Internat. J. Math. & Math. Sci.
VOL. 17 NO. 1 (1994) 147-154

ON THE APPLICATION OF NEWTON'S AND CHORD METHODS TO BIFURCATION PROBLEMS

M.B.M. ELGINDI

Department of Mathematics
University of Wisconsin - Eau Claire
Eau Claire, WI 54702-4004 U.S.A.

(Received November 5, 1992 and in revised form March 20, 1993)

Abstract. This paper is concerned with the applications of Newton's and chord methods in the computations of the bifurcation solutions in a neighborhood of a simple bifurcation point for prescribed values of the bifurcation parameter.

Key Words and Phrases: Simple bifurcation point; bifurcation from the trivial solution, Newton's and chord methods.

1980 Mathematics Subject Classification: 65J99, 58F14.

1. INTRODUCTION

Many problems in applications are formulated as

$$G(x,\lambda) = 0\,, \tag{1.1}$$

where $G: H \times \Re \rightarrow H$ is a smooth (nonlinear) map satisfying

$$G(0,\lambda) = 0\,; \tag{1.2}$$

H is a real Hilbert space and λ is a real parameter which often represents some physical quantity.

The solution curve $\Gamma_0 : x_0(\lambda) \equiv 0$ is called the trivial solution curve. A point $(0,\lambda_0) \in \Gamma_0$ is called a bifurcation point if there exists a smooth curve $\Gamma_1 : x_1 = x_1(\lambda)$ of nontrivial solutions of (1.1) which is defined in some neighborhood of $(0,\lambda_0)$ and passing through it. It follows from the Implicit Function Theorem that a necessary condition for $(0,\lambda_0)$ to be a bifurcation point is that the Frechet derivative $G_x(0,\lambda_0)$ is singular. On the other hand the singularity of $G_x(0,\lambda_0)$ is not a sufficient condition for $(0,\lambda_0)$ to be a bifurcation point. Some sufficient conditions for $(0,\lambda_0) \in \Gamma_0$ to be a "simple" bifurcation point will be stated in Section 2.

This paper is concerned with the numerical computations of the nontrivial solution curves of (1.1) in a neighborhood of a simple bifurcation point $(0,\lambda_0)$. Many excellent analytical and numerical treatments of this problem exist in the literature. The reader is referred to [1]-[7], and the references therein for an extensive account of the subject. In almost all the previous numerical works the parameter λ is treated as a variable and is determined along with the "state" variable x. However, in applications λ represents a physical parameter and it is often required to determine the state variable x for some given values of the parameter λ. In this paper we examine the applications of Newton's and chord methods in solving (1.1) for x while λ is kept fixed near a simple bifurcation point λ_0.

The rest of the paper is organized in three sections. In Section 2 we present some well known preliminary results, regarding the solution set of (1.1) which are based upon the Implicit Function Theorem and state a convergence theorem for Newton's and chord methods due to G. Moore [7]. In Section 3 we present and prove the convergence of some numerical schemes for computing the nontrivial solution curves of (1.1) in a neighborhood of a simple bifurcation point $(0,\lambda_0)$ for given values of the parameter λ. In section 4 we illustrate the use of the schemes developed in Section 3 by applying them to a finite dimensional numerical example.

2. PRELIMINARIES

All the results and proofs stated in this section are well known and we present them here for completeness. We present some sufficient conditions for a point $(0,\lambda_0)$ of Γ_0 to be a bifurcation point and state a basic convergence theorem for Newton's and chord methods.

We assume that the Frechet derivative $G_x^0 = G_x(0,\lambda_0)$ satisfies, for some $\lambda_0 \in \Re$, the conditions

$$\begin{aligned}
&\text{(a)}\quad N(G_x^0) \text{ is one-dimensional spanned by } \phi, \langle\phi,\phi\rangle = 1,\\
&\text{(b)}\quad N(G_x^{0*}) \text{ is one-dimensional spanned by } \psi \neq 0,\\
&\text{(c)}\quad R(G_x^0) = N(G_x^{0*})^\perp \text{ and } R(G_x^{0*}) = N(G_x^0)^\perp,\\
&\text{(d)}\quad a = \langle\psi, G_{x\lambda}^0\phi\rangle \neq 0,\\
&\text{(e)}\quad \langle\psi,\phi\rangle = 1,
\end{aligned} \tag{2.1}$$

