Internat. J. Math. & Math. Sci.
VOL. 17 NO. 2 (1994) 301-314

ON A GENERALIZATION OF HANKEL KERNEL

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(Received May 13, 1992 and in revised form January 14, 1993)

ABSTRACT. We consider an expression involving the Bessel function, the Neumann function and the MacDonald function and discover various combinations of these functions which are Fourier kernels or conjugate Fourier kernels. Also a large number of integration formulae are established involving these kernels.

KEY WORDS AND PHRASES. Fourier kernels, conjugate Fourier kernels, Mellin transforms, Bessel functions, Neumann functions, MacDonald functions, Elastic Plates
1992 AMS SUBJECT CLASSIFICATION CODES. 42A38

1. INTRODUCTION.

In a previous paper [1], we considered the manner in which Fourier Kernels may be generated as solutions of ordinary differential equations. We generated some previously known Fourier Kernels in this way and many others. One of the former ones was [2,9],

$$k(x) = \sqrt{x}\ [\sin \tfrac{\nu\pi}{2}\ J_\nu(x) + \cos \tfrac{\nu\pi}{2}\ (Y_\nu(x) + \tfrac{2}{\pi}\ K_\nu(x))]$$

involving the Bessel function J_ν, the Neumann function Y_ν, and the MacDonald function K_ν.

In this paper we follow a different line of thought. We inquire which expressions of the type

$$k(x) = \sqrt{x}\ [AJ_\nu(x) + BY_\nu(x) + CK_\nu(x)]$$

(for constant A, B, C) are Fourier kernels or have conjugate kernels of the same form. In this manner, we discover some new Fourier kernels and others which have a simple looking conjugate.

Also, we establish a large number of integration formulae, involving the function k(x) and its conjugate. Many of these formulae are believed to be unavailable in the literature. Throughout, we point out various known results as special cases of our general results.

2. PRELIMINARIES.

We shall mention below a few known results and definitions from the theory of Mellin transforms, which will be needed later. All of these results can be found in [3].
A function F(s), $s = c + it, -\infty < t < \infty, a < c < b$, is said to be the Mellin transform of f(x), if

$$\begin{aligned} F(s) &= \int_0^\infty f(x)\ x^{s-1}\ dx \\ &= \mathscr{M}\ [f(x);\ s]. \end{aligned}$$

And conversely, we call,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s}\, ds$$

$$= \mathscr{M}^{-1}[F(s);\, x]$$

the inverse Mellin transform of F(s).

An important result in the theory of Mellin transform, is the Parseval theorem:

If F(s) and K(s) are the Mellin transforms of the functions f(x) and k(x) respectively, then, under appropriate conditions.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)F(1-s)x^{-s}\, ds = \int_0^{\infty} k(xt)f(t)\, dt. \tag{2.1}$$

A direct consequence of the Parseval theorem is that if

$$K(s)F(1-s) = G(s), \tag{2.2}$$

where K(s), F(s) and G(s) denote the Mellin transforms of k(x), f(x) and g(x) respectively, then, in a suitable strip of the s–plane, we have

$$\int_0^{\infty} k(xt)f(t)\, dt = g(x). \tag{2.3}$$

Furthermore (2.3) implies (2.2), and we call g to be the k–transform of f. If further, the inversion formula

$$\int_0^{\infty} h(xt)g(t)\, dt = f(x), \tag{2.4}$$

involving the kernel h(x), holds, then k and h are said to be conjugate of each other. Also, their Mellin transforms satisfy the functional equation

$$K(s)H(1-s) = 1$$

in some strip of the s–plane,. If instead of (2.4), we have the inversion formula

$$\int_0^{\infty} k(xt)g(t)\, dt = f(x) \tag{2.5}$$

along with (2.3), then k is said to be self–conjugate or a Fourier kernel. Also its Mellin transform satisfies the equation

$$K(s)K(1-s) = 1. \tag{2.6}$$

Thus, if the equations (2.3) and (2.4) hold simultaneously, then we shall call k(x) and h(x), conjugate kernels. If, on the other hand, equations (2.3) and (2.5) hold, then k(x) is said to be a self–conjugate kernel.

Let k be a self–conjugate kernel. If for some suitable f,

$$\int_0^{\infty} k(xt)f(t)\, dt = \pm f(t),$$

then f is said to be an eigenfunction of the operator k, corresponding to the eigenvalue ±1 respectively. It should be noted that if the operator k is a Fourier–kernel, then it has only these two eigenvalues.

3. THE KERNELS.

We consider the function

$$k(x) = \sqrt{x}\,[AJ_\nu(x) + BY_\nu(x) + CK_\nu(x)],$$

where A, B and C are real constants. One can assign appropriate values to these constants so that k(x) is either self–conjugate or has a conjugate of the same type. Our first task will be to determine those values of A, B and C. The technique we shall employ to find those suitable values, consists of using results from Mellin transform theory.

The crucial part of our procedure is to express the function K(s), the Mellin transform of k(x), as a rational expression of Gamma functions.

