Internat. J. Math. & Math. Sci.
VOL. 17 NO. 3 (1994) 429-436

FIXED POINT OF NONEXPANSIVE TYPE AND K-MULTIVALUED MAPS

ABDUL LATIF

Department of Mathematics
Gomal University, D.I. Khan
Pakistan

TAQDIR HUSAIN

Department of Mathematics
McMaster University
Hamilton, Canada

and

ISMAT BEG

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

(Received August 10, 1992)

ABSTRACT. Some fixed point theorems for nonexpansive type and K-multivalued mappings are proved. Also the strong convergence of sequences of iterates of multivalued type maps is established.

KEY WORDS AND PHRASES. Normed spaces, contractive type, nonexpansive type, K-multivalued maps and fixed points.

1991 AMS SUBJECT CLASSIFICATION CODES. 47H04, 47H10, 54H25.

1. INTRODUCTION.

A single valued self-map of a metric space (X,d) is called contractive if $d(f(x),f(y)) \leq hd(x,y)$ for all x,y in X and for a fixed number $h, 0 \leq h < 1$. If

$$d(f(x),f(y)) \leq d(x,y), \tag{1.1}$$

then f is known as a single valued nonexpansive mapping. A classical theorem (also known as the Contraction Principle) asserts that each contractive self-map of a complete metric space has a unique fixed point. It is clear that in general a nonexpansive self-mapping of a complete metric space need not have a fixed point. However, for such mappings defined on convex sets in a Banach space, some interesting fixed point results have been obtained by Browder [2] and Kirk [12].

The notion of contractiveness and nonexpansiveness for multivalued maps has been extended in several ways and some fixed points of such multivalued functions have also been established. See, for example, [1], [8], [13].

The second author and Tarafdar [5] introduced the notion of a nonexpansive type multivalued map and proved a fixed point theorem on compact intervals of the real line, which has been extended by Husain and Latif ([6], [7]) in several directions.

Kannan ([9], [10]) has proved some fixed point theorems for single valued self-mappings f of a metric space (X,d) satisfying the property:

$$d(f(x),f(y)) \le \frac{1}{2}\{d(x,f(x))+d(y,f(y))\}. \tag{1.2}$$

We shall call a mapping satisfying (1.2) a K-mapping in the sequel. It is known [10] that conditions (1.1) and (1.2) are independent. Kannan [11] proves some fixed point theorems for K-mappings on certain subsets of Banach spaces.

In this paper, we prove some fixed point theorems (see section 2) for nonexpansive type multivalued maps which extend results in [7] and include the result of Dotson [4]. Section 3 deals with the notion of K-multivalued mappings which is a generalization of K-mappings and we prove some fixed point results for such mappings.

We recall the following notions needed in the sequel.

Let C be a nonempty subset of a metric space (X,d). A multivalued map $J: C \to 2^X$ (nonempty subsets of X) is called contractive type [7] if for all $x \in C, u_x \in J(x)$ there is a $u_y \in J(y)$ for all $y \in C$ such that

$$d(u_x,u_y) \le h\ d(x,y)$$

for a fixed real number $h, 0 \le h < 1$. This notion clearly generalizes the usual concept of contractive maps [7]. Further, if in the above inequality we have

$$d(u_x,u_y) \le d(x,y)$$

then J is called a nonexpansive type map. An element $x \in C$ is called a fixed point of J if $x \in J(x)$.

A Banach space X is said to satisfy Opial's condition [14] if for each $x \in X$ and for each sequence $\{x_n\}$ weakly convergent to x, the inequality

$$\liminf_n \| x_n - y \| > \liminf_n \| x_n - x \|$$

holds for all $x \neq y$. Every Hilbert space satisfies Opial's condition [14] and so does each $l_p(1 < p < \infty)$.

A subset C of a linear space X is said to be star-shaped if there is a $x_0 \in C$ such that $\{tx + (1-t)x_0 : 0 \le t \le 1\} \subset C$ for each $x \in C$. The element x_0 is called a star-centre for C. The class of star-shaped subsets of X includes convex subsets as a proper subclass. We denote $d(C) = \sup_{x,y \in C} \| x - y \|$ and $F(x,C) = \sup_{y \in C} \| x - y \|$. We know:

THEOREM 1.1 [7]. Let C be a nonempty closed subset of a complete metric space (X,d). Then each closed-valued contractive type multivalued mapping $J: C \to 2^C$ has a fixed point.