where $G_{x\lambda}^0 = G_{x\lambda}(0,\lambda_0)$ and the notations L^*, $N(L)$ and $R(L)$ denote the adjoint operator, the null and the range spaces of a linear operator L, respectively. Under the assumptions (a)-(e) of (2.1), $(0,\lambda_0)$ is a bifurcation point. To see this we decompose H as

$$H = \langle\phi\rangle \oplus R(G_x^{0*}). \tag{2.3}$$

using (2.3), each $x \in H$ can be written in the form

$$x = \varepsilon\phi + w, \tag{2.4}$$

for unique $\varepsilon \in \Re$ and $w \in R(G_x^{0*})$. Writing $\lambda = \lambda_0 + \mu$, using (2.4) and

$$H = \langle\psi\rangle \oplus R(G_x^0),$$

we can decompose equation (1.1) into the two equations

$$\begin{aligned}
&\text{(i)}\quad G(\varepsilon\phi + w, \lambda_0 + \mu) - \langle\psi, G(\varepsilon\phi + w, \lambda_0 + \mu)\rangle \frac{\psi}{\|\psi\|^2} = 0,\\
&\text{(ii)}\quad \langle\psi, G(\varepsilon\phi + w, \lambda_0 + \mu)\rangle = 0.
\end{aligned} \tag{2.5}$$

Let $K(w,\varepsilon,\mu)$ denote the left hand side of equation (i) of (2.5). Then $K(0,0,0) = 0$ and $K_w(0,0,0) = G_x^0$, which has bounded inverse from $R(G_x^{0*})$ to $R(G_x^0)$. This enables us to apply the Implicit Function Theorem to equation (i) of (2.5) to conclude the existence and uniqueness of a smooth function $w = w(\varepsilon,\mu)$ defined in some neighborhood M of (0,0), such that

$$K(w(\varepsilon,\mu),\varepsilon,\mu) = 0, \quad (\varepsilon,\mu) \in M.$$

It follows that, in M, equation (1.1) reduces to

$$g(\varepsilon,\mu) = \langle\psi, G(\varepsilon\phi + w(\varepsilon,\mu), \lambda_0 + \mu)\rangle = 0. \tag{2.6}$$

Define $h(\varepsilon,\mu) = \frac{1}{\varepsilon} g(\varepsilon,\mu)$. Since $h(0,0) = 0$ and $h_\mu(0,0) = a \neq 0$, we can apply the Implicit Function Theorem to conclude the existence and uniqueness of a smooth function $\mu(\varepsilon)$ defined in some neighborhood of $\varepsilon = 0$ such that

$$h(\varepsilon,\mu(\varepsilon)) = 0,$$

for each ε in that neighborhood.

The smoothness of $w(\varepsilon,\mu)$ and $\mu(\varepsilon)$ allow us to expand about $\varepsilon = 0$ to obtain the following expansions

$$w(\varepsilon,\mu) = \varepsilon\mu w_1 + \varepsilon^2 w_2 + \varepsilon 0((|\varepsilon| + |\mu|)^2), \tag{2.7}$$

$$\mu(\varepsilon) = -\frac{E_0}{a} - \frac{E_3 a + E_1 E_2 + E_2 E_0}{a^2}\varepsilon^2 + O(\varepsilon^3), \tag{2.8}$$

where

(a) $w_1 = A'\phi - G_x^{0^{-1}}\left(G_{x\lambda}^0\phi - a\dfrac{\psi}{\|\psi\|^2}\right),$

(b) $w_2 = A''\phi - G_x^{0^{-1}}\left(G_{xx}^0\phi\phi - \langle\psi, G_{xx}^0\phi\phi\rangle\dfrac{\psi}{\|\psi\|^2}\right),$

(c) $E_0 = \dfrac{1}{2}\langle\psi, G_{xx}^0\phi\phi\rangle,$

(d) $E_1 = \dfrac{1}{2}\langle\psi, G_{xx}^0\phi w_1\rangle + \dfrac{1}{2}\langle\psi, G_{xx}^0 w_1\phi\rangle + \dfrac{1}{2}\langle\psi, G_{xx\lambda}^0\phi\phi\rangle + \langle\psi, G_{x\lambda}^0 w_2\rangle,$

