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ON SOME REGULAR AND SINGULAR PROBLEMS OF BIRKHOFF INTERPOLATION

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ABSTRACT. Here we investigate the pure $(0,1,\cdots,r-2,r)$-interpolation problem on the zeros of $(1-x^2)\ P_n^{(\alpha)}\ (x)=(1-x^2)\ P_n^{(\alpha,\alpha)}\ (x)$, $\alpha>-1$, where $P_n^{(\alpha,\alpha)}\ (x)$ is the Jacobi polynomial of degree n with $\beta=\alpha$.

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1. INTRODUCTION.

Let k and l be natural numbers and let $E=E_\ell^k=(\epsilon_{ij})$ $(i=1,2,\cdots,k;j=0,1,\cdots,l-1)$ be a matrix with k rows and l $(l\geq k)$ columns having $\epsilon_{ij}=0$ or 1, which are such that $\sum\limits_{i,j}\epsilon_{ij}=1$ and no row is entirely composed of zeros. Let

$$x_1<x_2<\cdots<x_k \tag{1.1}$$

be increasing reals and $e_l^k=\{(i,j):\epsilon_{ij}=1\}$. The reals x_i and the incidence matrix E describe the interpolation problem

$$P^{(j)}(x_i)=y_i^{(j)},\ \text{for}\ (i,j)\ \epsilon\ e_\ell^k \tag{1.2}$$

where $y_i^{(j)}$ are prescribed and the problem is to find the polynomial $P(x)$ of degree $\leq l-1$, which satisfies the condition (1.2). If $y_i^{(j)}=0$ for $(i,j)\ \epsilon\ e_\ell^k$ then the problem (1.2) is the homogeneous interpolation problem. Let $X=\{x_i\}_1^k$ be the interpolation nodes. We say that (E,X) is regular if (1.2) has a unique solution for all choices of reals $y_i^{(j)}$, and singular otherwise. If $P^{(j)}(x_i)=0$ for $(i,j)\ \epsilon\ e_l^k$, then $P(x)$ is said to be annihilated by (E,X).

Turán and his associate [4] considered $E=E_{2n}^n$ with $x_1,x_2,\cdots,x_n$ as the zeros of $\pi_n(x)=(1-x^2)P'_{n-1}(x)$, where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1)=1$. Turán proved that (E,X) is regular if n is even and singular if n is odd. Later, Varma ([5], [6]); Anderson and Prasad [1]; and Prasad and Anderson [3] considered different incidence matrices. Recently, Bajpai and Saxena [2] proved the following:

THEOREM A. If E is the matrix of order $(n+2)\times(m+1)(n+2)$, $m\geq 2$, with rows $(\underbrace{1\ 1\ \cdots\ 1}_{m}\ 0\ 1\ 0\ \cdots\ 0\ )$ and X is the set of zeros of $(1-x^2)P_n(x)$, $P_n(x)$ being the Legendre polynomial of degree n, then:

(i) if m is even, (E,X) is singular, and

(ii) if m is odd, (E,X) is regular if n is even and singular if n is odd.

Let X be the set of the zeros $\{x_k\}_0^{n+1}$ of $(1-x^2)\ P_n^{(\alpha)}\ (x) = (1-x^2)\ P_n^{(\alpha,\alpha)}\ (x)$, $\alpha > -1$, where $P_n^{(\alpha,\alpha)}(x)$ is the Jacobi polynomial of degree n with $\beta = \alpha$, such that

$$-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1.$$

Our aim here is to prove the following:

THEOREM 1. Let X be the set of the zeros of $(1-x^2)P_n^{(\alpha)}(x), \alpha > -1$, and E be the incidence matrix given by

$$E = E_{(n+2)\times(m+1)(n+2)}^{n+2} = \begin{pmatrix} (1)_m & 0 & 1 & 0 & \cdot & \cdot & 0 \\ (1)_m & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (1)_m & 0 & 1 & 0 & \cdot & \cdot & 0 \end{pmatrix} \tag{1.3}$$

where $(1)_m$ means m entries of 1 in that row. Let m be an odd positive integer ≥ 3, and $-1 < \alpha < 1$, then:

(i) if n is odd then (E,X) is singular.

(ii) if n is even, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then (E,X) is singular and for all other values of $m-1-\alpha(m+2)$, (E,X) is regular.

(iii) if n is even and $\alpha = \frac{m-2}{m+2}$, then (E,X) is singular.