Now, making use of the Mellin transform of the functions $\sqrt{x}J_\nu(x)$, $\sqrt{x}Y_\nu(x)$ and $\sqrt{x}K_\nu(x)$, [4], the Mellin transform of k(x) is then given by

$$\begin{aligned} K(s) &= \mathcal{M}[k(x);\ s] \\ &= \frac{1}{\pi} 2^{s-\frac{1}{2}} \Gamma(\tfrac{1}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}s)\Gamma(\tfrac{1}{4} - \tfrac{1}{2}\nu + \tfrac{1}{2}s)[A \sin\pi(\tfrac{1}{4} - \tfrac{1}{2}\nu + \tfrac{1}{2}s) \\ &\qquad - B\cos\pi(\tfrac{1}{4} - \tfrac{1}{2}\nu + \tfrac{1}{2}s) + C], \end{aligned}$$

where $|\nu| - \frac{1}{2} < \operatorname{Re} s < 1$. In order to consolidate the bracketed terms into a single term, an appropriate choice for the constants is that

$$A = \cos\theta\pi, \qquad B = \sin\theta\pi, \qquad C = \frac{2}{\pi}\sin\alpha\pi,$$

where θ and α are arbitrary. Then one can write after some simplification,

$$K(s) = \frac{1}{\pi} 2^{s-\frac{1}{2}} \Gamma(\tfrac{1}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}s)\Gamma(\tfrac{1}{4} - \tfrac{1}{2}\nu + \tfrac{1}{2}s)\sin\frac{\pi}{4}(\tfrac{1}{2} - \nu - 2\theta + 2\alpha + s).$$
$$\sin\frac{\pi}{4}(\tfrac{3}{2} + \nu + 2\theta + 2\alpha - s).$$

Now using the functional equation

$$\Gamma(z)\,\Gamma(1-z) = \pi\operatorname{cosec}\pi z$$

and the duplication formula for $\Gamma(2z)$, we obtain,

$$K(s) = 2^{2s-1}\frac{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{1}{8}-\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{5}{8}-\frac{1}{4}\nu-A_1+\frac{1}{4}s)\Gamma(\frac{3}{8}+\frac{1}{4}\nu+A_1-\frac{1}{4}s)\Gamma(\frac{1}{8}-\frac{1}{4}\nu-B_1+\frac{1}{4}s)}\cdot$$
$$\frac{\Gamma(\frac{5}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{7}{8}+\frac{1}{4}\nu+B_1-\frac{1}{4}s)} \tag{3.1}$$

here $|\nu| - \frac{1}{2} < \operatorname{Re} s < 1$, $A_1 = \frac{1}{2}(\theta+\alpha)$ and $B_1 = \frac{1}{2}(\theta-\alpha)$. Also, the corresponding form of k(x), using the above values for A, B and C, is then

$$\begin{aligned} k(x) &= \mathcal{M}^{-1}[K(s):\ x] \\ &= \sqrt{x}\,[\cos\theta\pi\, J_\nu(x) + \sin\theta\pi\, Y_\nu(x) + \frac{2}{\pi}\sin\alpha\pi\, K_\nu(x)] \end{aligned} \tag{3.2}$$

To determine the function h(x), the conjugate of k(x), we consider the functional equation

$$H(s)K(1-s) = 1$$

where H(s) and K(s) are the Mellin transforms of h(x) and k(x) respectively. Whence,

$$H(s) = 2^{2s-1}\frac{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+A_1+\frac{1}{4}s)\Gamma(\frac{7}{8}-\frac{1}{4}\nu-A_1-\frac{1}{4}s)}{\Gamma(\frac{3}{8}+\frac{1}{4}\nu-\frac{1}{4}s)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)}\cdot$$
$$\cdot\frac{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+B_1+\frac{1}{4}s)\Gamma(\frac{3}{8}-\frac{1}{4}\nu-B_1-\frac{1}{4}s)}{\Gamma(\frac{3}{8}-\frac{1}{4}\nu-\frac{1}{4}s)\Gamma(\frac{7}{8}-\frac{1}{4}\nu-\frac{1}{4}s)} \tag{3.3}$$

Now in a suitable strip of the s–plane, we have

$$h(x) = \mathcal{M}^{-1}[H(s);\ x],$$

which can be shown, by complex integration, to be the sum of two hypergeometric series, eventually giving us the conjugate of the function k(x).

In the next two sections, we shall explore situations giving rise to four special cases. These cases are of particular interest since they lead to a simpler representation of the conjugate function h(x). In some instances h(x) coincides with k(x), defining a self-conjugate kernel. We shall discuss self-conjugate kernels first.

4. SELF-CONJUGATE KERNELS.

Let $\theta = \frac{1}{2}(1-\nu)$ and $\alpha = \frac{1}{2}(1+\nu)$. Then $A_1 = \frac{1}{2}$, $B_1 = -\frac{1}{2}\nu$ and from (3.1) and (3.3), we have

$$K_1(s) = H_1(s) = 2^{2s-1}\frac{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{5}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)\Gamma(\frac{7}{8}-\frac{1}{4}\nu-\frac{1}{4}s)} \tag{4.1}$$

where $|\nu|-\frac{5}{2} < \text{Re } s < 1$, and

$$K_1(s)K_1(1-s) = 1.$$

Therefore, from (3.2)

$$k_1(x) = \sqrt{x}\,[\sin\tfrac{1}{2}\nu\pi J_\nu(x) + \cos\tfrac{1}{2}\nu\pi(Y_\nu(x) + \tfrac{2}{\pi}K_\nu(x))] \tag{4.2}$$

and it defines a self-conjugate kernel, i.e. $k_1(x)$ is a Fourier kernel, [1]. An interesting special case of the above kernel occurs when $\nu = \pm\frac{1}{2}$, when $k_1(x)$ becomes [2],