(e) $E_2 = \langle\psi, G_{x\lambda}^0 w_1\rangle,$

(f) $E_3 = \dfrac{2}{3}\langle\psi, G_{xx}^0\phi w_2\rangle + \dfrac{1}{3}\langle\psi, G_{xx}^0 w_2\phi\rangle + \dfrac{1}{6}\langle\psi, G_{xxx}^0\phi\phi\phi\rangle,$

A' and A'' are parameters to be determined by the conditions $\langle\phi, w_1\rangle = 0$ and $\langle\phi, w_2\rangle = 0$, respectively, and G_{xx}^0 denotes $G_{xx}(0,\lambda_0)$, etc.

We gather the conclusions of the above computations in the following theorem.

2.1 Theorem: A point $(0,\lambda_0)$ on Γ_0 which satisfies conditions (a)-(e) of (2.1) is a bifurcation point. Furthermore, there exists exactly one bifurcation branch of solutions $(x(\varepsilon),\lambda(\varepsilon))$ passing through $(0,\lambda_0)$ which is (locally) given by

$$x(\varepsilon) = \varepsilon\phi + w(\varepsilon,\mu(\varepsilon)),$$

$$\lambda(\varepsilon) = \lambda_0 + \mu(\varepsilon),$$

where $w(\varepsilon,\mu(\varepsilon))$ and $\mu(\varepsilon)$ are given by (2.7) and (2.8), respectively.

We now state a convergence result for Newton's and chord method from [7].

2.2 Theorem: Let H_1 and H_2 be Hilbert spaces and $F(U,\delta)$ be a mapping from $H_1 \times \Re$ into H_2. Assume that F is continuously differentiable with respect to U and continuous with respect to δ. Let $U^0(\delta)$ be a continuous mapping from $\Re^+$ into H_1 such that for some $\delta_1 > 0$, $F_U(U^0(\delta),\delta)$ has a bounded inverse $T(\delta)$ for all $0 < \delta < \delta_1$ which satisfies

(a) $\|T(\delta)F(U^0(\delta),\delta)\| \le \eta(\delta),$

(b) $\|T(\delta)[F_U(U,\delta) - F_U(V,\delta)]\| \le \xi(\delta),$ for $U, V \in N_{2\eta(\delta)}(U^0(\delta)),$

where $N_{2\eta(\delta)}(U^0(\delta))$ is the neighborhood of $U^0(\delta)$ with radius $2\eta(\delta)$ and $\eta(\delta)$ and $\xi(\delta)$ satisfy $\eta(\delta)\xi(\delta) = O(\delta)$ as $\delta \to 0$. Then there exists $0 < \delta_2 \le \delta_1$ and a continuous mapping $U^*(\delta)$ from $(0,\delta_2)$ into H_1 which is the unique solution of $F(U,\delta) = 0$ in $N_{2\eta(\delta)}(U^0(\delta))$ and the Newton's iterates

$$U^{n+1} = U^n - F_U^{-1}(U^n,\delta)F(U^n,\delta),$$

converge to $U^*(\delta)$ for each $\delta \in (0,\delta_2)$.

Furthermore, if condition (b) is replaced by

(c) $\|T(\delta)\{F_U(U^0(\delta),\delta)(U-V) - [F(U,\delta) - F(V,\delta)]\}\| \le \xi(\delta)\|U-V\|,$ for

$U, V \in N_{2\eta(\delta)}(U^0(\delta)),$

where $\xi(\delta) = O(\delta)$ as $\delta \to 0$, then there exists $0 < \delta_2 \le \delta_1$ and a continuous mapping $U^*(\delta)$ from $(0,\delta_2)$ into H_1 such that $U^*(\delta)$ is the unique solution of $F(U,\delta) = 0$ in $N_{2\eta(\delta)}(U^0(\delta))$ and the chord iterates

$$U^{n+1} = U^n - F_U^{-1}(U^0(\delta),\delta)F(U^n,\delta),$$

converge to $U^*(\delta)$ for each $\delta \in (0,\delta_2)$.

