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FIRST PASSAGE PROCESSES IN QUEUING SYSTEM $M^X/G^r/1$ WITH SERVICE DELAY DISCIPLINE

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ABSTRACT. This article deals with a general single-server bulk queueing system with a server waiting until the queue will reach level r before it starts processing customers. If at least r customers are available the server takes a batch of the fixed size r of units for service. The input stream is assumed to be a compound Poisson process modulated by a semi-Markov process and with a multilevel control of service time.

The authors evaluate the steady state probabilities of the queueing processes with discrete and continuous time parameter preliminarily establishing necessary and sufficient conditions for the ergodicity of the processes. The authors use the recent results on the first excess level processes to explicitly find all characteristics of the named processes. Some characteristics of the input process, service cycle, intensity of the system, and both idle and busy periods are also found. The results obtained in the article are illustrated by numerous examples.

KEY WORDS AND PHRASES. Queueing process, embedded Markov chain, semi-Markov process, semi-regenerative process, modulated random measure, service cycle, idle period, busy period, output process, continuous time parameter process.

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1. INTRODUCTION.

Queueing systems, where the server takes batches of a fixed size, r (generally greater than 1), usually preclude that the server will wait for the queue length to accumulate so many customers if unavailable. Such models are discussed in the book of Chaudhry and Templeton [.], where the authors refereed works from sixties and seventies by different authors. Perhaps, Chaudhry and Templeton were the first to use modified Kendall's symbolic $M/G^r/1$ for these systems.

As it appeared from different sources, the evaluation of the steady state probability distribution for an embedded queueing process has a problem of dealing with roots of certain a analytic function which were difficult to find and whose existence were firmly related to a finite set of unknown probabilities. Although some recipes existed in the literature prior to this work but there are only those appropriate which dealt with very special systems. A recent work by Abolnikov and Dukhovny [2] enabled not only to supplement and refine the existing results but also to enlarge the class of systems to which this analysis can successfully be applied.

Dshalalow and Russell [7] were probably the first to employ the new techniques to a class of systems of type $M/G^r/1$ generalizing usual assumptions about this system by allowing the input stream to be modulated by a semi-Markov process and by implementing a service control. This work generalized models considered by various authors (see for instance, Chaudhry and Templeton [3,4]). By assuming that, in addition, the input stream is general bulk, we obviously have another problem of the *critical behavior* of the queueing process about level r. Indeed, in this case, when the server will start his service for the first time after being idle, the queue length is more likely to exceed than to exactly reach level r. The authors apply the recent results of the first passage problem obtained in Abolnikov and Dshalalow [1] to study the behavior of the queueing process at the instant of the *first passage time* of level r by the process. Thus, one of the central problems in the analysis of such queueing systems is another (auxiliary) process embedded in the queueing process over the successive instants off first passage times.

The authors establish necessary and sufficient conditions for the ergodicity of the queueing process with discrete and continuous time parameters and study its steady state behavior in both cases.

Due to the queue length dependent service delay discipline assumed in this article, an auxiliary random process describing the value of the first excess of the queue length above level $r-1$ appears to be one of the kernel components in the analysis of the queueing process. The authors study this process independent of its relation with the queueing system and obtain formulas for its distribution. Using these formulas the invariant probability measure of the embedded process is found in terms of generating functions and roots of a certain associated function in the unit disc of the complex plane.

The stationary distribution of the queueing process with continuous time parameter is derived by using semi-regenerative techniques. The authors also obtain the intensity of the input process, service cycle, a formula for the capacity of the system, the distribution of an idle period and the mean busy period. A number of various examples illustrate the results obtained in the article.

2. AN INFORMAL DESCRIPTION OF THE SYSTEM

Let $Q(t)$ give the total number of customers in the system at time t, and let the stochastic sequence $\{T_n;\ n = 0,1,...;\ T_0 = 0\}$ gives the successive instants of time when the server completes his service. Consider the embedded sequence $\{Q(T_n + 0);\ n = 0,1,...\}$ which gives the total number of the customers in the system immediately after a batch of processed units departs from the system. If at time $T_n + 0$ the queue length Q_n is greater than or equal to r, the server takes any group from the queue of a fixed size r and begin processing this group in accordance with an arbitrary distribution function, B_{Q_n}, generally dependent on Q_n. If $Q_n < r$ the server rests as long as the queue needs to accumulate its level to at least r. The server activity is fully restored at the instant of time, say $\mathcal{T}_n$, the queue length reaches or exceed level r. [As mentioned in the introduction, because customers arrive in groups the queue level at time $\mathcal{T}_n$ is more likely to exceed r than to reach it.]

In addition we assume that in the interval $(T_n, T_{n+1}]$ the input flow is a compound Poisson process with parameter $\lambda(Q_n)$. This assumption allows a greater flexibility of the system incorporating a natural reaction of the input flow on the state of the system.

In the next section we will study the behavior of another embedded process $Q(\mathcal{T}_n)$. We will be using some basic results on the first passage problem stated and developed in Abolnikov and Dshalalow [1].

3. PRELIMINARIES ON THE FIRST PASSAGE TIME PROBLEM PROCESSES

First we treat the process $\{Q(\mathcal{T}_n)\}$ without any connection to the queueing system. Because of its "conditionally-regenerative" properties we will study the point process $\mathcal{T}_n$ which can obviously be described by a certain auxiliary integer-valued renewal process $S = \{S_n = X_0 + X_1 + \ldots + X_n;\ n = 0,1,\ldots\}$ whose successive increments $X_1, X_2, \ldots$, give sizes of groups of the input process arriving at the system. Unlike the (usually regular) renewal process S itself, the concrete process $Q(\mathcal{T}_n)$ which S describes, is terminated for some n.