$$\frac{1}{\sqrt{\pi}}[\sin x - \cos x + e^{-x}].$$

Next we shall establish various integration formulae, involving the function $k_1(x)$. These formulae are derived as a result of suitable decomposition of the Mellin transform function $K_1(s)$. For instance let us define a function F, by

$$F(s) = 2^{-\frac{1}{2}+s}\frac{\Gamma(\frac{5}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)},$$

where $s = c + it, -\infty < t < \infty$ and $\frac{1}{2}-\nu < c < \frac{7}{2}-\nu$. Then from (4.1), we deduce that

$$K_1(s)F(1-s) = F(s). \tag{4.3}$$

Now since [4],

$$F(s) = \mathcal{M}[\frac{1}{\sqrt{\pi}}2^{\nu+1}x^{\frac{1}{2}-\nu}\sin(\tfrac{1}{2}x^2);\, s],$$

where $\nu-\frac{5}{2} < \text{Re } s < \nu+\frac{3}{2}$ then due to the result (2.3), the functional equation (4.3) implies that, on $\text{Re } s = \frac{1}{2}$,

$$\frac{1}{\sqrt{\pi}}2^{\nu+1}\int_0^\infty t^{\frac{1}{2}-\nu}\sin(\tfrac{1}{2}t^2)k_1(xt)\,dt = \frac{1}{\sqrt{\pi}}2^{\nu+1}x^{\frac{1}{2}-\nu}\sin(\tfrac{1}{2}x^2),\ |v| < 3 \tag{4.4}$$

Hence $\frac{1}{\sqrt{\pi}}2^{\nu+1}x^{\frac{1}{2}-\nu}\sin(\frac{1}{2}x^2)$ is an eigenfunction of the operator $k_1(x)$, corresponding to the eigenvalue 1. Letting $\nu = \frac{1}{2}$, gives the special case

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \sin(\tfrac{1}{2}t^2)(\sin xt - \cos xt + e^{-xt})dt = \sin(\tfrac{1}{2}x^2). \tag{4.5}$$

Letting $\nu = 1$ and 2 in (4.4), we obtain two more interesting special cases, which are

$$\int_0^\infty t^{-\frac{1}{2}}\sin(\tfrac{1}{2}t^2)\sqrt{xt}\,J_1(xt)\,dt = x^{-\frac{1}{2}}\sin(\tfrac{1}{2}x^2), \qquad [5, p. 19(16)], \tag{4.6}$$

and

$$-\int_0^\infty t^{-\frac{3}{2}}\sin(\tfrac{1}{2}\, t^2)\,\sqrt{xt}\,(Y_2(xt) + \tfrac{2}{\pi}\, K_2(xt))dt = x^{-\frac{3}{2}}\sin(\tfrac{1}{2}\, x^2). \tag{4.7}$$

Next, if we define F by

$$F(s) = 2^{-\frac{1}{2}+s}\,\Gamma(\tfrac{5}{8}+\tfrac{1}{4}\,\nu+\tfrac{1}{4}\,s)\Gamma(\tfrac{5}{8}-\tfrac{1}{4}\,\nu+\tfrac{1}{4}\,s),$$

where Re s $> -\frac{5}{2} + |\nu|$, then from (4.1), we have

$$K_1(s)F(1-S) = F(s), \quad |\nu| - \tfrac{5}{2} < \text{Re } s < 1. \tag{4.8}$$

And since,

$$F(s) = \mathscr{M}[x^{\frac{5}{2}}\, K_{\frac{\nu}{2}}\,(\tfrac{1}{2}\, x^2);\ s]\ ,$$

then (4.8), due to (2.1), implies

$$\int_0^\infty t^{\frac{5}{2}}K_{\frac{\nu}{2}}\,(\tfrac{1}{2}\, t^2)k_1(xt)\ dt = x^{\frac{5}{2}}\, K_{\frac{\nu}{2}}\,(\tfrac{1}{2}\, x^2),\ |\nu| < 3, \tag{4.9}$$

giving another eigenfunction $x^{\frac{5}{2}}K_{\frac{\nu}{2}}\,(\frac{1}{2}\, x^2)$ of the operator $k_1(x)$.

Again letting $\nu = 1$ and 2, we obtain special cases of (4.9), which are respectively

$$\int_0^\infty t^2\, e^{-\frac{1}{2}t^2}\, J_1\,(xt)dt = x\, e^{-\frac{1}{2}x^2}, \qquad [5,\ p.\ 19(8)], \tag{4.10}$$

and

$$-\int_0^\infty t^3\, K_1\,(\tfrac{1}{2}\, t^2)\,(Y_2(xt) + \tfrac{2}{\pi}\, K_2(xt))dt = x^2\, K_1\,(\tfrac{1}{2}\, x^2). \tag{4.11}$$

In general, in order that f(x) should be an eigenfunction of the operator k_1 corresponding to the eigenvalue 1, F(s), $s = \sigma + i\tau$, the Mellin transform of f should be of the form

$$F(s) = 2^{s-\frac{1}{2}}\,\Gamma(\tfrac{5}{8}+\tfrac{\nu}{4}+\tfrac{s}{4})\ \Gamma(\tfrac{5}{8}-\tfrac{\nu}{4}+\tfrac{s}{4})\ \Psi(s),$$

where $\Psi(s) = \Psi(1-s)$. The eigenfunctions mentioned above in (4.4) and (4.9) are special cases when

$$\Psi(s) = \frac{1}{\Gamma(\frac{7}{8}+\frac{\nu}{4}-\frac{s}{4})\ \Gamma(\frac{5}{8}+\frac{\nu}{4}+\frac{s}{4})}$$

and when

$$\Psi(s) = 1$$

respectively.

