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EXTENSIONS OF HARDY-LITTLEWOOD INEQUALITIES

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ABSTRACT. For a function $f \in H^p(B_n)$, with $f(0) = 0$, we prove

① If $0 < p \leqslant s$, then

$$\int_0^1 r^{-1}\left(\log \frac{1}{r}\right)^{s\beta-1} M_p^s(r, R^\beta f)dr \leqslant \| f \|_p^{s-p} \| f \|_{p,s,\beta}^p$$

② If $s \leqslant p < \infty$, then

$$\| f \|_{p,s,\beta}^p \leqslant \| f \|_p^{p-s} \int_0^1 r^{-1}\left(\log \frac{1}{r}\right)^{s\beta-1} M_p^s(r, R^\beta f)dr$$

where $R^\beta f$ is the fractional derivative of f. These results generalize the known cases $s = 2, \beta = 1$([1]).

KEY WORDS AND PHRASES. $H^p(B_n)$ space, fractional derivative.
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1. INTRODUCTION.

Let C^n denote the n-dimensional vector space over C. Let B_n denote the unit ball in C^n with boundary ∂B_n and let σ denote the rotation-invariant positive measure on ∂B_n for which $\sigma(\partial B_n) = 1$.

We assume that f is holomorphic in B_n. Let $R^\beta f(z) = \sum_{a\geqslant 0} |\alpha|^\beta a_a z^a$ be the fractional derivative of $f(z) = \sum_{a\geqslant 0} a_a z^a (\beta > 0)$.

For $0 < p, s, \beta < \infty$, we set

$$M_p^p(r, f) = \int_{\partial B_n} |f(r\zeta)|^p d\sigma(\zeta)$$

and

$$\| f \|_{p,s,\beta}^p = \int_0^1 \int_{\partial B_n} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s \left(\log \frac{1}{r}\right)^{s\beta-1} r^{-1} d\sigma(\zeta)dr$$

As usual, for $0 < p < \infty$, $H^p(B_n)$ denotes the space of holomorphic functions on B_n for which the means $M_p(r, f)$ are bounded and the norm of $f \in H^p(B_n)$ is defined by

$$\| f \|_p = \sup_{0<r<1} M_p(r,f).$$

Throughout this note, we assume that $f \in H^p(B_n)$, with $f(0) = 0$.

In [1], Hardy—Littlewood proved the following well-known theorem about $H^p(B_1)$.

THEOREM HL. If $0 < p \leqslant 2$, $f \in H^p(B_1)$, then

$$\int_0^1 (1-r) M_p^2(r,f')dr < \infty \qquad (*)$$

If $2 \leqslant p < \infty$, then $(*)$ implies $f \in H^p(B_1)$.

In this note, we generalize these results to the unit ball B_n, with a new and short proof. That is, we prove the following

THEOREM. ① If $0 < p \leqslant s$, then

$$\int_0^1 r^{-1}\left(\log \frac{1}{r}\right)^{s\beta-1} M_p^s(r,R^\beta f)dr \leqslant \| f \|_p^{s-p} \| f \|_{p,s,\beta}^p$$

② If $s \leqslant p < \infty$, then

$$\| f \|_{p,s,\beta}^p \leqslant \| f \|_p^{p-s} \int_0^1 r^{-1}\left(\log \frac{1}{r}\right)^{s\beta-1} M_p^s(r,R^\beta f)dr$$

Set $s = 2, \beta = 1$ in the Theorem; by the following

LEMMA. For $0 < p < \infty$, then

$$\| f \|_p^p = p^2 \| f \|_{p,2,1}^p$$

we have the following corollary, which extends Theorem HL (note that for $\zeta \in B_n$, $R^1 f(\lambda\zeta) = \lambda f'_\zeta(\lambda)$, where $f_\zeta(\lambda) = f(\lambda\zeta)$, $\lambda \in B_1$, and $r\log\frac{1}{r} \sim 1-r$)

COROLLARY. ① If $0 < p \leqslant 2$, then

$$\int_0^1 r^{-1}\left(\log \frac{1}{r}\right) M_p^2(r,R^1 f)dr \leqslant \frac{1}{p^2} \| f \|_p^2$$

② If $2 \leqslant p < \infty$, then

$$\| f \|_p^2 \leqslant p^2 \int_0^1 r^{-1}\left(\log \frac{1}{r}\right) M_p^2(r,R^1 f)dr$$