In this section we will discuss the "critical behavior" of a compound Poisson process Z determined by a Poisson process $\tau = \{\tau_n = t_0 + t_1 + \ldots + t_n; n \geq 0,\ t_0 = 0\}$ on $\mathbb{R}_+$ marked by a discrete-valued delayed renewal process $S = \{S_n = X_0 + X_1 + \ldots + X_n; n \geq 0\}$ on $\Psi = \{0,1,\ldots\}$. We assume that the processes τ and S are independent. We also assume that inter-renewal times $t_n = \tau_n - \tau_{n-1}$, are described in terms of its common Laplace-Stieltjes transform $e(\theta) = E[e^{-\theta t_n}] = \frac{\lambda}{\lambda + \theta}$, $n = 1,2,\ldots$.

For convenience we agree to set $\mathcal{T}_n = T_n$, as long as $X_0 = Q_n \geq r$.

For a fixed integer $r \geq 1$ we will be interested in the behavior of the process S and some related processes about level r.

The following terminology is introduced and will be used throughout the paper.

3.1 DEFINITIONS.

(i) For each n the random variable $\nu_n = inf\{k \geq 0: S_k \geq r\}$ is called the *index of the the first excess* (above level $r-1$).

(ii) The random variable S_{ν_n} is called the *level of the first excess* (above $r-1$).

(iii) The random variable τ_{ν_n} is known as the *first passage time* of S of level r.

(iv) The random variable $\mathfrak{I}_n = S_{\nu_n} - S_0$ is called the *increment of the input process over the time interval* $[T_n, \mathcal{T}_n]$, or shortly, the *total increment*.

Let

(3.1a) $$\gamma^{(i)}(\theta,z) = E^i[e^{-\theta\tau_{\nu_0}}z^{\nu_0}],\ \mathcal{G}^{(i)}(\theta,z) = E^i[e^{-\theta\tau_{\nu_0}}z^{S_{\nu_0}}],$$
$$G_i(\theta,z) = \sum_{j\geq 0}E^i[e^{-\theta\tau_j}z^{S_j}I_{U_{r-1}}(S_j)],$$

where $U_p = \{0,1,\ldots,p\}$ and I_A is the indicator function of a set A. We call $G_i(\theta,z)$ the *generator of the first excess level*. We will also use the following functionals of marginal processes:

(3.1b) $$\gamma^{(i)}(z) = \gamma^{(i)}(0,z),$$

(3.1c) $$\mathcal{G}^{(i)}(z) = \mathcal{G}^{(i)}(0,z),$$

(3.1d) $$G_i(z) = G_i(0,z).$$

It is readily seen that $G_i(z)$ is a polynomial of $(r-1)$th degree.

We formulate the main theorems from Abolnikov and Dshalalow [1] and give formulas for the joint distributions of the first passage time and the random variables listed in 3.1 (*i-iii*).

3.2 THEOREM. *The functional $\gamma^{(i)}(\theta,z)$ (of the first passage time and of the index of the first excess level) satisfies the following formula:*

(3.2a) $$\gamma^{(i)}(\theta,z) = \begin{cases} e(\theta)z\mathfrak{D}_x^{r-i-1}\left\{\dfrac{1-a(x)}{(1-x)(1-ze(\theta)a(x))}\right\}, & i<r \\ 1, & i\geq r, \end{cases}$$

where

(3.2b) $$\mathfrak{D}_x^k = \lim_{x\to 0}\frac{1}{k!}\frac{\partial^k}{\partial x^k},\ k\geq 0.$$

Specifically, the Laplace-Stieltjes transform of the first passage time, $\gamma^{(i)}(\theta,1)$, is as follows:

(3.2c) $$\gamma^{(i)}(\theta,1) = \begin{cases} e(\theta)\mathfrak{D}_x^{r-i-1}\left\{\dfrac{1-a(x)}{(1-x)(1-e(\theta)a(x))}\right\}, & i<r \\ 1, & i\geq r. \end{cases}$$

From formula (3.2a) we immediately obtain that the mean value of the index of the first excess equals

(3.3) $$\bar{\gamma}_r^{(i)} = \begin{cases} \mathfrak{D}_x^{r-i-1}\left\{\dfrac{1}{(1-x)[1-a(x)]}\right\}, & i<r \\ 0, & i\geq r. \end{cases}$$

From (3.2a) we also obtain the mean value of the first passage time:

(3.4) $$E^i[\tau_{\nu_0}] = \frac{1}{\lambda}\bar{\gamma}_r^{(i)}.$$

3.5 THEOREM. *The generator $G_i(\theta,z)$ of the first excess level can be determined from the following formula:*

(3.5a) $$G_i(\theta,z) = \begin{cases} z^i\mathfrak{D}_x^{r-i-1}\left\{\dfrac{1}{(1-x)[1-e(\theta)a(xz)]}\right\}, & i<r \\ 0, & i\geq r. \end{cases}$$

The rationale behind the use of the term "generator of the first excess level" comes from the following main result.

3.6 THEOREM. *The functional $\mathcal{G}^{(i)}(\theta,z)$ (of the first passage time and of the first excess level) can be determined from the formula*

(3.6a) $$\mathcal{G}^{(i)}(\theta,z) = z^i - [1-e(\theta)a(z)]G_i(\theta,z).$$

3.7 REMARK. To obtain the functionals of the marginal processes defined in (3.1b-

3.1d) we set $e(\theta) = 1$ in formulas (3.2a), (3.5a) and (3.6a).