THEOREM 1.2 [7]. Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition. Then each compact-valued nonexpansive type multivalued mapping $J: C \to 2^C$ has a fixed point.

2. NONEXPANSIVE TYPE MULTIVALUED MAPS.

Now we extend Theorem 1.2. But first, we show

THEOREM 2.1. Let C be a nonempty closed star-shaped subset of a Banach space X and $J: C \to 2^C$ a compact-valued nonexpansive type multivalued mapping. If $J(C)$ is bounded and $(I-J)C$ is closed, then J has a fixed point.

PROOF. Consider a sequence of positive numbers $\{t_n\}$ converging to 1 and $0 < t_n < 1$ for all

$n \geq 1$. Let x_0 be a start-centre of C. For each $n \geq 1$, define the multivalued mapping J_n of C into 2^C by setting:

$$\begin{aligned} J_n(x) &= t_n J(x) + (1 - t_n)x_0 \\ &= \{t_n u + (1 - t_n)x_0 : u \in J(x)\}. \end{aligned}$$

For each $n \geq 1, J_n$ is a closed valued contractive type multivalued mapping. Therefore, Theorem 1.1 implies that for each $n \geq 1$, there exists a $x_n \in C$ such that $x_n \in J_n(x)$. From the definition of $J_n(x)$, there is a $u_n \in J(x_n)$, $n \geq 1$ such that

$$x_n = t_n u_n + (1 - t_n)x_0.$$

and so

$$x_n - u_n = (1 - t_n)(x_0 - u_n).$$

Since $J(C)$ is bounded, due to the fact that $t_n \to 1$ as $n \to \infty$, we have

$$(x_n - u_n) \to 0 \text{ as } n \to \infty.$$

Since $(I - J)C$ is closed, $0 \in (I - J)C$. Hence there is a point $x \in C$ such that $x \in J(x)$.

THEOREM 2.2. Let C be a nonempty weakly closed star-shaped subset of a Banach space X which satisfies Opial's condition. Let $J: C \to 2^C$ be a compact valued nonexpansive type mapping and let $J(C) \subseteq M$ for some weakly compact subset M of X, then J has a fixed point.

PROOF. As we have shown in the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ in C such that

$$y_n = x_n - u_n \to 0 \text{ as } n \to \infty, u_n \in J(x_n).$$

Since the sequence $\{x_n - y_n\} \subset M$ and M is weakly compact, we can find a weakly convergent subsequence $\{x_m - y_m\}$ of $\{x_n - y_n\}$. Let $z = w - \lim_m (x_m - y_m)$. Clearly $z \in M$. Since $y_m \to 0$, it follows that $z = w - \lim_m x_m \in C$ because C is weakly closed.

Now for each $m \geq 1, x_m - y_m = u_m \in J(x_m)$ and J being a nonexpansive type map, there is $v_m \in J(z)$ such that

$$\| x_m - (y_m + v_m) \| \leq \| x_m - z \|.$$

Since $\{v_m\}$ is contained in the compact set $J(z)$, there is a subsequence of $\{v_m\}$, also denoted by $\{v_m\}$, converging to $v \in J(z)$. Therefore

$$y_m + v_m \to v \text{ as } m \to \infty.$$

It follows that

$$\lim_m \inf \| x_m - v \| \leq \lim_m \inf \| x_m - z \|.$$

Since $x_m \to z$ weakly, using the Opial's condition, we have $z = v \in J(z)$.

COROLLARY 2.3. Let C be a nonempty closed convex subset of a reflexive Banach space X satisfying the Opial's condition. If $J(C)$ is bounded, then each compact valued nonexpansive type map $J: C \to 2^C$ has a fixed point.

COROLLARY 2.4. Let C be a nonempty closed convex subset of a Hilbert space H. If $J(C)$ is bounded, then each compact valued nonexpansive type map $J: C \to 2^C$ has a fixed point.