Now we define functions F and G, by

$$F(s) = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{5}{8}-\frac{1}{4}\ \nu+\frac{1}{4}\ s)}{\Gamma(\frac{7}{8}-\frac{1}{4}\ \nu-\frac{1}{4}\ s)}$$

and

$$G(s) = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{5}{8}+\frac{1}{4}\ \nu+\frac{1}{4}\ s)}{\Gamma(\frac{7}{8}+\frac{1}{4}\ \nu-\frac{1}{4}\ s)}$$

Then from (4.1), we have the functional equation

$$K_1(s)F(1-s) = G(s). \tag{4.12}$$

Since,

$$F(s) = \mathscr{M}[x^{\frac{3}{2}} J_{\frac{1}{2}-\frac{\nu}{2}}(\tfrac{1}{2}x^2); s], \; \nu - \tfrac{5}{2} < \text{Re } s < \tfrac{3}{2}$$

and

$$G(s) = \mathscr{M}[x^{\frac{3}{2}} J_{\frac{1}{2}+\frac{\nu}{2}}(\tfrac{1}{2}x^2); s], \; -\nu - \tfrac{5}{2} < \text{Re } s < \tfrac{3}{2},$$

hence due to the result (2.1), (4.12) implies the equation

$$\int_0^\infty t^{\frac{3}{2}} J_{\frac{1}{2}-\frac{\nu}{2}}(\tfrac{1}{2}t^2)k_1(xt)dt = x^{\frac{3}{2}} J_{\frac{1}{2}+\frac{\nu}{2}}(\tfrac{1}{2}x^2), \; |\nu| < 3. \tag{4.13}$$

Furthermore, k_1 is self-conjugate, therefore, the inversion formula gives

$$\int_0^\infty t^{\frac{3}{2}} J_{\frac{1}{2}+\frac{\nu}{2}}(\tfrac{1}{2}t^2)k_1(xt)dt = x^{\frac{3}{2}} J_{\frac{1}{2}-\frac{\nu}{2}}(\tfrac{1}{2}x^2). \tag{4.14}$$

This establishes the pair $x^{\frac{3}{2}} J_{\frac{1}{2}-\frac{\nu}{2}}(\frac{1}{2}x^2)$, $x^{\frac{3}{2}} J_{\frac{1}{2}+\frac{\nu}{2}}(\frac{1}{2}x^2)$ as k_1-transforms of each other.

Some special cases of (4.13), when $\nu = 0$, 1 and 2, are respectively

$$\int_0^\infty t \sin(\tfrac{1}{2}t^2)\,(Y_0(xt) + \tfrac{2}{\pi}K_0(xt))dt = \sin(\tfrac{1}{2}x^2) \tag{4.15}$$

$$\int_0^\infty t^2 J_0(\tfrac{1}{2}t^2)\,J_1(xt)dt = xJ_1(\tfrac{1}{2}x^2), \quad [6, \text{ p. } 215(3)], \tag{4.16}$$

and

$$\int_0^\infty t\cos(\tfrac{1}{2}t^2)\,(Y_2(xt) + \tfrac{2}{\pi}K_2(xt))dt = \cos(\tfrac{1}{2}x^2) - (\tfrac{1}{2}x^2)^{-1}\sin(\tfrac{1}{2}x^2). \tag{4.17}$$

Next, if we put

$$\theta = -\tfrac{1}{2}(1+\nu) \text{ and } \alpha = \tfrac{1}{2}(1+\nu)$$

in (3.1) and (3.3), then

$$K_2(s) = H_2(s) = 2^{2s-1}\frac{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{1}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{3}{8}+\frac{1}{4}\nu-\frac{1}{4}s)\Gamma(\frac{3}{8}-\frac{1}{4}\nu-\frac{1}{4}s)}$$

satisfying the equation

$$K_2(s)K_2(1-s) = 1, \quad |\nu| - \tfrac{1}{2} < \text{Re } s < 1.$$

Also from (3.2), we have

$$k_2(x) = \mathscr{M}^{-1}[K_2(s); x] = -\sqrt{x}\,[\sin\tfrac{1}{2}\nu\pi\, J_\nu(x) + \cos\tfrac{1}{2}\nu\pi(Y_\nu(x) - \tfrac{2}{\pi}K_\nu(x))], \tag{4.18}$$

and it defines a self-conjugate kernel. Note that if $\nu = \frac{1}{2}$, then we obtain

$$k_2(x) = \frac{1}{\sqrt{\pi}}(\cos x - \sin x + e^{-x}),$$

an interesting special case [2].

Various integration formulae, involving the function $k_2(x)$, are given below, again as a result of different decompositions of the function $K_2(s)$. First, we define F by

$$F(s) = 2^{-\frac{1}{2}+s} \frac{\Gamma(\frac{1}{8} - \frac{1}{4}\nu + \frac{1}{4}s)}{\Gamma(\frac{3}{8} + \frac{1}{4}\nu - \frac{1}{4}s)},$$

Then

$$K_2(s)F(1-s) = F(s). \tag{4.19}$$

Now,

$$F(s) = \mathcal{M}[\frac{2}{\sqrt{\pi}} x^{\frac{1}{2}-\nu} \cos(\tfrac{1}{2}x^2);s], \; \nu - \tfrac{1}{2} < \text{Re } s < \nu + \tfrac{3}{2}.$$