3.8 COROLLARY. *The generating function* $\mathcal{G}^{(i)}(z)$ *of the first excess level is determined by the following formula:*

$$\text{(3.8a)} \qquad \mathcal{G}^{(i)}(z) = \begin{cases} z^i \mathfrak{D}_x^{r-i-1}\left\{\dfrac{a(z)-a(xz)}{(1-x)(1-a(xz))}\right\}, & i < r \\ z^i, & i \geq r. \end{cases}$$

By using change of variables in (3.8a) we can transform it into an equivalent expression

$$\text{(3.8b)} \qquad \mathcal{G}^{(i)}(z) = \begin{cases} z^r \mathfrak{D}_x^{r-i-1}\left\{\dfrac{a(z)-a(x)}{(z-x)[1-a(x)]}\right\}, & i < r \\ z^i \qquad\qquad , & i \geq r. \end{cases}$$

3.9 COROLLARY.

$$\text{(3.9a)} \qquad \bar{\mathcal{G}}_r^{(i)} = E^i[S_{\nu_0}] = i + \alpha \bar{\gamma}_r^{(i)},$$

where $\alpha = E[X_1]$.

Specifically, the mean value $\bar{\mathfrak{J}}_r^{(i)} = E^i[\mathfrak{J}_0]$ of the total increment is then

$$\text{(3.9b)} \qquad \bar{\mathfrak{J}}_r^{(i)} = \alpha \bar{\gamma}_r^{(i)}.$$

3.10 REMARK. Now we notice that the above results can be applied to our queueing system, where in formulas (3.2a)-(3.9a) we supply $a(z)$ with subscript i.

4. PROJECTIVE OPERATORS TECHNIQUES

In this section a formula for the generating function of the process $\{S_{\nu_n}\}$ will be derived in another form.

Let f be an analytic function in the annulus $A(0,0,1) = \{z \in \mathbb{C}: 0 < \|z\| < 1\}$ and continuous on the boundary of the unit disc, $\Gamma = \{\|z\| = 1\}$. Then f equals its Laurent series $\mathcal{L}f$ for all points $z \in A(0,0,1)$. Denote T^+f the tame part and T^-f the principal part of that Laurent series. We mention a few properties of the operators T^+ and T^-:

[$P1$] If f is analytic in $\Gamma^+ = B(0,1) = \{z \in \mathbb{C}: \|z\| < 1\}$ and continuous on Γ then

$$T^+f(z) = f(z) \text{ and } T^-f(z) = 0,\ z \in \bar{B}(0,1) = \{z \in \mathbb{C}: \|z\| \leq 1\}.$$

[$P2$] If f is analytic in $\Gamma^- = \{z \in \mathbb{C}: \|z\| > 1\}$ and continuous on Γ then

$$T^+f(z) = f(\infty) \text{ and } T^-f(z) = f(z) - f(\infty),\ z \in \Gamma \cup \Gamma^-.$$

[$P3$] $T^+T^- = T^-T^+ = \Theta$ (zero operator), $T^+T^+ = T^+$, $T^-T^- = T^-$.

[$P4$] If $f(z) = \sum_{n=0}^{\infty} f_n z^n$ then

$$(P4a) \qquad \sum_{k=0}^{r-1} f_k z^k = z^r T^- z^{-r} f(z)$$

$$(P4b) \qquad \sum_{k \geq r} f_k z^k = z^r T^+ z^{-r} f(z)$$

[$P5$] Let X be a random variable valued in $\mathbb{N}_0$ such that $E[z^X] = a(z)$ and let $A = \{0,1,\ldots,r-1\}$. Denote $A^c = \{r, r+1, \ldots\}$. Then from [$P4$] it follows that

$$(P5a) \qquad E[z^X I_A(X)] = z^r T^- z^{-r} a(z)$$

$$(P5b) \qquad E[z^X I_{A^c}(X)] = z^r T^+ z^{-r} a(z)$$

where I_A denotes the indicator function of set A and the operator $(f)^+$ denotes $sup\{f,0\}$.

Note that applications of the operators T^+ and T^- are especially useful when the function f is continuous on Γ and meromorphically extendible to Γ^+ or Γ^- (*cf*. Dukhovny [8]).

Now let us return to the process S introduced in section 3.

4.1 PROPOSITION. *The generating function* $\mathcal{G}_r^{(i)}(z)$ *is determined by the following formula:*

(4.1a)
$$\mathcal{G}_r^{(i)}(z) = z^i T^+ a_i(z) T^- \left\{ z^{i-r} \frac{1 - a_i^{r-i}(z)}{1 - a_i(z)} \right\}, \ i < r .$$

PROOF. Applying $(P5a)$ and $(P5b)$ we obtain

(4.1b)
$$E^i[z^{S_{k+1,0}} I_A(S_{k0}) I_{A^c}(S_{k+1,0})] = z^r T^+ a_i(z) T^- z^{i-r} a_i^k(z).$$

Now (4.1a) follows from (4.1b) and the fact that

$$\mathcal{G}_r^{(i)}(z) = \sum_{k=1}^{r-i} E^i[z^{S_{k0}} I_A(S_{k-1,0}) I_{A^c}(S_{k0})] = \sum_{k=0}^{r-i-1} z^i T^+ a_i(z) T^- z^{i-r} a_i^k(z). \qquad \square$$

(AS.1) We additionally assume that the greatest common divisor of all values of X_{1,Q_0} (taken with positive probability) is 1. [Observe that a bulk input with constant size of batches will be excluded from this class.]

The following result gives an elegant computational formula for $\mathcal{G}_r^{(i)}$ for a broad class of special cases restricted by (AS.1).