THEOREM 2. Let X be the set of the zeros of $(1-x^2)P_n^{(\alpha)}(x), \alpha > -1$, and E be the incidence matrix given by (1.1). Let m be an even positive integer ≥ 2, and $-1 < \alpha < 1$, then:

(i) If n is odd then (E,X) is singular.

(ii) If n is even and $\alpha = \frac{m-2}{m+2}, (0 \leq \alpha < 1)$, then (E,X) is singular.

(iii) If n is even and $\alpha \neq \frac{m-2}{m+2}$, then (E,X) is singular if $m-1-\alpha(m+2)$ is an odd positive integer and regular otherwise.

2. SOME LEMMAS.

Here we state and prove a few lemmas.

LEMMA 1. If $w_n(x) = P_n^{(\alpha)}(x), \alpha > -1, \lambda_r(x) = [(1-x^2)w_n^2(x)]^r, r = 1,2,\cdots$ and $\{x_k\}_1^n$ are the zeros of $w_n(x)$ then:

$$\left[w_n^{2r}\ (x)\right]_{x=x_k}^{(2r)} = (2r)\ !\ [w_n'\ (x_k)]^{2r} \tag{2.1}$$

$$\begin{aligned}\left[w_n^{2r}\ (x)\right]_{x=x_k}^{(2r+1)} &= 2r\ (2r+1)\ !\ (\alpha+1)\ x_k\ (1-x_k^2)^{-1}\ [w_n'\ (x_k)]^{2r} \\ &= 2r\ (2r+1)\ x_k\ (\alpha+1)\ (1-x_k^2)^{-1}\ [w_n^{2r}\ (x)]_{x=x_k}^{(2r)}\end{aligned} \tag{2.2}$$

$$\lambda_r^{(i)}\ (x_k) = \begin{cases} 0, & i = 0,\ 1, \cdots,\ 2r-1 \\ (1-x_k^2)^r\ (2r)\ !\ [w_n'\ (x_k)]^{2r}, & i = 2r, \end{cases} \tag{2.3}$$

$$\begin{aligned}\lambda_r^{(2r+1)}\ (x_k) &= 2r\ (2r+1)\ !\ \alpha x_k\ (1-x_k^2)^{r-1}\ [w_n'\ (x_k)]^{2r} \\ &= 2r(2r+1)\alpha x_k(1-x_k^2)^{-1}\lambda_r^{(2r)}(x_k).\end{aligned} \tag{2.4}$$

The proof is obvious.

LEMMA 2. Let $\delta_{2r}(x) = (1-x^2)^{2r} = (x^2-1)^{2r}$, $r = 1,\ 2, \cdots$.

Then:

$$\delta_{2r}^{(i)}\ (\pm 1) = \begin{cases} 0, & i = 0,\ 1, \cdots,\ 2r-1 \\ (2r)\ !\ 2^{2r}, & i = 2r \end{cases} \tag{2.5}$$

$$\delta_{2r}^{(2r+1)}(1) = 2^{2r}(2r+1)!\,r = -\delta_{2r}^{(2r+1)}(-1) \tag{2.6}$$

$$\delta_{2r}^{(2r+1)}(1) = r\,(2r+1)\,\delta_{2r}^{(2r)}(1) \tag{2.7}$$

and

$$\delta_{2r}^{(2r+1)}(-1) = -r\,(2r+1)\,\delta_{2r}^{(2r)}(-1). \tag{2.8}$$

The proof is obvious.

LEMMA 3. Let $F_n(x) = [(1-x^2)w_n(x)]^m q_{n+1}(x)$ be a polynomial of degree $\le (n+2)(m+1)-1$, where $q_{n+1}(x)$ is a polynomial of degree $\le n+1$, and let

$$F_n^{(m+1)}(x_k) = 0, k = 0, 1, 2, \cdots, n+1.$$

Then, $q_{n+1}(x)$ satisfies the following conditions:

$$(1-x_k^2)\,q'_{n+1}(x_k) + m(\alpha-1)\,x_k\,q_{n+1}(x_k) = 0,\ k = 1,\ 2, \cdots, n;\ \alpha > -1, \tag{2.9}$$

$$2q'_{n+1}(1) + m\left[\frac{n(n+2\alpha+1)}{1+\alpha}+1\right]q_{n+1}(1) = 0, \tag{2.10}$$

$$2q'_{n+1}(-1) - m\left[\frac{n(n+2\alpha+1)}{1+\alpha}+1\right]q_{n+1}(-1) = 0. \tag{2.11}$$