Then due to (2.1), the functional equation (4.19), on Re $s = \frac{1}{2}$, implies that

$$\int_0^\infty t^{\frac{1}{2}-\nu} \cos(\tfrac{1}{2}t^2) k_2(xt)dt = x^{\frac{1}{2}-\nu} \cos(\tfrac{1}{2}x^2), \; |\nu| < 1. \tag{4.20}$$

i.e. $x^{\frac{1}{2}-\nu} \cos(\frac{1}{2}x^2)$ is an eigenfunction of the operator $k_2(x)$, defined by (4.18).
A special case can be derived from (4.20) by setting $\nu = \frac{1}{2}$,

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \cos(t^2/2)(\cos xt - \sin xt + e^{-xt})dt = \cos(x^2/2). \tag{4.21}$$

It is interesting to compare this result with (4.5).
Another eigenfunction of the kernel $k_2(x)$, can be obtained by letting

$$F(s) = 2^{-\frac{1}{2}+s} \Gamma(\tfrac{1}{8}+\tfrac{1}{4}\nu+\tfrac{1}{4}s) \, \Gamma(\tfrac{1}{8}-\tfrac{1}{4}\nu+\tfrac{1}{4}s)$$

then,

$$K_2(s)F(1-s) = F(s)$$

where

$$F(s) = \mathcal{M}[x^{\frac{1}{2}} K_{\frac{\nu}{2}}(\tfrac{1}{2}x^2);s], \; \text{Re } s > |\nu| - \tfrac{1}{2},$$

implies

$$\int_0^\infty t^{\frac{1}{2}} K_{\frac{\nu}{2}}(\tfrac{1}{2}t^2) k_2(xt)dt = x^{\frac{1}{2}} K_{\frac{\nu}{2}}(\tfrac{1}{2}x^2), \; |\nu| < 1. \tag{4.22}$$

Letting $\nu = 0$ and $\frac{1}{2}$, (4.22) reduces to, respectively,

$$\int_0^\infty t \, K_0(\tfrac{1}{2}t^2) \, (Y_0(xt) - \tfrac{2}{\pi} K_0(xt))dt = - K_0(\tfrac{1}{2}x^2) \tag{4.23}$$

and

$$\frac{1}{\sqrt{\pi}} \int_0^\infty t^{\frac{1}{2}} K_{\frac{1}{4}}(\tfrac{1}{2}t^2) \, (\cos xt - \sin xt - e^{-xt})dt = x^{\frac{1}{2}} K_{\frac{1}{4}}(\tfrac{1}{2}x^2). \tag{4.24}$$

As before, the eigenfunctions f of the operator k_2, can be characterized, by expressing its

Mellin transform as

$$F(s) = \Gamma(\tfrac{1}{8}+\tfrac{\nu}{4}+\tfrac{s}{4})\,\Gamma(\tfrac{1}{8}-\tfrac{\nu}{4}+\tfrac{s}{4})\,\Psi(s),$$

where $\Psi(s) = \Psi(1-s)$. Letting

$$\Psi(s) = 1$$

and

$$\Psi(s) = \frac{1}{\Gamma(\tfrac{3}{8}+\tfrac{\nu}{4}-\tfrac{s}{4})\,\Gamma(\tfrac{3}{8}-\tfrac{\nu}{4}-\tfrac{s}{4})},$$

give us the eigenfunctions mentioned above in (4.20) and (4.22) respectively. Finally, we define functions F and G, by

$$F(s) = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{3}{8}-\frac{1}{4}\nu-\frac{1}{4}s)}$$

and

$$G(s) = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{3}{8}+\frac{1}{4}\nu-\frac{1}{4}s)}$$

then

$$K_2(s)F(1-s) = G(s) \tag{4.25}$$

where

$$F(s) = \mathscr{M}[x^{\frac{3}{2}} J_{-\frac{1}{2}-\frac{\nu}{2}}(\tfrac{1}{2}x^2);\, s],\ \nu-\tfrac{1}{2} < \text{Re } s < \tfrac{3}{2}$$

and

$$G(s) = \mathscr{M}[x^{\frac{3}{2}} J_{-\frac{1}{2}+\frac{\nu}{2}}(\tfrac{1}{2}x^2);\, s],\ -\nu-\tfrac{1}{2} < \text{Re } s < \tfrac{3}{2}.$$

Hence from (4.25), we obtain, on Re $s = \frac{1}{2}$,

$$\int_0^\infty t^{\frac{3}{2}} J_{-\frac{1}{2}-\frac{\nu}{2}}(\tfrac{1}{2}t^2)k_2(xt)dt = x^{\frac{3}{2}} J_{-\frac{1}{2}+\frac{\nu}{2}}(\tfrac{1}{2}x^2),\ \nu < 1 \tag{4.26}$$

and conversely,

$$\int_0^\infty t^{\frac{3}{2}} J_{-\frac{1}{2}+\frac{\nu}{2}}(\tfrac{1}{2}t^2)k_2(xt)dt = x^{\frac{3}{2}} J_{-\frac{1}{2}-\frac{\nu}{2}}(\tfrac{1}{2}x^2),\ \nu > -1. \tag{4.27}$$