4.2 COROLLARY. *Let* (X) *be as in assumption* $(AS.1)$. *Then formula* (4.1a) *is reduced to one of the two equivalent expressions below*

(4.2a)
$$\mathcal{G}_r^{(i)}(z) = z^i + z^i(1 - a_i(z)) T^- \left\{ \frac{1 - z^{i-r}}{1 - a_i(z)} \right\}, \ i < r,$$

(4.2b)
$$\mathcal{G}_r^{(i)}(z) = z^r + z^r(1 - a_i(z)) T^+ \left\{ \frac{z^{i-r} - 1}{1 - a_i(z)} \right\}, \ i < r .$$

PROOF. From (4.1a) we have

$$\mathcal{G}_r^{(i)}(z) = z^r T^+ a_i(z) T^- \left\{ z^{i-r} \frac{1 - a_i^{r-i}(z)}{1 - a_i(z)} \right\} = z^r T^+ a_i(z) T^- \left\{ \frac{z^{i-r} - 1}{1 - a_i(z)} + \frac{1 - z^{i-r} a_i^{r-i}(z)}{1 - a_i(z)} \right\}.$$

Under assumption (AS.1), it follows from [2] (for $m = 0$) that the only root of the function $1 - a_i(z)$ in $\bar{B}(0,1)$ is 1. Therefore, $\dfrac{1 - z^{i-r} a_i^{r-i}(z)}{1 - a_i(z)}$ is analytic in Γ^+ and continuous on Γ. Hence we obtain from $[P1]$ that

(4.2c)
$$\mathcal{G}_r^{(i)}(z) = z^r T^+ a_i(z) T^- \left\{ \frac{z^{i-r} - 1}{1 - a_i(z)} \right\}.$$

Now the statement of the corollary follows from (4.2c) and properties $[P1$-$P4]$. $\square$

5. FORMAL DESCRIPTION OF THE SYSTEM

We begin this section with definition a modulated process introduced and studied in Dshalalow [6]. All stochastic processes below will be considered on a probability space $\{\Omega, \mathcal{F}, (P^x)_{x \in \Psi}\}$, with $\Psi = \{0,1,\ldots\}$.

Let $\{T_n;\ n = 0,1,\ldots;\ T_0 = 0\}$ be a point process and let $\xi(t)$ be an integer-valued jump process with successive jumps at T_n (we allow $\xi(T_n) = \xi(T_{n+1})$ with positive probability). Let $\{\tau_k;\ k \in \mathbb{N}\}$ be a non-stationary orderly Poisson point process with intensity function $\lambda(t)$. Then we call the doubly stochastic Poisson point process with intensity $\lambda(\xi(t))$ the *Poisson process modulated by the jump process* $\{\xi(t)\}$ and it is denoted by $\{\tau_n^\xi\}$. Denote $N^\xi(\cdot)$ the associated counting measure. A compound modulated process is defined as follows. Let $\{X\} = \{X_{1\xi(t)}, X_{2\xi(t)},\ldots\}$ be a doubly stochastic sequence of random variables such that given a fixed value of $\xi(t)$ $\{X\}$ is a sequence of independent identically distributed random variables. Then the *compound Poisson modulated process* is defined as $Z^{\xi}(t) = \sum_{i=1}^{N^{\xi}(t)} X_{i\xi(t)}$.

Let $\{Q(t): t \geq 0\} \to \Psi = \{0,1,...\}$ be a stochastic process describing the number of units at time t in a single-server queueing system with an infinite waiting room. Following the introduction, $\{T_n; n \in \mathbb{N}_0, T_0 = 0\}$ is the sequence of successive completions of service and $Q_n = Q(T_n + 0)$.

INPUT. Let $C(\cdot)$ be the counting measure associated with the point process $\{T_n\}$. Define $\xi(t) = Q(T_{C(t)} + 0)$, $t \geq 0$. Then the input is a compound Poisson process *modulated by* $\{\xi(t)\}$ according to the above definition.

Assume that the input is a compound Poisson process modulated by $\{\xi(t)\}$, where $X_{i\xi(t)}$ is ith batch size of the input flow which depends on $\xi(t)$. Thus, in our case $\{X\}$ is an integer-valued doubly stochastic process describing the sizes of groups of entering units. We denote $a_{\xi(t)}(z) = E[z^{X_{i\xi(t)}}]$, $i = 1,2,...$, the generating function of ith component of the process $\{X\}$.

SERVICE TIME AND SERVICE DISCIPLINE. At time $T_n + 0$ the server takes a batch of units of size r from the queue and serves it during a random length of time σ_{n+1} if the queue length Q_n is at least r. Otherwise, the server idles until the queue length for the first time reaches or exceeds the level r. Let $\gamma_n = inf\{k \in \mathbb{N}: \tau_k^{\xi} \geq T_n\}$, $n \in \mathbb{N}_0$. Then the size of the first group after T_n (which arrives at the instant of time $\tau_{\gamma_n}^{\xi}$) is X_{γ_n, Q_n}. For a convenience in notation, we reset the first index-counter of the process $\{X\}$ on 1 after time t hits T_n. Therefore, in the light of the new notation, $X_{1Q_n}, X_{2Q_n},...$ will denote the sizes of successive groups of units arriving at the system past time T_n. Let $S_{kn} = X_{0Q_n} + X_{1Q_n} + ... + X_{kQ_n}$, where $X_{0Q_n} = Q_n$. Then, given Q_n, $\{S_{kn};\ k \in \mathbb{N}_0\}$ is an integer-valued delayed renewal process. Denote $\nu_n = inf\{k \geq 0:\ S_{kn} \geq r\}$ the random index when the process $\{S_{kn}\}$ first reaches or exceeds level r given that the queue length is Q_n. If $Q_n \geq r$, $T_{n+1} - T_n$ coincides with length of service time σ_{n+1} of the $n+1$st batch. If $Q_n < r$ the interval $(T_n, T_{n+1}]$ contains the waiting time for $X_{1Q_n} + ... + X_{\nu_n Q_n}$ units to arrive and the actual service time σ_{n+1}. In both cases we assume that σ_{n+1} has a probability distribution function $B_{Q_n} \in \{B_0, B_1,...\}$, where the latter is a given sequence of arbitrary distribution functions with finite means $\{b_0, b_1,...\}$.