3. NEWTON'S AND CHORD METHODS

In [6], Decker and Keller introduced a method for constructing the bifurcation branches near a simple bifurcation point $(0,\lambda_0)$. In that paper the equation

$$G(\varepsilon\phi + w, \lambda_0 + \mu) = 0, \tag{3.1}$$

was replaced by an "inflated" system

$$F(y,\varepsilon) \equiv \begin{bmatrix} G(\varepsilon\phi + w, \lambda_0 + \mu) \\ \langle \phi, w \rangle \end{bmatrix} = 0, \tag{3.2}$$

where $y = \begin{pmatrix} w \\ \mu \end{pmatrix} \in H \times \Re$. Using the initial guess $y^0 = 0$, they proved the convergence of Newton's and chord iterates of (3.2) to a nontrivial solution $y(\varepsilon)$ of (3.1) for each given ε in $0 < |\varepsilon| \le \varepsilon_0$, provided ε_0 is small enough. Thus, in this method both the state variable x and the bifurcation parameter λ are treated as unknowns and are approximated as functions of the parameter ε. This method cannot be used in practical problems where it is required to approximate the state variable x for some given values of λ. On the other hand it is not possible in general to parameterize x using the bifurcation parameter λ; for example; in the case when nontrivial solutions only occur at the bifurcation point. This shows that some additional assumptions are to be imposed.

In addition to conditions (a)-(e) of (2.1) we will assume that either

(Non-degenerate case) $$E_0 \neq 0 \tag{3.3}$$

(Degenerate case) $$E_0 = 0, \quad E_3 \neq 0, \quad \frac{a}{E_3} < 0, \tag{3.4}$$

hold, and show that the methods of [6] can be modified to approximate the nontrivial solutions of (3.1) with λ being fixed near the bifurcation point λ_0.

Let H_1 denote the Hilbert space $H \times \Re$ with inner product

$$\left\langle \begin{pmatrix} w_1 \\ \varepsilon_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ \varepsilon_2 \end{pmatrix} \right\rangle = \langle w_1, w_2 \rangle + \varepsilon_1\varepsilon_2,$$

and define $F: H_1 \times \Re \to H_1$ by

$$F(U,\lambda) = \begin{bmatrix} G(\varepsilon\phi + w, \lambda) \\ \langle \phi, w \rangle \end{bmatrix}, \tag{3.5}$$

for $U = \begin{pmatrix} w \\ \varepsilon \end{pmatrix} \in H_1, \lambda \in \Re$. For fixed λ, using (2.7) and (2.8) we define the initial guesses

$$U^0(\lambda) = \begin{pmatrix} 0 \\ \bar{\varepsilon} \end{pmatrix}, \quad \bar{\varepsilon} = \frac{-a(\lambda - \lambda_0)}{E_0}, \quad \lambda \in \Re, \tag{3.6}$$

$$U^0(\lambda) = \begin{pmatrix} \bar{\varepsilon}^2 w_2 \\ \bar{\varepsilon} \end{pmatrix}, \quad \bar{\varepsilon} = \pm\sqrt{\frac{-a(\lambda - \lambda_0)}{E_3}}, \quad \lambda \ge \lambda_0, \tag{3.7}$$

for the non-degenerate case (3.3) and degenerate case (3.4), respectively.

For a given λ near λ_0 we will prove that both of Newton's and chord iterates with initial guess (3.6) or (3.7), according to whether condition (3.3) and (3.4) hold, converge to a unique nontrivial solution U^* of the equation

$$\alpha = \frac{\beta \pm \sqrt{\beta^2 - 4\xi}}{2},$$

where $\beta = b_{11}(\bar{\varepsilon}) + b_{22}(\bar{\varepsilon})$ and $\xi = b_{11}(\bar{\varepsilon})b_{22}(\bar{\varepsilon}) - b_{12}(\bar{\varepsilon})b_{21}(\bar{\varepsilon})$. From (3.8) we have