Putting $\nu = 0$ and $\frac{1}{2}$, in (4.26), we obtain respectively,

$$\int_0^\infty t\cos(\tfrac{1}{2}t^2)\,(Y_0(xt) - \tfrac{2}{\pi}K_0(xt))dt = -\cos(\tfrac{1}{2}x^2) \tag{4.28}$$

and

$$\frac{1}{\sqrt{\pi}}\int_0^\infty t^{\frac{3}{2}} J_{-\frac{3}{4}}(\tfrac{1}{2}t^2)\,(\cos xt - \sin xt + e^{-xt})dt = x^{\frac{3}{2}} J_{-\frac{1}{4}}(\tfrac{1}{2}x^2). \tag{4.29}$$

5. CONJUGATE KERNELS.

If we now put $\theta = -\frac{1}{2}\nu$ and $\alpha = 1 + \frac{1}{2}\nu$ in the equations (3.1) and (3.3), we obtain

$$K_3(s) = 2^{2s-1}\frac{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{5}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)\Gamma(\frac{3}{8}-\frac{1}{4}\nu-\frac{1}{4}s)}$$

and

$$H_3(s) = 2^{2s-1}\frac{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\Gamma(\frac{1}{8}-\frac{1}{4}\nu+\frac{1}{4}s)}{\Gamma(\frac{3}{8}+\frac{1}{4}\nu-\frac{1}{4}s)\Gamma(\frac{7}{8}-\frac{1}{4}\nu-\frac{1}{4}s)}$$

so that

$$H_3(s)K_3(1-s) = 1.$$

Then, from (3.2),

$$k_3(x) = \sqrt{x}\,[\cos\tfrac{1}{2}\nu\pi\, J_\nu(x) - \sin\tfrac{1}{2}\nu\pi\,(Y_\nu(x) + \tfrac{2}{\pi}K_\nu(x))], \tag{5.1}$$

where $|\nu| - \frac{1}{2} < \text{Re } s < 1$. It is now an easy matter to evaluate $h_3(x)$, which is the conjugate of $k_3(x)$, and is given by

$$\begin{aligned} h_3(x) &= \mathcal{M}^{-1}[H_3(s):x] \\ &= \sqrt{x}\,[\cos\tfrac{1}{2}\nu\pi J_\nu(x) - \sin\tfrac{1}{2}\nu\pi(Y_\nu(x) - \tfrac{2}{\pi}K_\nu(x))], \end{aligned} \tag{5.2}$$

for $|\nu| - \frac{1}{2} < \text{Re } s < 1$. Thus we have established a pair of conjugate kernels $k_3(x)$ and $h_3(x)$. As a special case when $\nu = \pm\frac{1}{2}$, we have pairs of conjugate kernels,

$$\frac{1}{\sqrt{\pi}}(\cos x + \sin x \mp e^{-x}) \tag{5.3}$$

By employing the technique of the previous sections, we arrive at the following integration formulae involving the kernel $k_3(x)$. Integrals involving $h_3(x)$ can easily be obtained by the inversion formulae (2.4). We believe that these are all new results.

$$\int_0^\infty \sqrt{t}\, J_{-\frac{\nu}{2}}(\tfrac{1}{2}t^2)k_3(xt)dt = \sqrt{x}\, J_{\frac{\nu}{2}}(\tfrac{1}{2}x^2), \qquad \nu \le 2 \tag{5.4}$$

$$\int_0^\infty \sqrt{t}\, J_{\frac{\nu}{2}}(\tfrac{1}{2}t^2)\, h_3(xt)dt = \sqrt{x}\, J_{-\frac{\nu}{2}}(\tfrac{1}{2}x^2), \qquad \nu \ge -2 \tag{5.5}$$

If $\nu = 2n$, $n = 0, 1, 2,\ldots$, then [5, p. 56(1)], the equation (5.5) gives,

$$\int_0^\infty t\, J_n(\tfrac{1}{2}t^2)\, J_{2n}(xt)dt = J_n(\tfrac{1}{2}x^2). \tag{5.6}$$

If $\nu = 1$, then from (5.4) and (5.5), we have respectively,

$$-\int_0^\infty \cos(\tfrac{1}{2}t^2)\,(Y_1(xt) + \tfrac{2}{\pi}K_1(xt))dt = \tfrac{1}{x}\sin(\tfrac{1}{2}x^2), \tag{5.7}$$

and

$$-\int_0^\infty \sin(\tfrac{1}{2}t^2)\,(Y_1(xt) - \tfrac{2}{\pi}K_1(xt))dt = \tfrac{1}{x}\cos(\tfrac{1}{2}x^2). \tag{5.8}$$

Also,
$$\int_0^\infty t^{\frac{1}{2}-\nu}\cos(\tfrac{1}{2}t^2)k_3(xt)dt = x^{\frac{1}{2}-\nu}\sin(\tfrac{1}{2}x^2),\ -\tfrac{1}{2} < \nu < 2. \tag{5.9}$$

If $\nu = 0$, (5.9) gives, [5, p. 38(40)],

$$\int_0^\infty t\cos(\tfrac{1}{2}t^2)\,J_0(xt)dt = \sin(\tfrac{1}{2}x^2) \tag{5.10}$$

It is also easy to establish that

$$\int_0^\infty t^{\frac{3}{2}}K_{\frac{1}{2}+\frac{1}{2}\nu}(\tfrac{1}{2}t^2)k_3(xt)dt = x^{\frac{3}{2}}K_{\frac{1}{2}+\frac{1}{2}\nu}(\tfrac{1}{2}x^2),\ |\nu| < 2. \tag{5.11}$$