THEOREM 2.5. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $J: C \to 2^C$ be a compact-valued nonexpansive type map and $J(C)$ bounded. Assume

$$J_n(x) = t_n J(x) + (1 - t_n)x_0,$$

where $0 < t_n < 1, t_n \to 1$ and x_0 is an arbitrary point in C. If $x_n \in J_n(x_n)$, then $\{x_n\}$ converges strongly to a fixed point of J.

PROOF. Since $\{x_n\}$ is bounded, there is a weakly convergence subsequence of $\{x_n\}$. We denote the subsequence also by $\{x_n\}$ for convenience. Clearly $z = w - \lim_n x_n \in C$. Moreover, $z \in J(z)$ (see the proof of Theorem 2.2). To show that $\{x_n\}$ converges strongly to z, we note that $x_n \in J_n(x_n)$ and so there is a $u_n \in J(x_n)$ such that

$$x_n = t_n u_n + (1 - t_n)x_0.$$

For convenience we can take $x_0 = 0$ because otherwise the similar arguments can be used. Note: $x_0 \in C$ and $u_n \in J(x_n) \subset C$ imply $\| u_n - x_0 \|$ is bounded and so

$$\| x_n - u_n \| = | t_n - 1 | \ \| u_n - x_0 \| \to 0 \text{ as } t_n \to 1.$$

But then

$$\| z - x_n/t_n \|^2 = \| z - u_n \|^2$$

$$\leq \| z - x_n \|^2 + \| x_n - u_n \|^2 + 2 < z - x_n, x_n - u_n > .$$

It further implies that

$$\lim_{n\to\infty} \| z - x_n/t_n \|^2 \leq \lim_{n\to\infty} \| z - x_n \|^2.$$

So there is a positive integer N such that

$$\| z - x_n/t_n \|^2 \leq \| z - x_n \|^2, \qquad n \geq N,$$

hence

$$t_n^2 \| z \|^2 + \| x_n \|^2 - 2\, t_n < z, x_n > \ \leq t_n^2 [\, \| z \|^2 + \| x_n \|^2 - 2 < z, x_n >].$$

and so

$$\| x_n \|^2 \leq \frac{2t_n}{1 + t_n} < z, x_n > \ \leq \ < z, x_n > .$$

Thus

$$\| x_n \| \ \leq \ < z, \frac{x_n}{\| x_n \|} > \ \leq \ \| z \| \ \| \frac{x_n}{\| x_n \|} \| \ = \ \| z \| .$$

Now

$$\begin{aligned} \| z \|^2 \qquad & \geq \| x_n \|^2 = \| x_n - z + z \|^2 \\ & = \| x_n = z \|^2 + \| z \|^2 + 2 < x_n - z, z > \end{aligned}$$

which gives $\| x_n - z \| \to 0$ as $n \to \infty$.

REMARK. Theorem 2.5 includes the result of Browder [3] and contains a special case of Singh and Watson [15].

3. K-MULTIVALUED MAPPINGS.

Let C be a nonempty subset of a normed linear space X. We say a mapping $J: C \to 2^C$ is K-multivalued if for each $x \in C$, $u_x \in J(x)$ there is a $u_y \in J(y)$ for all $y \in C$ such that

$$\| u_x - u_y \| \leq \tfrac{1}{2}\{ \| x - u_x \| + \| y - u_y \| \}.$$

Clearly this notion generalizes the usual concept of K-mapping [9, 10].

THEOREM 3.1. Let C be a nonempty subset of a normed linear space X. Let $J: C \to 2^X$ be a K-multivalued mapping. Suppose

$$\sup_{x \in A} F(x, Jx) < \tfrac{1}{2} d(A)$$

for every closed J-invariant star-shaped subset A of C with nonzero diameter. If there exists a minimal closed J-invariant star-shaped subset M of C such that the image of its star-centre is a singleton set, then J has a unique fixed point.