(a) $\quad F_U(\bar{\varepsilon})\Phi_1(\bar{\varepsilon}) = b_{11}(\bar{\varepsilon})\Phi_1(\bar{\varepsilon}) + b_{12}(\bar{\varepsilon})\Phi_2(\bar{\varepsilon})$

Differentiating equation (a) with respect to $\bar{\varepsilon}$, setting $\bar{\varepsilon} = 0$ and using (3.10) we obtain

(b) $$\begin{bmatrix} G_{xx}^0\phi\phi - \frac{E_0}{a}G_{x\lambda}^0\phi \\ 0 \end{bmatrix} + F_U^0\dot{\Phi}_1(0) = \dot{b}_{11}(0)\Phi_1 + \dot{b}_{12}(0)\Phi_2,$$

for the non-degenerate case (3.3), and

(c) $$\begin{bmatrix} G_{xx}^0\phi\phi \\ 0 \end{bmatrix} + F_U^0\dot{\Phi}_1(0) = \dot{b}_{11}(0)\Phi_1 + \dot{b}_{12}(0)\Phi_2,$$

for the degenerate case (3.4). The solvability conditions of (b) and (c) are

(d) $\quad \dot{b}_{12}(0) = E_0,$

and

(e) $\quad \dot{b}_{12}(0) = \dot{b}_{11}(0) = 0,$

respectively. From (d) it follows that the eigenvalues of $B(\bar{\varepsilon})$ are of the form (3.12) with $\gamma = \frac{1}{2}$. It follows from (e) that

(f) $$\dot{\Phi}_1(0) = \begin{pmatrix} 2w_2 \\ \eta \end{pmatrix},$$

for some constant η. Differentiating equation (a) once more and setting $\bar{\varepsilon} = 0$ gives

$$\begin{bmatrix} 4G_{xx}^0\phi w_2 + 2G_{xx}^0 w_2\phi + G_{xxx}^0\phi\phi\phi - \frac{2E_3}{a}G_{x\lambda}^0\phi \\ 0 \end{bmatrix} + F_U^0\ddot{\Phi}_1(0) = \ddot{b}_{11}(0)\Phi_1 + \ddot{b}_{12}(0)\Phi_2,$$

whose solvability condition is

$$\ddot{b}_{12}(0) = 4E_3,$$

and this proves that the eigenvalues of $B(\bar{\varepsilon})$ in the degenerate case are of the form (3.12) with $\gamma = 1$.

The following result follows from Lemma 3.1.

3.2 Lemma: For small enough $\bar{\varepsilon} > 0$ the linear operator $F_U(\bar{\varepsilon})$ has a bounded inverse such that

$$\| F_U^{-1}(\bar{\varepsilon}) \| = 0\left(\frac{1}{\bar{\varepsilon}^\gamma}\right),$$

where γ is as in Lemma 3.1.

To examine the convergence of Newton's and chord iterates we need to estimate $\| F_U^{-1}(\bar{\varepsilon})F(\bar{\varepsilon}) \|$. This is done in the following Lemma.

3.3 Lemma: For small enough $\bar{\varepsilon} > 0$ we have

$$\| F_U^{-1}(\bar{\varepsilon})F(\bar{\varepsilon}) \| = 0(\bar{\varepsilon}^{1+\gamma})$$

where γ is as in Lemma 3.1.

Proof: We note that for the non-degenerate case

$$F(\bar{\varepsilon}) = \begin{bmatrix} \bar{\varepsilon}^2\left(\frac{1}{2}G_{xx}^0\phi\phi - \frac{E_0}{a}G_{x\lambda}^0\phi\right) \\ 0 \end{bmatrix} + 0(\bar{\varepsilon}^3)$$

and for the degenerate case

$$F(U,\lambda) = 0 .$$

We will denote $F(U^0,\lambda), F_U(U^0,\lambda)$ and $F_U(U^0,\lambda_0)$ by $F(\bar{\varepsilon}), F_U(\bar{\varepsilon})$ and F_U^0, respectively.