PROOF. Let $m = 2r$. Then

$$F_n(x) = \lambda_r(x)\left[(1-x^2)^r q_{n+1}(x)\right].$$

On using Leibnitz's formula and Lemma 1 one can easily see that for $k = 1, 2, \cdots, n$,

$$F_n^{(2r+1)}(x_k) = (2r+1)\,\lambda_r^{(2r)}(x_k)\,(1-x_k^2)^{r-1}\left[(1-x_k^2)\,q'_{n+1}(x_k) + 2rx_k(\alpha-1)q_{n+1}(x_k)\right]. \tag{2.12}$$

To evaluate $F_n^{(2r+1)}(\pm 1)$ we proceed as follows:

$$F_n(x) = \delta_{2r}(x)\left\{[w_n(x)]^{2r}q_{n+1}(x)\right\}.$$

Now, making use of Leibnitz formula and Lemma 2, we get

$$F_n^{(2r+1)}(1) = \delta_{2r}^{(2r+1)}(1)[w_n(1)]^{2r}q_{n+1}(1) +$$
$$+\binom{2r+1}{2r}\delta_{2r}^{(2r)}(1)\left\{2r[w_n(1)]^{2r-1}w'_n(1)q_{n+1}(1) + [w_n(1)]^{2r}q'_{n+1}(1)\right\}. \tag{2.13}$$

We know that

$$(1-x^2)w''_n(x) - 2(\alpha+1)xw'_n(x) + n(n+2\alpha+1)w_n(x) = 0 \tag{2.14}$$

hence

$$w'_n(1) = \frac{n(n+2\alpha+1)}{2(\alpha+1)}w_n(1), w_n(1) = \binom{n+\alpha}{n} \tag{2.15}$$

So, from (2.13) and (2.15) it follows that

$$F_n^{(2r+1)}(1) = 2^{2r-1}(2r+1)![w_n(1)]^{2r}\left\{2q'_{n+1}(1) + 2r\left[1+\frac{n(n+2\alpha+1)}{1+\alpha}\right]q_{n+1}(1)\right\}. \tag{2.16}$$

We also know that

$$w_n(-1) = (-1)^n w_n(1), \tag{2.17}$$

$$w_n'(-x) = (-1)^{n+1} w_n'(x). \tag{2.18}$$

Further, using Leibnitz formula, Lemma 2, (2.17) and (2.18) one can easily verify that

$$F_n^{(2r+1)}(-1) = 2^{2r-1}(2r+1)![w_n(1)]^{2r}\left\{2q_{n+1}'(-1) - 2r\left[1 + \frac{n(n+2\alpha+1)}{1+\alpha}\right]q_{n+1}(-1)\right\}. \tag{2.19}$$

Next, let $m = 2r+1$. We now write

$$F_n(x) = \lambda_r(x)[(1-x^2)^{r+1} w_n(x) q_{n+1}(x)].$$

Again, on using Leibnitz formula, Lemma 1 and (2.14), it follows that for $k = 1, 2, \cdots, n$,

$$F_n^{(2r+2)}(x_k) = (2r+1)(2r+2)(1-x_k^2)^r w_n'(x_k)\,\lambda_r^{(2r)}(x_k)\Big[(1-x_k^2)\,q_{n+1}'(x_k) + (2r+1)(\alpha-1)x_k q_{n+1}(x_k)\Big]. \tag{2.20}$$

Further, to compute $F_n^{(2r+2)}(\pm 1)$, we write

$$F_n(x) = \delta_{2r}(x)\Big[(1-x^2) w_n^{2r+1}(x) q_{n+1}(x)\Big]$$

and use Leibnitz formula to get

$$F_n^{(2r+1)}(x) = \sum_{i=0}^{2r+2} \binom{2r+2}{i} \delta_{2r}^{(i)}(x)\Big[(1-x^2) w_n^{2r+1}(x) q_{n+1}(x)\Big]^{(2r+1-i)}. \tag{2.21}$$