PROOF. If $d(M) = 0$, then the point in M is a fixed point of J. Suppose $d(M) > 0$. Let x_0 be a star-centre of M so that

$$\{tx + (1-t)x_0 : 0 \le t \le 1\} \subset M$$

for each $x \in M$. Let $J(x_0) = \{u_{x_0}\}$. Since J is a K-multivalued mapping, for each $x \in M$ and each $u \in J(x)$,

$$\begin{aligned} \| u - u_{x_0} \| &\le \tfrac{1}{2} \{ \| x - u \| + \| x_0 - u_{x_0} \| \} \\ &\le \tfrac{1}{2} \{ \sup_{x \in M} F(x, Jx) + \sup_{x \in M} F(x, Jx) \} \\ &\le \sup_{x \in M} F(x, Jx) = \nu \qquad \text{(say).} \end{aligned}$$

Thus $J(M)$ is contained in a closed sphere S with centre u_{x_0} and radius ν. Clearly $M \cap S$ is a closed J-invariant star-shaped subset of C. By minimality of M, we have $M \subset S$ and so for each $x \in M$, $\| u - u_{x_0} \| \le \nu$.

Define

$$M' = \{ y \in M : \tfrac{1}{2} \ \| x - y \| \le \nu \}.$$

Clearly M' is a nonempty closed subset of M and $u_{x_0} \in M'$. If $y \in M'$ and $v \in J(y) \subset M$, then for each $x \in M$,

$$\| x - v \| \le \| x - u_{x_0} \| + \| u_{x_0} - v \| \le 2\nu.$$

This shows that $J(y) \subset M'$. Since y is arbitrary, we have $J(M') \subset M'$. Finally, for $y \in M'$, $x \in M$, $t \in [0,1]$, we have

$$\| ty + (1-t)u_{x_0} - x \| \le t \| y - u_{x_0} \| + \| u_{x_0} - x \| \le 2\nu,$$

which implies that $ty + (1-t)u_{x_0} \in M'$, for each $y \in M'$ and $t \in [0,1]$. From our hypothesis we have $d(M') \le 2\nu < d(M)$, which shows that M' is a proper closed J-invariant star-shaped subset of M. This contradicts the minimality of M and the uniqueness of the fixed point is easily established.

THEOREM 3.2. Let C, A and J be as in Theorem 3.1. Suppose

$$\sup_{x \in A} F(x, J(x)) < \frac{1}{2n} d(A). \tag{3.2.1}$$

If there exists a minimal closed J-invariant star-shaped subset M of C, then J has a unique fixed point.

PROOF. As before, if $d(M) = 0$, then the point in M is a fixed point of J. Suppose $d(M) > 0$ and let x_0 be the star-centre of M. Since M is J-invariant and J is a K-multivalued map, for $x \in M, u \in J(x) \subset M$, there is $v \in J(x_0) \subset M$ such that

$$\begin{aligned} \| u - v \| &\le \tfrac{1}{2} \{ \| x - u \| + \| x_0 - v \| \} \\ &\le \sup_{x \in M} F(x, J(x)) = \nu \qquad \text{(say, as in Theorem 3.1).} \end{aligned}$$

Thus, for each $x \in M$, there exist a positive integer n and $v_{x_0} \in J(x_0)$ such that for all

$w \in J(x)$, $\| w - v_{x_0} \| \leq n\nu$.
Hence $J(M)$ is contained in a closed sphere S with centre v_{x_0} and radius $n\nu$. Similarly, as before $M \cap S$ is a closed J-invariant star-shaped subset of C. By minimality of M, it follows that $M \subset S$. Thus for each $x \in M$, $\| x - v_{x_0} \| \leq n\nu$.
If we set

$$M' = \{y \in M: \frac{1}{2n} \| x - y \| \leq \nu\},$$

then as before M' is a nonempty closed J-invariant star-shaped proper subset of M, which contradicts the minimality of M and the proof follows.

THEOREM 3.3. Let C be a nonempty convex bounded subset of a uniformly convex Banach space X and $J: C \to 2^X$ a K-multivalued mapping which satisfies the inequality (3.2.1). If C is J-invariant and there exists a minimal closed J-invariant star-shaped subset M of C, then for any arbitrary point x_0 of C, the sequence $\{x_n\}$ generated from x_0 by

$$x_{n+1} = \frac{x_n + u_n}{2}, u_n \in J(x_n),$$

converges strongly to the fixed point of J.