We observe that

$$F_U^0 = \begin{bmatrix} G_x^0 & 0 \\ \langle \phi, \cdot \rangle & 0 \end{bmatrix},$$

and that Lemma 5.6 [6] implies that $F_U^0, F_U(\bar{\varepsilon})$ and $F_U^*(\bar{\varepsilon})$ satisfy the properties

(i) F_U^0 is a Fredholm operator with zero index, where $N(F_u^0)$ and $N(F_U^{0*})$ are spanned by $\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\Psi_1 = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$, respectively,

(ii) the zero eigenvalue of F_U^0 has algebraic multiplicity two,

(iii) there exist smooth functions $\Phi_1(\bar{\varepsilon}), \Phi_2(\varepsilon), \Psi_1(\bar{\varepsilon}), \Psi_2(\bar{\varepsilon})$,

$$B(\bar{\varepsilon}) = \begin{bmatrix} b_{11}(\bar{\varepsilon}) & b_{12}(\bar{\varepsilon}) \\ b_{21}(\bar{\varepsilon}) & b_{22}(\bar{\varepsilon}) \end{bmatrix} \quad \text{and} \quad \hat{B}(\bar{\varepsilon}) = \begin{bmatrix} \hat{b}_{11}(\bar{\varepsilon}) & \hat{b}_{12}(\bar{\varepsilon}) \\ \hat{b}_{21}(\bar{\varepsilon}) & \hat{b}_{22}(\bar{\varepsilon}) \end{bmatrix},$$

defined for $|\bar{\varepsilon}| < \delta$, for some $\delta > 0$, such that

$$\begin{bmatrix} F_U(\bar{\varepsilon}) & 0 \\ 0 & F_U(\bar{\varepsilon}) \end{bmatrix} \begin{bmatrix} \Phi_1(\bar{\varepsilon}) \\ \Phi_2(\bar{\varepsilon}) \end{bmatrix} = B(\bar{\varepsilon}) \begin{bmatrix} \Phi_1(\bar{\varepsilon}) \\ \Phi_2(\bar{\varepsilon}) \end{bmatrix}, \tag{3.8}$$

$$\begin{bmatrix} F_U(\bar{\varepsilon}) & 0 \\ 0 & F_U(\bar{\varepsilon}) \end{bmatrix} \begin{bmatrix} \Psi_1(\bar{\varepsilon}) \\ \Psi_2(\bar{\varepsilon}) \end{bmatrix} = \hat{B}(\bar{\varepsilon}) \begin{bmatrix} \Psi_1(\bar{\varepsilon}) \\ \Psi_2(\bar{\varepsilon}) \end{bmatrix}, \tag{3.9}$$

$$\Phi_1(0) = \Phi_1, \quad \Phi_2(0) = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \quad \Psi_1(0) = \Psi_1, \quad \Psi_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$B(0) = \hat{B}(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{3.10}$$

(iv) there exists $\delta > 0$ such that for each $\bar{\varepsilon}$ in $|\bar{\varepsilon}| < \delta$ the restriction of the linear operator $F_U(\bar{\varepsilon})$ to the subspace $N(\bar{\varepsilon})$ spanned by $\Phi_1(\bar{\varepsilon})$ and $\Phi_2(\bar{\varepsilon})$ has two eigenvalues which are the same as those for $B(\bar{\varepsilon})$. Furthermore, H_1 can be decomposed into

$$H_1 = N(\bar{\varepsilon}) \oplus H_1(\bar{\varepsilon}), \tag{3.11}$$

where

$$H_1(\bar{\varepsilon}) = \{U \in H_1 : \langle \Psi_i(\bar{\varepsilon}), U \rangle = 0, i = 1,2\},$$

and the restriction of $F_U(\bar{\varepsilon})$ to $H_1(\bar{\varepsilon})$ has bounded inverse.

It follows from property (iv) above that in order to examine the rate at which $\| F_U^{-1}(\bar{\varepsilon}) \|$ tends to infinity as $\bar{\varepsilon}$ tends to zero it is enough to examine the behavior of the two eigenvalues of $B(\bar{\varepsilon})$. Those eigenvalues are studied in the following lemma.

3.1 Lemma: The two eigenvalues of $B(\bar{\varepsilon})$ have the form

$$\alpha = C\bar{\varepsilon}^{\gamma} + O((\bar{\varepsilon}^{\gamma})^2), \tag{3.12}$$

where C is constant and $\gamma = \frac{1}{2}$ or 1 according to whether condition (3.3) or (3.4) is satisfied, respectively.