On simplification using Lemma 2, (2.21) yields

$$F_n^{(2r+2)}(1) = -(2r+2)!\,2^{2r}\,w_n^{2r+1}(1)\left[2q_{n+1}'(1) + (2r+1)\left\{\frac{n(n+2\alpha+1)}{\alpha+1} + 1\right\} q_{n+1}(1)\right], \tag{2.22}$$

$$F_n^{(2r+2)}(-1) = (-1)^n (2r+2)!\,2^{2r} w_n^{2r+1}(-1)\left[2q_{n+1}'(-1) - (2r+1)\left\{\frac{n(n+2\alpha+1)}{\alpha+1} + 1\right\} q_{n+1}(-1)\right]. \tag{2.23}$$

Hence the conditions

$$F^{(m+1)}(x_k) = 0, \; k = 0, 1, 2, \cdots, n+1$$

along with (2.12), (2.16), (2.19), (2.20), (2.22) and (2.23) imply (2.9), (2.10) and (2.11) for m even or odd. This completes the proof of Lemma 3.

LEMMA 4. Let $q_{n+1}(x)$ be a polynomial of degree $\leq n+1$ which satisfies the following $n+2$ conditions:

$$(1-x_k^2)q_{n+1}'(x_k) + m(\alpha-1)x_k q_{n+1}(x_k) = 0, k = 1, 2, \cdots, n; \alpha > -1, \tag{2.24}$$

$$2q_{n+1}'(1) + m\left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1\right] q_{n+1}(1) = 0, \tag{2.25}$$

$$2q_{n+1}'(-1) - m\left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1\right] q_{n+1}(-1) = 0. \tag{2.26}$$

Then $q_{n+1}(x)$ satisfies the following equation:

$$(1-x^2)\,q_{n+1}'(x) + m(\alpha-1)\,x q_{n+1}(x) = c[x^2 - \Delta(\alpha)]\,w_n(x), \tag{2.27}$$

where c is an arbitrary constant and

$$\Delta(\alpha) = \frac{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha}\left[\binom{n+\alpha}{n}\left\{\frac{m(1-\alpha)+2}{1-\alpha}\right\}+1\right]+\left[2\binom{n+\alpha}{n}-1\right]}{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha}\left[\binom{n+\alpha}{n}\left\{\frac{m(1-\alpha)+2}{1-\alpha}\right\}+1\right]-\left[2\binom{n+\alpha}{n}+1\right]}, \; \alpha \neq 1. \tag{2.28}$$

PROOF. Due to (2.24), it follows that

$$(1-x^2)q'_{n+1}(x)+m(\alpha-1)xq_{n+1}(x)=(cx^2+dx+e)w_n(x), \tag{2.29}$$

where c, d and e are constants. From (2.29), (2.15) and (2.17) we see that

$$m(\alpha-1)q_{n+1}(1)=(c+d+e)\binom{n+\alpha}{n}, \tag{2.30}$$

$$-m(\alpha-1)q_{n+1}(-1)=(c-d+e)(-1)^n\binom{n+\alpha}{n}. \tag{2.31}$$

Also, on differentiating (2.29) we have

$$(1-x^2)q''_{n+1}(x)+[m(\alpha-1)-2]xq'_{n+1}(x)+m(\alpha-1)q_{n+1}(x)= \\ (cx^2+dx+e)w'_n(x)+(2cx+d)w_n(x). \tag{2.32}$$

Hence, from (2.32) we conclude that

$$[m(\alpha-1)-2]q'_{n+1}(1)+m(\alpha-1)q_{n+1}(1)=(c+d+e)w'_n(1)+(2c+d)w_n(1), \tag{2.33}$$

$$-[m(\alpha-1)-2]q'_{n+1}(-1)+m(\alpha-1)q_{n+1}(-1)=(c-d+e)w'_n(-1)+(-2c+d)w_n(-1). \tag{2.34}$$

Further, from (2.30), (2.31), (2.15), (2.17), (2.18), (2.33) and (2.34) it follows that

$$[m(\alpha-1)-2]\ q'_{n+1}(1)=(c-e)\binom{n+\alpha}{n}+(c+d+e)\ \frac{n(n+2\alpha+1)}{2(\alpha+1)}, \tag{2.35}$$

$$[m(\alpha-1)-2]\ q'_{n+1}(-1)=(-1)^n\left[(c-e)\binom{n+\alpha}{n}+(c-d+e)\ \frac{n(n+2\alpha+1)}{2(\alpha+1)}\right]. \tag{2.36}$$

