transformations over integral domains, *Linear and Multilinear Algebra*, to appear.