Letting $\nu = 0$, we have, [5, p. 29(10)],

$$\int_0^\infty te^{-\frac{1}{2}t^2}J_0(xt)dt = e^{-\frac{1}{2}x^2} \tag{5.12}$$

Our last pair of conjugate kernels is obtained if we set $\alpha = 0$ in (3.1) and (3.3). Then for $|\nu| - \frac{1}{2} < \text{Re } s < 1$,

$$K_4(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu+\theta-\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}\nu-\theta+\frac{1}{2}s)},$$

and for $0 < \text{Re } s < 1$, $|\nu + 2\theta| < \frac{3}{2}$,

$$H_4(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\theta+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}\nu-\theta-\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}\nu-\frac{1}{2}s)}$$

so that

$$H_4(s)K_4(1-s) = 1.$$

Thus, in an appropriate strip of the s–plane, from (3.2),

$$\begin{aligned} k_4(x) &= \mathcal{M}^{-1}[K_4(s):x] \\ &= \sqrt{x}\,[\cos\theta\pi\,J_\nu(x) + \sin\theta\pi\,Y_\nu(x)]. \end{aligned} \tag{5.13}$$

Also, using complex integration, one can find that [4p. 353(43)],

$$\begin{aligned} h_4(x) &= \mathcal{M}^{-1}[H_4(s):x] \\ &= \sqrt{2}\,G^{11}_{13}\left[\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} a_1 \\ b_1, b_2, b_3 \end{matrix}\right], \end{aligned} \tag{5.14}$$

where $a_1 = b_1 = \frac{1}{4} + \frac{1}{2}\nu + \theta$, $b_2 = \frac{1}{4} - \frac{1}{2}\nu$, $b_3 = \frac{1}{4} + \frac{1}{2}\nu$, and G being the Meijer's G–function.

Note that alternatively, [4,p. 379],

$$h_4(x) = \frac{\sqrt{x}}{\Gamma(1+\theta)\Gamma(1+\nu+\theta)}\left[\tfrac{1}{2}x\right]^{\nu+2\theta}{}_1F_2(1;1+\theta,\ 1+\nu+\theta;\ -\tfrac{1}{4}x^2) \tag{5.15}$$

It is now easy to see that if $\theta = \frac{1}{2}$, then we obtain

$$k_4(x) = \sqrt{x}\,Y_\nu(x) \tag{5.16}$$

and [4, p.380]

$$h_4(x) = \sqrt{x}\, H_\nu(x), \tag{5.17}$$

a pair of well-known conjugate kernels, H_ν being Struve's function, [3, p. 215(2)].

Finally we shall list a few integration formulae involving the operator $k_4(x)$. Integrals involving $h_4(x)$, can be written by the usual inversion formulae of the type (2.4).

$$\int_0^\infty t^{\frac{1}{2}+\theta} J_{\nu+\theta}(t) k_4(xt)dt = \frac{2^{1+\theta}}{\Gamma(-\theta)} (1-x^2)^{-1-\theta} x^{\frac{1}{2}-\nu} H(1-x), \ \nu+\theta+1>0 \tag{5.18}$$

where $-1 \leq \theta < 0$ and H is the Heaviside function.
If $\theta = -\frac{1}{2}$, then [6, p. 272(4)],

$$\int_0^\infty \sqrt{xt}\, J_{\nu-\frac{1}{2}}(t)\, Y_\nu(xt)dt = -\sqrt{\frac{2}{\pi}}\, x^{\frac{1}{2}-\nu}(1-x^2)^{-\frac{1}{2}} H(1-x). \tag{5.19}$$

Also,

$$\int_0^\infty (t)^{\frac{1}{2}+\nu+\theta} J_\theta(t) k_4(xt)dt = \frac{2^{1+\nu+\theta}}{\Gamma(-\nu-\theta)} x^{\frac{1}{2}+\nu} (1-x^2)^{-(1+\nu+\theta)} H(1-x), \ \nu+\theta<0, \ \theta \geq -1. \tag{5.20}$$

Let $\theta = -1$, we get [5, p. 48(7)],

$$\int_0^\infty t^\nu J_1(t)\, J_\nu(xt)dt = \frac{2^\nu}{\Gamma(1-\nu)} x^\nu (1-x^2)^{-\nu} H(1-x) \tag{5.21}$$

We also have,

$$\frac{2^{1-\theta}}{\Gamma(\theta)} \int_1^\infty t^{\frac{1}{2}+\nu} (t^2-1)^{\theta-1} k_4(xt)dt = x^{\frac{1}{2}-\theta} J_{\nu+\theta}(x), \ 0 < \theta < \frac{3}{4}-\frac{1}{2}\nu. \tag{5.22}$$

If $\theta = \frac{1}{2}$, then [5, p. 102(29)],

$$\sqrt{\frac{2}{\pi}} \int_1^\infty t^{1+\nu} (t^2-1)^{-\frac{1}{2}} Y_\nu(xt)dt = x^{-\frac{1}{2}} J_{\nu+\frac{1}{2}}(x) \tag{5.23}$$

Finally,

$$\int_0^\infty \frac{t^{\frac{1}{2}+\nu+2\theta}}{a^2+t^2} k_4(xt)dt = a^{\nu+2\theta}\sqrt{x}\, K_\nu(ax), \ -1 < \theta < \frac{3}{4}-\frac{1}{2}\nu, \tag{5.24}$$

and generally, [8, p.424(2)],

$$\int_0^\infty \frac{t^{\frac{1}{2}+\nu+2\theta}}{(a^2+t^2)^{m+1}} k_4(xt)dt = \frac{(-1)^m}{m!2^m} \sqrt{x} \left[\frac{1}{a}\frac{d}{da}\right]^m \left[a^{\nu+2\theta} K_\nu(ax)\right] \tag{5.25}$$

Now letting $\theta = 0$ and $\frac{1}{2}$, (5.24) yields respectively, [5, p. 23(12)], [5, p.99(15)],

$$\int_0^\infty \frac{t^{\nu+1}}{a^2+t^2} J_\nu(xt)dt = a^\nu K_\nu(ax) \tag{5.26}$$

$$\int_0^\infty \frac{t^{\nu+2}}{a^2+t^2} Y_\nu(xt)dt = a^{\nu+1}K_\nu(ax). \tag{5.27}$$

We wish to note that, to our knowledge, all the results for which we have not given references from the literature, appear to be new. Our method, therefore, has yielded a large number of new integration formulae.