Consequently, on substituting the values of $q_{n+1}(1)$, $q'_{n+1}(1)$, $q_{n+1}(-1)$ and $q'_{n+1}(-1)$ from the above equations into (2.25) and (2.26) and simplifying we get

$$\left[\left\{2\binom{n+\alpha}{n}-1\right\}c-d-\left\{2\binom{n+\alpha}{n}+1\right\}e\right] \\ +(c+d+e)\left[\frac{n(n+2\alpha+1)}{1+\alpha}+1\right]\left[\binom{n+\alpha}{n}\left\{m+\frac{2}{1-\alpha}\right\}+1\right]=0, \tag{2.37}$$

$$\left[\left\{2\binom{n+\alpha}{n}-1\right\}c+d-\left\{2\binom{n+\alpha}{n}+1\right\}e\right] \\ +(c-d+e)\left[\frac{n(n+2\alpha-1)}{1+\alpha}+1\right]\left[\binom{n+\alpha}{n}\left\{m+\frac{2}{1-\alpha}\right\}+1\right]=0. \tag{2.38}$$

Now, from (2.37) and (2.38) we see that $d=0$ and

$$\left[2\left(\frac{n+\alpha}{n}\right)-1\right]c-\left[2\binom{n+\alpha}{n}+1\right] \\ +(c+e)\left[\frac{n(n+2\alpha+1)+1+\alpha}{1+\alpha}\right]\left[\binom{n+\alpha}{n}\left\{\frac{m(1-\alpha)+2}{1-\alpha}\right\}+1\right]=0$$

which, on simplification, yields

$$e=-\frac{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha}\left[\binom{n+\alpha}{n}\left\{\frac{m(1-\alpha)+2}{1-\alpha}\right\}+1\right]+\left[2\binom{n+\alpha}{n}-1\right]}{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha}\left[\binom{n+\alpha}{n}\left\{\frac{m(1-\alpha)+2}{1-\alpha}\right\}+1\right]-\left[2\binom{n+\alpha}{n}+1\right]}\,c \tag{2.39}$$

or, using (2.28) we have

$$e = -\Delta(\alpha)c. \tag{2.40}$$

This completes the proof of Lemma 4.

3. PROOF OF THEOREM 1 AND THEOREM 2.

Let E be the incidence matrix given by (1.3) and let X be the set of zeros of $(1-x^2)P_n^{(\alpha)}(x) = (1-x^2)w_n(x)$. Let $F_n(x)$ be a polynomial of degree $\le (n+2)(m+1)-1$ annihilated by (E,X). We have to ascertain if $F_n(x)$ is identically zero. Since

$$F_n(x_k) = F_n'(x_k) = F_n''(x_k) = \cdots = F_n^{(m-1)}(x_k) = 0,\ k = 0,\ 1, \cdots, n+1,$$

$$F_n(x) = [(1-x^2)w_n(x)]^m q_{n+1}(x),$$

where $q_{n+1}(x)$ is a polynomial of degree $\le n+1$. Further, since we have required that

$$F_n^{(m+1)}(x_k) = 0,\ k = 0,\ 1, \cdots, n+1,$$

on account of Lemma 4, $q_{n+1}(x)$ satisfies the following equation:

$$(1-x^2)q_{n+1}'(x) + m(\alpha-1)xq_{n+1} = c[x^2 - \Delta(\alpha)]w_n(x), \tag{3.1}$$

where c is a numerical constant. Let

$$q_{n+1}(x) = \sum_{k=0}^{n+1} a_k w_k(x). \tag{3.2}$$

Further, it is well-known that

$$(1-x^2)w_n'(x) = -nxw_n(x) + (n+\alpha)w_{n-1}(x). \tag{3.3}$$

Now, from (3.1), (3.2) and (3.3), on simple computations, it follows that

$$\sum_{k=1}^{n+2} a_{k-1}\,[m(\alpha-1)-k+1]\,\frac{k(k+2\alpha)}{(k+\alpha)(2k+2\alpha-1)}\,w_k(x)$$

$$+\sum_{k=0}^{n} a_{k+1}(k+\alpha+1)\left[1+\frac{m(\alpha-1)-k-1}{2k+2\alpha+3}\right]w_k(x) = c[x^2-\Delta(\alpha)]w_n(x). \tag{3.4}$$

Also, we know that

$$xw_n(x) = \frac{(n+1)(n+2\alpha+1)}{(2n+2\alpha+1)(n+\alpha+1)}\,w_{n+1}(x) + \frac{(n+\alpha)}{(2n+2\alpha+1)}\,w_{n-1}(x). \tag{3.5}$$