Finally, denoting $V_n = Z^{\xi}(\sigma_n)$ we obtain the following relation for process $\{Q_n\}$:

(5.1)
$$Q_{n+1} = \begin{cases} S_{\nu_n} - r + V_{n+1}, & Q_n < r \\ Q_n - r + V_{n+1}, & Q_n \geq r. \end{cases}$$

6. EMBEDDED PROCESS

From relation (6.1) and the nature of the input process it follows that the process $\{\Omega, \mathcal{F}, (P^x)_{x\in\Psi}, Q(t); t \geq 0\} \to \Psi$ has at T_n, $n \geq 1$, the locally strong Markov property (see definition A.3 in Appendix) and that $\{\Omega, \mathcal{F}, (P^x)_{x\in\Psi}, Q_n; n \in \mathbb{N}_0\} \to \Psi$ is a homogeneous Markov chain with transition probability matrix $A = (a_{ij})$. Let $A_i(z)$ denotes the generating function of ith row of matrix A. Since $A_i(z) = E^i[z^{Q_1}]$ we obtain from (6.1) that

(6.1)
$$A_i(z) = g_i(z) z^{-r} \mathcal{G}_r^{(i)}(z),\ i \in \Psi,$$
where

(6.1a)
$$g_i(z) = \beta_i(\lambda_i - \lambda_i a_i(z)),$$

$\beta_i(\theta)$, $Re(\theta) \geq 0$, is the Laplace-Stieltjes transform of the probability distribution function B_i.

For analytical convenience and without considerable loss of generality we can drop the modulation of the input process and service control when the queue length exceeds a fixed (perhaps very large) level N. In other words, we assume that

(AS2) $B_i(x) = B(x), \beta_i(\theta) = \beta(\theta), g_i(z) = g(z), b_i = b, \lambda(i) = \lambda_i = \lambda, a_i(z) = a(z), a_i = a, i > N,$ $N \geq r-1$, where $\alpha_i = a_i'(1)$, $i \in \Psi$.

Given assumption (AS2), it can be shown that the transition probability matrix A is reduced to a form of the $\Delta_{r,N}$-matrix introduced and studied in [2]. According to theorem A.1 (by Abolnikov and Dukhovny, see Appendix), the Markov chain $\{Q_n\}$ is recurrent-positive if and only if

(6.2) $$\frac{d}{dz} A_i(z)\Big|_{z=1} < \infty, \; i = 0,1,\ldots,N,$$

and

(6.2a) $$\frac{d}{dz} g(z)\Big|_{z=1} < r.$$

Condition (6.2) is obviously met and condition (6.2a) is equivalent to

(6.2b) $$\rho = \lambda\alpha b < r.$$

Therefore, given that $\rho < r$, the Markov chain $\{Q_n\}$ is ergodic. Let $P = (p_x; x \in \Psi)$ be the invariant probability measure of operator A and let $P(z)$ be the generating function of the components of vector P. Denote $\bar{\Gamma}^+ = \{z \in \mathbb{C}: \| z \| \leq 1\}$. Now we formulate the main result of this section.

6.3 THEOREM. *The embedded queueing process* $\{Q_n\}$ *is ergodic if and only if* $\rho < r$. *Under this condition,* $P(z)$ *is determined by the following formula:*

(6.3a) $$P(z) = \frac{\sum_{i=0}^{N} p_i\left\{g_i(z)\mathfrak{G}_r^{(i)}(z) - z^i g(z)\right\}}{z^r - g(z)},$$

Probabilities $p_0, \ldots, p_N$ *form the unique solution of the following system of linear equations:*

(6.3b) $$\sum_{i=0}^{N} p_i \frac{d^k}{dz^k}\{z^{-r}g_i(z)\mathfrak{G}_r^{(i)}(z) - z^i\}\Big|_{z=z_s} = 0, \; k = 0,\ldots,k_s - 1, \; s = 1,\ldots,S,$$

(6.3c) $$\sum_{i=0}^{N} p_i[\rho_i - \rho + \bar{G}_r^{(i)}] = r - \rho,$$

where z_s *are the roots of* $z^{N+1} - z^{N+1-r}g(z)$ *in the region* $\bar{\Gamma}^+\backslash\{1\}$ *with their multiplicities* k_s *such that* $\sum_{s=1}^{S} k_s = N$, *and where*

(6.3d) $$\rho_i = \lambda_i\alpha_i b_i, \qquad \textit{and}$$

(6.3e) $$\bar{G}_r^{(i)} = \begin{cases} \frac{d}{dz}\mathfrak{G}_r^{(i)}(z)\Big|_{z=1} - i, & i < r \\ 0, & i \geq r \end{cases}$$

PROOF. Formula (6.3a) follows from $P(z) = \sum_{i \in \Psi} p_i A_i(z)$ and (6.1-6.1a). It is easy to modify formula (6.3a) into

(6.3f) $$\sum_{i=N+1}^{\infty} p_i z^{i-N-1} = \frac{\sum_{i=0}^{N} p_i\{g_i(z)z^{-r}\mathfrak{G}_r^{(i)}(z) - z^i\}}{z^{N+1} - z^{N+1-r}g(z)},$$

so that the function in the left-hand side of (6.3f) is analytic in $\Gamma^+ = \{z \in \mathbb{C}: \| z \| < 1\}$ and continuous on $\Gamma = \partial\Gamma^+$. According to theorem A.2, for $\rho < r$, the function $z \mapsto z^r - g(z)$ has exactly r zeros in $\bar{\Gamma}^+$ (counting with their multiplicities); all zeros located on the boundary Γ (including 1), are simple. Therefore, the denominator in the right-hand side of (6.3f) must have exactly N roots in the region $\bar{\Gamma}^+\backslash\{1\}$. This fact along with $(P,1) = 1$ (which yields (6.3c)) leads to equations (6.3b-6.3d).