Proof: The eigenvalues of $B(\bar{\varepsilon})$ are given by

$$F(\bar{\varepsilon}) = \begin{bmatrix} \frac{\bar{\varepsilon}^3}{6}\left(4G^0_{xx}\phi w_2 + 2G^0_{xx}w_2\phi + G^0_{xxx}\phi\phi\phi - \frac{6E_3}{a}G^0_{x\lambda}\phi\right) \\ 0 \end{bmatrix} + O(\bar{\varepsilon}^4).$$

It follows from these relations that

$$\langle \Psi_i(\bar{\varepsilon}), F(\bar{\varepsilon})\rangle = O(\bar{\varepsilon}^3), \quad i = 1,2,$$

for the non-degenerate case, and

$$\langle \Psi_i(\bar{\varepsilon}), F(\bar{\varepsilon})\rangle = O(\bar{\varepsilon}^4), \quad i = 1,2,$$

for the degenerate case. From (3.11), $F(\bar{\varepsilon})$ can be written (uniquely) as

$$F(\bar{\varepsilon}) = \delta_1\Phi_1(\bar{\varepsilon}) + \delta_2\Phi_2(\bar{\varepsilon}) + h_1(\bar{\varepsilon}),$$

for some constants δ_1, δ_2 and some $h_1(\bar{\varepsilon}) \in H_1(\bar{\varepsilon})$. Furthermore, the constants δ_1 and δ_2 are determined by the relations

$$\delta_1 = \langle \Psi_2(\bar{\varepsilon}), F(\bar{\varepsilon})\rangle,$$

$$\delta_2 = \langle \Psi_1(\bar{\varepsilon}), F(\bar{\varepsilon})\rangle.$$

These relations and the fact that $F_U(\bar{\varepsilon})$ has a bounded inverse on $H_1(\bar{\varepsilon})$ imply the estimate

$$\begin{aligned} \| F_U^{-1}(\bar{\varepsilon})F(\bar{\varepsilon})\| &\le \| F_U^{-1}(\bar{\varepsilon})(\delta_1\phi_1(\bar{\varepsilon}) + \delta_2\Phi_2(\bar{\varepsilon}))\| + \| F_U^{-1}(\bar{\varepsilon})h_1(\bar{\varepsilon})\| \\ &\le O(\bar{\varepsilon}^{2+\gamma}) + O(\bar{\varepsilon}^{1+\gamma}) \\ &= O(\bar{\varepsilon}^{1+\gamma}) \end{aligned}$$

It follows from Lemmas 3.2 and 3.3 that the functions $\eta(\bar{\varepsilon})$ and $\xi(\bar{\varepsilon})$ of Theorem 2.2 satisfy

$$\eta(\bar{\varepsilon}) = O(\bar{\varepsilon}^{1+\gamma}) \quad \text{and} \quad \xi(\bar{\varepsilon}) = O(\bar{\varepsilon}),$$

for both the nondegenerate and degenerate cases. This proves the following theorem.

3.4 Theorem. There exists $\delta > 0$ such that for λ in $0 < |\lambda - \lambda_0| < \delta$, both of Newton's and chord iterates with initial guess $U^0(\bar{\varepsilon})$ converge to a unique solution U^* of the equation

$$F(U,\lambda) = 0.$$

4. NUMERICAL EXAMPLE

In this section we illustrate the use of the numerical schemes developed in Section 3 by applying them to approximate the nontrivial solutions of the (finite dimensional) equation

$$G\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \lambda\right) = \begin{bmatrix} x_2^4 + x_1^3 + \frac{3}{2}x_1x_2^2 - \lambda x_1 \\ x_1^4 + x_2^3 + \frac{1}{2}x_1^2x_2 + (1-\lambda)x_2 \end{bmatrix} = 0, \tag{4.1}$$

in a neighborhood of the simple bifurcation point (0,0), for several prescribed values of the bifurcation parameter λ.