Repeated application of (3.5) in (3.4), on simplification, yields

$$\sum_{k=0}^{n} a_{k+1}\frac{(k+\alpha+1)\,[k+\alpha(m+2)+2-m]}{(2k+2\alpha+3)}\,w_k(x)$$

$$+\sum_{k=1}^{n+2} a_{k-1}\frac{[m(\alpha-1)-k+1]k(k+2\alpha)}{(k+\alpha)\,(2k+2\alpha-1)}\,w_k(x)$$

$$= Aw_{n-2}(x) + Bw_n(x) + Cw_{n+2}(x), \tag{3.6}$$

where

$$A = \frac{(n+\alpha)\,(n+\alpha-1)}{(2n+2\alpha+1)\,(2n+2\alpha-1)}\,c, \tag{3.7}$$

$$B = \left[\frac{(n+1)\,(n+2\alpha+1)}{(2n+2\alpha+1)\,(2n+2\alpha+3)} + \frac{n(n+2\alpha)}{(2n+2\alpha+1)\,(2n+2\alpha-1)} - \Delta(\alpha)\right]c, \tag{3.8}$$

$$C = \frac{(n+1)\,(n+2\alpha+1)\,(n+2)\,(n+2\alpha+2)}{(n+\alpha+1)\,(n+\alpha+2)\,(2n+2\alpha+1)\,(2n+2\alpha+3)}\,c. \tag{3.9}$$

Consequently, we obtain

$$\frac{(1+\alpha)\,[\alpha(m+2)+2-m]}{2\alpha+3}\,a_1 = 0,$$

$$\frac{(k+1+\alpha)\,[k+\alpha(m+2)+2-m]}{(2k+2\alpha+3)}\,a_{k+1} + \frac{k(k+2\alpha)\,[m(\alpha-1)-k+1]}{(k+\alpha)\,(2k+2\alpha-1)}\,a_{k-1} = 0,$$

$$k = 1, 2, \cdots, n-4, n-3,$$

$$\frac{(n+\alpha-1)\,[n+\alpha(m+2)-m]}{(2k+2\alpha-1)}\,a_{n-1} + \frac{[m(\alpha-1)-n+3]\,(n-2)\,(n+2\alpha-2)}{(n+\alpha-2)\,(2n+2\alpha-5)}\,a_{n-3} = A,$$

$$\frac{(n+\alpha)\,[n+\alpha(m+2)+1-m]}{(2n+2\alpha+1)}\,a_n + \frac{[m(\alpha-1)-n+2]\,(n-1)\,(n+2\alpha-1)}{(n+\alpha-1)\,(2n+2\alpha-3)}\,a_{n-2} = 0,$$

$$\frac{(n+\alpha+1)\,[n+\alpha(m+2)+2-m]}{(2n+2\alpha+3)}\,a_{n+1} + \frac{[m(\alpha-1)-n+1]\,n(n+2\alpha)}{(n+\alpha)\,(2n+2\alpha-1)}\,a_{n-1} = B,$$

$$\frac{(n+2\alpha+1)\,(n+1)\,[m(\alpha-1)-n]}{(n+\alpha+1)\,(2n+2\alpha+1)}\,a_n = 0,$$

and

$$\frac{(n+2)\,(n+2\alpha+2)\,[m(\alpha-1)-n-1]}{(n+\alpha+2)\,(2n+2\alpha+3)}\,a_{n+1} = C.$$

Let m be an odd positive integer ≥ 3:

(i) If n is odd, $-1 < \alpha < 1$, and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_3 = a_1 = 0$$

and

$$a_0 = a_2 = \cdots\cdot = a_{m-3-\alpha(m+2)} = 0$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \cdots\cdot, a_{n+1}$ are note necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero. If n is odd, $-1 < \alpha < 1$, and α is such that $m-1-\alpha(m+2)$ is an even negative integer then

$$a_n = a_{n-2} = \cdots\cdots = a_3 = a_1 = 0$$

and $a_0, a_2, \cdots, a_{n-3}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero.
If n is odd, $-1 < \alpha < 1$, and α is such that $m-1-\alpha(m+2)$ is an odd integer or a fraction then

$$a_n = a_{n-2} = \cdots = a_3 = a_1 = 0$$

but $a_0, a_2, \cdots, a_{n-3}, a_{n-1}$ and a_{n+1} are not all zero. Hence, $q_{n+1}(x)$ is not identically zero.
So, it follows that (E, X) is singular if n is odd.