The uniqueness of $p = \{p_0,\ldots,p_N\}$ follows from the following considerations. Suppose that the system of equations (6.3b-6.3e) has another solution $p^* = \{p_i^*;\ i = 0,...,N\}$ which we substitute into (6.3a) to obtain the generating function $P^*(z)$. Then, $P^*(z)$ is analytic in Γ^+ and continuous on Γ. Therefore, $P^* = \{p_i^*; i \in \Psi\} \in (l^1, \|\cdot\|)$. Obviously, equations $P^*(z) = \sum_{i\in\Psi} p_i^* A_i(z)$ and $P^*(z) = \dfrac{\sum_{i=0}^N p_i^*\left\{g_i(z)\mathfrak{G}_r^{(i)}(z) - z^i g(z)\right\}}{z^r - g(z)}$ are equivalent. The last equation is also equivalent to $P^* = P^*A$. Since p^* satisfies (6.3c) it follows that $(P^*,1) = 1$. Thus, the system of equations $x = xA$, $(x,1) = 1$ has two different solutions from $(l^1, \|\cdot\|)$ which is impossible (*cf.* Gihman and Skorohod [9], theorem 15, p. 108). □

6.4 LEMMA. *The expected number of units* $\bar{\mathfrak{I}}_r^{(i)} = \bar{\mathfrak{G}}_r^{(i)} - i$ *that arrive during an idle period (of the server) started with the queue length equals i (defined in formula* (6.3e)) *can be represented in the form*

$$\bar{\mathfrak{I}}_r^{(i)} = \alpha_i \bar{g}_i^{(r)},\ i \in \Psi. \tag{6.4a}$$

PROOF. Lemma 6.4 is just another interpretation of formula (3.9b). □

6.5 REMARK. Consider the following auxiliary process. Let $\{\Omega, \mathfrak{F}, (P^x)_{x\in\Psi}, Y_n;\ n \in \mathbb{N}_0\} \to \Psi = \mathbb{N}_0$ (abbreviated $\{Y\}$) be a homogeneous irreducible and aperiodic Markov chain with transition probability matrix $A = \{a_{ij};\ i,j \in \Psi\}$ and the generating function $A_i(z)$ of ith row of A. Assume that $A_i(z) = z^{(i-r)^+} g(z)$, $i \in \Psi$, where g is analytic in Γ^+ and continuous on Γ. Then, according to Abolnikov and Dukhovny [2], the process $\{Y\}$ is ergodic if and only if $g'(1) < r$. Let $\delta = (\delta_x; x \in \Psi)$ be the invariant probability measure of operator A and let $\delta(z)$ be the generating function of δ. Then $\delta(z) = \sum_{i\in\Psi} \delta_i A_i(z)$ and after some algebraic transformations we get

$$\delta(z) = g(z)R(z)[z^r - g(z)]^{-1}, \text{ where} \tag{6.5a}$$

$$R(z) = \sum_{i=0}^{r-1} \delta_i (z^r - z^i). \tag{6.5b}$$

Modifying (6.5a) we obtain

$$\sum_{i=0}^{\infty} \delta_i z^{(i-r)^+} = \frac{R(z)}{z^r - g(z)}. \tag{6.5c}$$

According to theorem A.2 (in Appendix), the function $z \mapsto z^r - g(z)$ has exactly r roots (counted with their multiplicities) in $\bar{\Gamma}^+$. Since the function in the left-hand side of (6.5c) is analytic in Γ^+ and continuous on Γ the polynomial $R(z)$ should have the same r roots (given that $g'(1) < r$). Taking into account that the left-hand side of (6.5c) equals 1 for $z = 1$, we get

$$R'(1) = \sum_{i=0}^{r-1} \delta_i (r - i) = r - g'(1). \tag{6.5d}$$

The polynomial R, which determines the generating function $\delta(z)$, can uniquely be restored after we find all roots of $z^r - g(z)$ in $\bar{\Gamma}^+$ and satisfy (6.5d). In general, the roots can be evaluated numerically (in some cases also analytically). Alternatively, the roots, and therefore the polynomial R, can be obtained from computer simulation of the Markov chain $\{Y\}$.

7. APPLICATIONS AND EXAMPLES

We now will consider several special cases of our system. All of them can be analyzed with the help of the general theorem 6.3 and by using appropriate numerical methods. However, in the particular cases considered below, it turns out to be possible to develop a more direct approach and thereby obtain more convenient formulas for generating functions of the corresponding queueing processes. In the first case we drop the modulation of the input process and service control, thereby assuming that $\lambda_i = \lambda$, $a_i(z) = a(z)$, and $B_i = B, i \in \Psi$. The following proposition states that in this case the generating function $P(z)$ can be expressed in terms of the stationary probabilities for the auxiliary Markov chain $\{Y\}$ introduced in 6.5.

7.1 PROPOSITION. *Given that $\rho < r$, the generating function $P(z)$ of the invariant probability measure P of operator A for the embedded process $\{Q_n\}$ in a bulk queueing system with queue length dependent service delay and without modulation and service control, can be determined from the following formula:*

$$P(z) = \frac{(1-a(z))g(z)R(z)}{\alpha(1-z)(z^r - g(z))} = \frac{1-a(z)}{\alpha(1-z)}\delta(z), \tag{7.1a}$$

where $\delta(z)$ is found by formula (6.1) *and $R(z)$ is defined in* (6.2).