PROOF. The existence of the fixed point of J in C is given by Theorem 3.2. Let $w \in C$ and $w \in J(w)$. Since C is convex and J-invariant, $x_n \in C$ and by definition of J there is a $u_n \in J(x_n) \subset C$ such that

$$\| w - u_n \| \leq \frac{1}{2} \| x_n - u_n \| \leq \frac{1}{2}[\| x_n - w \| + \| w - u_n \|], \tag{3.3.1}$$

which shows that for all $n \geq 1$, $\| w - u_n \| \leq \| w - x_n \|$.

Consider the sequence $\{u_n - x_n\}$. Two cases arise:

CASE I. There exists an $\epsilon > 0$ such that $\| u_n - x_n \| \geq \epsilon$ for all $n > N$. Then

$$\| (w - x_n) - (w - u_n) \| = \| u_n - x_n \| \geq \epsilon.$$

Since X is uniformly convex, we have

$$\| w - x_{n+1} \| \leq \frac{1}{2} [\| w - x_n \| + \| w - u_n \|]$$
$$\leq \delta \; max \{ \| w - x_n \|, \| w - u_n | \}, 0 < \delta < 1, n > N.$$

As C is bounded, so are $\{ \| w - x_n \| \}$ and $\{ \| w - u_n \| \}$ and hence using the inequality (3.3.1), we have

$$\| w - x_{n+1} \| \leq \delta | w - x_n \|, 0 < \delta < 1, n > N.$$

Therefore, $\{ \| w - x_n \| \}, n > N$, is a monotonic decreasing sequence tending to zero and so $\{x_n\}$ converges to $w \in J(w)$.

CASE II. There exists a sequence of integers $\{n_k\}$ such that

$$\lim_{k \to \infty} \| u_{n_k} - x_{n_k} \| = 0.$$

Since

$$\| w - u_{n_k} \| \leq \frac{1}{2} \| u_{n_k} - x_{n_k} \|,$$

we have $\lim_{k \to \infty} u_{n_k} = w$ and $\lim_{k \to \infty} x_{n_k} = w$.
and

$$\| w - x_{n+1} \| = \| w - \frac{x_n + u_n}{2} \| \leq \| w - x_n \|.$$

which implies $\lim_{n \to \infty} x_n = w \in J(w)$.

ACKNOWLEDGEMENT. The first author is grateful to NAHE for a grant Under National Collaboration Programme. The second author's research was partially supported by an NSERC grant.

REFERENCES

1. BEG, I. & AZAM, A., Fixed points of asymptotically regular multivalued mappings, *J. Austral. Math. Soc.* (Series A, to appear).
2. BROWDER, F.E., Non-expansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci.* **54** (1965), 1041-1044.
3. BROWDER, F.E., Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* **24** (1967), 82-90.
4. DOTSON, W.G., JR., Fixed point theorems for nonexpansive mappings on star-shaped subsets of Banach spaces, *J. London Math. Soc.* **4** (1972), 408-410.
5. HUSAIN, T. & TARAFDAR, E., Fixed point theorems for multivalued mappings of nonexpansive type, *Yokohama Math. J.* **28** (1980), 1-6.
6. HUSAIN, T. & LATIF, A., Fixed points of multivalued nonexpansive maps I, *Math. Japonica* **33** (1988), 385-391.
7. HUSAIN, T. & LATIF, A., Fixed points of multivalued nonexpansive maps II, *Internat. J. Math. & Math. Sci.* **14** (1991), 421-430.
8. KANEKO, H., A report on general contractive type conditions for multivalued mappings, *Math. Japonica* **33** (1988), 543-550.
9. KANNAN, R., Some results on fixed points, *Bull. Cal. Math. Soc.* **60** (1968), 71-76.
10. KANNAN, R., Some results on fixed points II, *Amer. Math. Monthly* **76** (1969), 405-408.
11. KANNAN, R., Some results on fixed points III, *Fund. Math.* **LXX** (1971), 169-177.
12. KIRK, W.A., A fixed point theorem for nonexpansive mappings which do not increase distance, *Amer. Math. Monthly* **72** (1965), 1004-1006.
13. LAMIDOZO, L., Multivalued nonexpansive mappings and Opial's condition, *Proc. Amer. Math. Soc.,* **38** (1973), 285-292.
14. OPIAL, Z., Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591-597.
15. SINGH, S.P. & WATSON, B., On approximating fixed points, *Proc. of Symposia in Pure Math.* **45** (1986), 393-395.