2. PROOF OF THE MAIN RESULTS.

PROOF of the Theorem. Let $0 < p \leqslant s$. Assume without loss of generality that $\| f \|_p \neq 0$. Set $\mu(\zeta) = \dfrac{|f(r\zeta)|^p}{\| f \|_p^p}$, then $\int_{\partial B_n} \mu(\zeta)d\sigma(\zeta) \leqslant 1$; we have, by Jensen's inequality, for each r,

$$\left(\int_{\partial B_n} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s d\sigma(\zeta)\right)^{p/s}$$

$$= \left(\| f \|_p^p \int_{\partial B_n} \left|\frac{R^\beta f(r\zeta)}{f(r\zeta)}\right|^s \mu(\zeta)d\sigma(\zeta)\right)^{p/s}$$

$$\geqslant \| f \|_p^{p^2/s} \int_{\partial B_n} \left|\frac{R^\beta f(r\zeta)}{f(r\zeta)}\right|^p \mu(\zeta)d\sigma(\zeta)$$

$$= \| f \|_p^{p^2/s-p} \int_{\partial B_n} |R^\beta f(r\zeta)|^p d\sigma(\zeta)$$

$$= \| f \|_p^{p^2/s-p} M_p^p(r,R^\beta f)$$

So

$$\int_{\partial B_n} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s d\sigma(\zeta) \geqslant \| f \|_p^{s-p} M_p^s(r,R^\beta f)$$

Therefore

$$\|f\|_p^{s-p}\|f\|_{p,s,\beta}^s=\|f\|_p^{s-p}\int_0^1\int_{\partial B_n}|f(r\zeta)|^{p-s}|R^\beta f(r\zeta)|^s\left(\log\frac{1}{r}\right)^{s\beta-1}r^{-1}d\sigma(\zeta)dr$$

$$\geqslant\int_0^1 r^{-1}\left(\log\frac{1}{r}\right)^{s\beta-1}M_p^s(r,R^\beta f)dr$$

The case $p\geqslant s$ is treated in a similar way to obtain, for each r,

$$\int_{\partial B_n}|f(r\zeta)|^{p-s}|R^\beta f(r\zeta)|^s d\sigma(\zeta)\leqslant\|f\|_p^{p-s}M_p^s(r,R^\beta f)$$

So

$$\|f\|_{p,s,\beta}^p=\int_0^1\int_{\partial B_n}|f(r\zeta)|^{p-s}|R^\beta f(r\zeta)|^s\left(\log\frac{1}{r}\right)^{s\beta-1}r^{-1}d\sigma(\zeta)dr$$

$$\leqslant\|f\|_p^{p-s}\int_0^1 r^{-1}\left(\log\frac{1}{r}\right)^{s\beta-1}M_p^s(r,R^\beta f)dr$$

This completes the proof of the Theorem.

Now, we use the technique of [2] to give the proof of the lemma.

For $\zeta\in B_n$, $R^1 f(\lambda\zeta)=\lambda f'_\zeta(\lambda)$, where $f_\zeta(\lambda)=f(\lambda\zeta)$, $\lambda\in B_1$.

By the Hardy—Stein identity for one complex variable ([3]) we have

$$M_p^p(r,f_\zeta)=\frac{p^2}{2\pi}\int_0^r\int_0^{2\pi}|f_\zeta(\rho e^{i\theta})|^{p-2}|f'_\zeta(\rho e^{i\theta})|^2\left(\log\frac{r}{\rho}\right)\rho d\theta d\rho$$

$$=\frac{p^2}{2\pi}\int_0^r\int_0^{2\pi}|f(\rho\zeta e^{i\theta})|^{p-2}|R^1 f(\rho\zeta e^{i\theta})|^2\left(\log\frac{r}{\rho}\right)\rho^{-1}d\theta d\rho$$

Integrating with respect to $d\sigma(\zeta)$, using the Fubini theorem and the formula

$$\int_{\partial B_n}g(\zeta)d\sigma(\zeta)=\frac{1}{2\pi}\int_{\partial B_n}d\sigma(\zeta)\int_0^{2\pi}g(e^{i\theta}\zeta)d\theta,\qquad g\in L^1(\sigma)$$

(see [4, P. 15]), we have

$$M_p^p(r,f)\doteq p^2\int_0^r\int_{\partial B_n}|f(\rho\zeta)|^{p-2}|R^1 f(\rho\zeta)|^2\left(\log\frac{r}{\rho}\right)\rho^{-1}d\sigma(\zeta)d\rho$$

Letting $r\to 1$, we obtain the Lemma.

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