PROOF. In the assumptions of the proposition, formula (6.3a) for $P(z)$ reduces to

$$P(z) = \frac{g(z)\sum_{i=0}^{r-1} p_i\{\mathfrak{G}_r^{(i)}(z) - z^i\}}{z^r - g(z)}, \tag{7.1b}$$

which due to (3.6a) leads to

$$P(z) = \frac{g(z)(1-a(z))}{\alpha(1-z)} \; \frac{\alpha(z-1)\sum_{i=0}^{r-1} p_i\, G_{ir}(z)}{z^r - g(z)}. \tag{7.1c}$$

The first factor $z \mapsto \dfrac{g(z)[1-a(z)]}{a(1-z)}$ in (6.6c) has the following obvious properties: It is analy-tic in Γ^+ and continuous on Γ, it is valued 1 at $z = 1$ and it does not vanish at all roots of $z^r - g(z)$ inside $\bar{\Gamma}^+$. The latter property and analyticity of $P(z)$ yield that the function

$$z \mapsto \frac{\alpha(z-1)\sum_{i=0}^{r-1} p_i G_{ir}(z)}{z^r - g(z)} \tag{7.1d}$$

is analytic in Γ^+ and continuous on Γ and it takes on value 1 at $z = 1$. Thus, the numerator of (7.1d) must have the same roots with their multiplicities as the denominator in Γ^+ and it assumes value $r - g'(1)$ at $z = 1$. Since $G_{ir}(z)$ is a $(r-1)$th degree polynomial the numerator of (7.1d) is a rth degree polynomial. Therefore, from the considerations in 6.5 it follows that the numerator of (7.1d) must be equal to $R(z)$ defined in (6.5b). □

In the second example we assume that $B_i(x) = F * B(x)$, $i < r$, $B_i(x) = B(x)$, $i \geq r$, and $\lambda_i = \lambda$, $a_i(z) = a(z)$, $i \in \Psi$. Let $\varphi(\theta)$ denote the Laplace-Stieltjes transform of the (arbitrary) probability distribution function F and let $H(z) = \varphi(\lambda - \lambda a(z))$.

7.2 PROPOSITION. *Under the above assumptions, the generating function $P(z)$ of the invariant probability measure P of operator A is determined by the expression*

$$P(z) = g(z)\frac{1-a(z)}{\alpha(1-z)} \; \frac{L(z)R(z)}{z^r - g(z)}, \tag{7.2a}$$

where the function L is an analytic in Γ^+ and continuous on Γ, and $L(1) = 1$.

PROOF. In this case formula for $P(z)$ yields

$$P(z) = g(z)\frac{1-a(z)}{\alpha(1-z)} \; \frac{\alpha(z-1)\sum_{i=0}^{r-1} p_i\left(\dfrac{1-H(z)}{1-a(z)} z^i + H(z)\, G_{ir}(z)\right)}{z^r - g(z)}.$$

Repeating the same arguments as in proposition 7.1 we deduce that the function

$$L(z) = \frac{1}{R(z)}\, \alpha(z-1)\sum_{i=0}^{r-1} p_i\left(\frac{1-H(z)}{1-a(z)} z^i + H(z)\, G_{ir}(z)\right) \tag{7.2b}$$

is analytic in Γ^+, continuous on Γ and equals 1 at $z = 1$. □

7.3 EXAMPLE. Preserving all assumptions in proposition 6.7, we additionally assume that F is an exponential distribution with parameter γ. Then L in (7.2b) is

$$L(z) = \frac{\gamma}{\gamma + \lambda - \lambda a(z)} \, \frac{1}{R(z)} \left\{ \alpha(z-1)\sum_{i=0}^{r-1} p_i\left(G_{ir}(z) + \lambda\gamma^{-1} z^i\right)\right\}. \tag{7.3a}$$

Following the arguments of the proof of proposition 7.1, we conclude that the expression in braces equals $R(z)$, thereby reducing $L(z)$ to $\gamma(\gamma+\lambda-\lambda a(z))^{-1}$. Replacing $g(z)R(z)(z^r-g(z))^{-1}$ by $\delta(z)$ in (7.1a) we finally obtain that

$$P(z)=\delta(z)\frac{\gamma[1-a(z)]}{\alpha(1-z)[\gamma+\lambda-\lambda a(z)]}. \tag{7.3b}$$

A special case with an orderly modulated Poisson input process was studied earlier by Dshalalow and Russell [7]. In other words, we assume that $a_j(z)=z$, otherwise retaining all other assumptions made in (AS2). Then, the version of theorem 6.3 can be formulated as follows.

7.4 COROLLARY. *The embedded queueing process* $\{Q_n\}$ *is ergodic if and only if* $\rho<r$. *Under this condition,* $P(z)$ *is determined from the following formula*

$$P(z)=\frac{\sum_{i=0}^{N}p_i\left\{z^r\hat{g}_i(z)-z^ig(z)\right\}}{z^r-g(z)}, \tag{7.4a}$$

where

$$\hat{g}_i(z)=z^{(i-r)^+}g_i(z), \tag{7.4b}$$

and operator $(f)^+=\sup\{f,0\}$. *Probabilities* $p_0,\dots,p_N$ *form the unique solution of the following system of linear equations:*

$$\sum_{i=0}^{N}p_i\frac{d^k}{dz^k}\{\hat{g}_i(z)-z^i\}\Big|_{z=z_s}=0,\ k=0,\dots,k_s-1,\ s=1,\dots,S, \tag{7.4c}$$

$$\sum_{i=0}^{N}p_i[(\rho_i-\rho)+(r-i)^+]=r-\rho, \tag{7.4d}$$

where $\rho_i=\lambda_ib_i$ *and* z_s *are the roots of* $z^{N+1}-z^{N+1-r}g(z)$ *in the region* $\bar{\Gamma}^+\backslash\{1\}$ *with their multiplicities* k_s *such that* $\sum_{s=1}^{S}k_s=N$.