Using the same notations as in Section 3 we note that the inflated system corresponding to (4.1) is

$$F\left(\begin{pmatrix} w_1 \\ w_2 \\ \varepsilon \end{pmatrix}, \lambda\right) = \begin{bmatrix} w_2^4 + (w_1+\varepsilon)^3 + \frac{3}{2}(w_1+\varepsilon)w_2^2 - \lambda(w_1+\varepsilon) \\ (w_1+\varepsilon)^4 + w_2^3 + \frac{1}{2}(w_1+\varepsilon)^2w_2 + (1-\lambda)w_2 \\ w_1 \end{bmatrix} = 0, \tag{4.2}$$

and, for a given λ near 0, the initial guesses for the nontrivial solutions are given by

$$\begin{pmatrix} w_1^{(0)} \\ w_2^{(0)} \\ \varepsilon^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm\sqrt{\lambda} \end{pmatrix}, \tag{4.3}$$

for both Newton's and chord methods.

In Table 4.1 we present the numerical results obtained by applying Newton's and chord methods to (4.2) using the initial guesses (4.3). In this table $(w_1^{(i)}, w_2^{(i)}, \varepsilon^{(i)})$, $i = 1,2$, denote the two nontrivial solutions of (4.2) and N and C denote the number of iterations required for the convergence of Newton's and chord methods, respectively.

4.1 Remark. As expected, Table 4.1 shows that the number of iterations needed for the convergence of the chord method is larger than that for Newton's methods and that both of them increase as λ increases away from the bifurcation point.

4.2 Remark. Observe that the numerical schemes developed in Section 3 may be used to approximate the bifurcation solutions corresponding to different values of λ near the bifurcation point in parallel.

λ	$(w_1^{(1)}, w_2^{(1)}, \varepsilon^{(1)})$	$(w_1^{(2)}, w_2^{(2)}, \varepsilon^{(2)})$	N	C
.01	(0, −1.00(−4), 1.00(−1))	(0, −1.00(−4), −1.00(−1))	2	2
.05	(0, −2.56(−3), 2.24(−1))	(0, −2.56(−3), −2.24(−1))	3	3
.10	(0, −1.05(−2), 3.16(−1))	(0, −1.05(−2), −3.16(−1))	3	4
.15	(0, −2.40(−2), 3.86(−1))	(0, −2.40(−2), −3.86(−1))	4	5
.20	(0, −4.32(−2), 4.44(−1))	(0, −4.32(−2), −4.44(−1))	4	7
.25	(0, −6.75(−2), 4.93(−1))	(0, −6.75(−2), −4.93(−1))	4	9
.30	(0, −9.60(−2), 5.35(−1))	(0, −9.60(−2), −5.35(−1))	5	12
.35	(0, −1.28(−1), 5.70(−1))	(0, −1.28(−1), −5.70(−1))	5	19
.40	(0, −1.61(−1), 6.00(−1))	(0, −1.61(−1), −6.00(−1))	5	28

TABLE 4.1

ACKNOWLEDGEMENT. The author is grateful to Mrs. Sue Johnson for her conscientious and painstaking typing of the manuscript of this paper.

REFERENCES

1. CRANDALL, M. G. and RABINOWITZ, P. H., "Bifurcation from Simple Eigenvalues," *J. Funct. Anal.,* Vol. 8, 1971, 321-340.
2. KELLER, H. B. and LANGORD, W. F., "Iterations, Perturbations, and Multiplicities for Nonlinear Bifurcation Problems," *Arch. Rat. Mech. Anal.,* Vol. 48, 1972, 83-108.
3. CHOW, S. N. and HALE, J. K., *Methods of Bifurcation Theory,* Springer-Verlag, 1982.
4. ALLGOWER, E. L. and BOHMER, K., "Resolving Singular Nonlinear Equations," *Rocky Mountain J. of Math.,* 18, 1988, 225-268.
5. MEI, Z., "A Numerical Approximation for Simple Bifurcation Problems," *Numer. Func. Anal. and Optimiz.,* 10, 1989, 383-400.
6. DECKER, D. W. and KELLER, H. B., "Path Following Near Bifurcation," *Comm. on Pure and Applied Math.,* Vol. 34, 1981, 149-175.
7. MOORE, G., "The Application of Newton's Method to Simple Bifurcation and Turning Point problems," Ph.D. Thesis, University of Bath, 1979.