(ii) If n is even, $-1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \cdots\cdots = a_{m-3-\alpha(m+2)} = 0,$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \cdots, a_{n+1}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero. If n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, α is such that $m-1-\alpha(m+2)$ is an odd negative integer then

$$a_n = a_{n-2} = \cdots\cdots = a_2 = a_0 = 0.$$

Noting that $a_1 = 0$ we conclude that

$$a_1 = a_3 = \cdots\cdot = a_{n-3} = 0.$$

Recalling the equations for a_{n-1} and a_{n+1} and substituting the values of A, B, and C it can be easily verified that $c = 0$. So, a_{n-1} and a_{n+1} are also zero. Hence, $q_{n+1}(x)$ is identically zero. If n is even, $-1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Hence, $q_{n+1}(x)$ is identically zero.
If n is even, $-1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an even negative integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Hence, $q_{n+1}(x)$ is identically zero.
If n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is a fraction then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Hence, $q_{n+1}(x)$ is identically zero. Consequently, if n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then (E, X) is singular and for all other values of $m-1-\alpha(m+2), (E, X)$ is regular.

(iii) If n is even, $\alpha = \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is a negative integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and $a_1, a_3, \cdots, a_{n-3}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero.
If n is even, $\alpha = \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \cdots\cdots = a_{m-3-\alpha(m+2)} = 0$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \cdots, a_{n-1}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero. If n is even, $\alpha = \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and $a_1, a_3, \cdots, a_{n-3}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero.
Consequently, in this case, (E, X) is singular.
This completes the proof of Theorem 1.

Next, let m be an even positive integer ≥ 2, and $-1 < \alpha < 1$:

(i) If n is odd then

$$a_n = a_{n-2} = \cdots\cdot = a_3 = a_1 = 0$$

but not all $a_0, a_2, \cdots, a_{n+1}$ are zero. Hence, $q_{n+1}(x)$ is not identically zero. So, (E, X) is singular.

(ii) If n is even and $\alpha = \frac{m-2}{m+2}, 0 \leq \alpha < 1$, then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

but a_1, $a_3, \cdots, a_{n-3}$ are not necessarily zero. Hence $q_{n+1}(x)$ is not identically zero. Consequently, (E, X) is singular.

(iii) If n is even, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \cdots\cdots = a_{m-3-\alpha(m+2)} = 0,$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \cdots\cdot, a_{n+1}$ are not necessarily zero. Hence $q_{n+1}(x)$ is not identically zero.
If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is an odd negative integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and since $k+\alpha(m+2)+2-m$ is never zero for even values of k hence

$$a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

So, $q_{n+1}(x)$ is identically zero.
If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and also $a_1 = 0$ so that $k+\alpha(m+2)+2-m$ is never zero for even values of k hence

$$a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Consequently, $q_{n+1}(x)$ is identically zero.
If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is an even negative integer then

$$a_n = a_{n-2} = \cdots\cdot = a_2 = a_0 = 0$$

and also $a_1 = 0$ so that $k+\alpha(m+2)+2-m$ is never zero hence

$$a_1 = a_3 = a_5 = \cdots\cdot = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Consequently, $q_{n+1}(x)$ is identically zero.

If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is a fraction then

$$a_n = a_{n-2} = \cdots\cdot = a_0 = 0,$$

$a_1 = 0$, so $a_1 = a_3 = a_5 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0$.

Therefore, $q_{n+1}(x)$ is identically zero. Consequently, (E, X) is singular if $m-1-\alpha(m+2)$ is an odd positive integer and regular otherwise. This completes the proof of Theorem 2.

In conclusion, it is worthwhile to mention that H. Windauer [7] has also considered the modified $(0,1,\cdots,r-2,r)$ – interpolation problem on the zeros of $(1-x^2)P_n^{(\alpha)}(x), \alpha > -1$, and $(0,1,\cdots,r-2,r)$ – interpolation problem on the zeros of $P_n^{(\alpha)}(x), \alpha > -1$. As is evident, we have addressed the $(0,1,\cdots,r-2,r)$ – interpolation problem on the zeros of $(1-x^2)P_n^{(\alpha)}(x), \alpha > -1$.

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