6. SOME APPLICATIONS

Since the kernels in this paper are also solutions of a Fourth Order ordinary differential equation [1], it is expected that our results will find applications in situations which involve such differential equations. One such situation was encountered in [1]. We point out some more below.

If we consider the problem of finding solutions of

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0 \text{ in } 0 < x < \infty, \quad t > 0 \tag{6.1}$$

or of

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]^2 u + \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{in} \quad 0 < r < \infty, \quad t > 0 \tag{6.2}$$

which solutions are bounded at infinity, and which satisfy the conditions

$$(1) \qquad u = \frac{\partial u}{\partial x} = 0 \text{ (or } u = \frac{\partial u}{\partial r} = 0\text{) at } x=0 \text{ (or at } r=0) \tag{6.3}$$

or the conditions

$$(2) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0 \text{ (or } \nabla^2 u = \frac{\partial}{\partial r}\nabla^2 u = 0\text{) at } x=0 \text{(or at } r=0) \tag{6.4}$$

respectively, where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$,

then we encounter the kernels introduced in this paper. If these solutions are subject to the initial conditions

$$u = g_1(\xi) \tag{6.5a}$$

and

$$\frac{\partial u}{\partial t} = g_2(\xi) \tag{6.5b}$$

at $t=0$ in $0 < \xi < \infty$, where ξ is either x or r depending upon whether we are dealing with equation (6.1) or equation (6.2), then the solution is

$$u(\xi,t) = \int_0^\infty k(\lambda\xi)[A(\lambda)\cos(\lambda^2 t) + \frac{B(\lambda)}{\lambda}\sin(\lambda^2 t)]d\lambda \tag{6.6}$$

where

$$g_1(\xi) = \int_0^\infty A(\lambda)k(\lambda\xi)d\lambda \tag{6.7}$$

and

$$g_2(\xi) = \int_0^\infty \lambda B(\lambda)k(\lambda\xi)d\lambda \tag{6.8}$$

where k is an appropriate kernel. If k is self–conjugate then the solution of equations (6.7) and (6.8) is

$$A(\lambda) = \int_0^\infty g_1\,(\xi)k(\lambda\xi)d\xi \tag{6.9}$$

$$B(\lambda) = \frac{1}{\lambda}\int_0^\infty g_2(\xi)k(\lambda\xi)d\xi \tag{6.10}$$

and substitution in equation (6.6) gives u. The following cases should be noted:

(1) If $\nu=\frac{1}{2}$ and the conditions are $u = \frac{\partial u}{\partial x} = 0$ at the origin, then equation (6.6) gives the deflection of a vibrating semi-infinite elastic rod which is clamped at one end (the origin) and is subject to the initial conditions (6.5). In this case $k = k_1(x) = \frac{1}{\sqrt{\pi}}(\sin x - \cos x + e^{-x})$ which is self-conjugate.

(2) If $\nu = \frac{1}{2}$ and the conditions are $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0$ at the origin, then equation (6.6) gives the deflection of a vibrating semi-infinite elastic rod which is free at one end (the origin) and is subject to the initial conditions (6.5). In this case $k = k_2(x) = \frac{1}{\sqrt{\pi}}(\cos x - \sin x + e^{-x})$ which is self-conjugate.

(3) If $\nu=0$ and the conditions are $u = \frac{\partial u}{\partial r} = 0$ at $r=0$, then equation (6.6) gives the deflection of a (symmetrically) vibrating infinite elastic plate which is clamped at the origin and is subject to the initial conditions (6.5).
In this case

$$k = \frac{k_1(r)}{\sqrt{r}} = Y_0(r) + \frac{2}{\pi}K_0(r) \tag{6.11}$$

so that equations (6.7) and (6.8) become

$$\sqrt{\xi}\,g_1(\xi) = \int_0^\infty \frac{A(\lambda)}{\sqrt{\lambda}}\,k_1(\lambda\xi)d\lambda \tag{6.12}$$

and

$$\sqrt{\xi}\,g_2(\xi) = \int_0^\infty \sqrt{\lambda}\,B(\lambda)k_1(\lambda\xi)d\lambda. \tag{6.13}$$

Since k_1 is self conjugate, these equations are easily inverted and then substitution gives u.

It is interesting to note that in case 3, in the case of a vibrating infinite plate clamped at the origin, the vertical force exerted by the clamp on the plate is given by

$$\lim_{r\to 0}\int_0^{2\pi} -\frac{\partial}{\partial r}(\nabla^2 u)r d\theta = 8\int_0^\infty \lambda^2[A(\lambda)\cos(\lambda^2 t) + \frac{B(\lambda)}{\lambda}\sin(\lambda^2 t)]d\lambda. \tag{6.14}$$

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