8. ANALYSIS OF THE QUEUEING PROCESS WITH CONTINUOUS TIME PARAMETER

In this section our main objective is the stationary distribution of the queueing process with continuous time parameter.

8.1 DEFINITIONS.

(*i*) Let $\beta_j=E^j[T_1]$ (the mean sojourn time of the process $\{\xi(t)\}$ in state $\{j\}$) and $\beta=(\beta_j;j\in\Psi)^{\mathrm{T}}$. Then we will call the value $\boldsymbol{P\beta}$ the *mean service cycle of the system*, where $\boldsymbol{P}$ denotes the stationary probability distribution vector of the embedded queueing process $\{Q_n\}$.

(*ii*) Let $\boldsymbol{\alpha}=(\alpha_x;x\in\Psi)^{\mathrm{T}}$, $\boldsymbol{\lambda}=(\lambda_x;x\in\Psi)^{\mathrm{T}}$ and let $\boldsymbol{\rho}=\boldsymbol{\alpha}*\boldsymbol{\beta}*\boldsymbol{\lambda}$ be the Hadamard (entry-wise) product of vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$. We call the scalar product $\boldsymbol{P\rho}$ the *intensity of the system.*

Observe that the notion of the "intensity of the system" (frequently called the *offered load* in queueing theory) goes back to the classical $M/G/1$ system, when $\boldsymbol{P\rho}$ reduces to $\rho=\lambda b$. It is noteworthy that in many systems the intensity of the system and the server capacity coincide.

8.2 PROPOSITION. *Given the equilibrium condition* $\rho<r$, *the mean service cycle can be determined from the following expression:*

$$\boldsymbol{P\beta}=b+\sum_{j=0}^{N}p_j(b_j-b+\tfrac{1}{\lambda_j}\bar{\gamma}_r^{(j)}) \tag{8.2a}$$

PROOF. Obviously, $\beta_j=b_j+\bar{\gamma}_r^{(j)}/\lambda_j$. The statement follows after elementary algebraic transformations. □

8.3 THEOREM. *Given the equilibrium condition* $\rho<r$, *the intensity of the system* $\boldsymbol{P\rho}$

and server capacity coincide and equal r.

PROOF. According to the description of our model the server capacity is r. Now, the statement of the theorem follows from definition 8.1 (*ii*), equation (6.3c) and lemma 6.4. □

From the discussion in section 3 and from definition A.4, it follows that $\{\Omega, \mathfrak{F}, (P^x)_{x \in \Psi}, Q(t);\ t \geq 0\} \rightarrow (\Psi, \mathfrak{B}(\Psi))$ is a semi-regenerative process with conditional regenerations at points T_n, $n = 0,1,\ldots$, $T_0 = 0$. $\{\Omega, \mathfrak{F}, (P^x)_{x \in \Psi}, (Q_n, T_n):\ n = 0,1,\ldots\} \rightarrow (\Psi \times \mathbb{R}_+, \mathfrak{B}(\Psi \times \mathbb{R}_+))$ is the associated Markov renewal process. Let $\mathcal{I}(t)$ denote the corresponding semi-Markov kernel. Under a very mild restriction to the probability distribution functions B_i, we can assume that the elements of $\mathcal{I}(t)$ are not step functions which would imply that $\{Q_n, T_n\}$ is aperiodic. By proposition 8.2, the mean service cycle $P\beta$, which is also the mean inter-renewal time of the Markov renewal process, is obviously finite. Therefore, following definition A.5, the Markov renewal process is ergodic given the condition $\rho < r$.

It also follows that the jump process $\{\Omega, \mathfrak{F}, (P^x)_{x \in \Psi}, \xi(t);\ t \geq 0\} \rightarrow \Psi$, defined in section 5, is the minimal semi-Markov process associated with Markov renewal process $\{Q_n, T_n\}$ and therefore, following the definition at the beginning of section 5, the input process $\{\Omega, \mathfrak{F}, (P^x)_{x \in \Psi}, Z^{\xi}([0,t]);\ t \geq 0\} \rightarrow \Psi$ is a compound Poisson process modulated by the semi-Markov process ξ.

8.4 DEFINITION. Let

$$\delta_{xs}(t) = P^x\{Z^{\xi}([0,t]) = s,\ T_1 > t\} \tag{8.4a}$$

Then, given that $\xi(0) = x$ and that $Z^{\xi}(t)$ is not modulated by a new value of $\xi(t)$, the input process takes on the value

$$Z^x([0,t]) = \sum_{i=1}^{N^x(t)} X_{ix}, \tag{8.4b}$$

where $N^x(\cdot)$ is the Poisson counting measure with parameter λ_x. We call $\{Z^x([0,t])\}$ defined in (8.4b) the *x-partial process* (*of the compound Poisson process modulated by* ξ).

Therefore, by (8.4b),

$$\delta_{xs}(t) = P\{Z^x([0,t]) = s\}. \tag{8.4c}$$

Let $K(t) = (K_{jk}(t); j, k \in \Psi)$ be the semi-regenerative kernel (see definition A.6). The following proposition obviously holds true.

8.5 PROPOSITION. *The semi-regenerative kernel satisfies the following equations:*

(8.5a)

$$K_{jk}(t) = \delta_{j,k-j}(t), 0 \leq j \leq k < r,$$

$$K_{jk}(t) = \sum_{s=r-j}^{k-j} \int_0^t q_j(s+j, t-u)\, \delta_{j,k-j-s}(u)(1 - B_j(u))\, du, 0 \leq j < r \leq k$$

$$K_{jk}(t) = \delta_{j,k-j}(t)[1 - B_j(t)], r \leq j \leq k$$

$$K_{jk}(t) = 0,\ 0 \leq k < j,$$

where $\delta_{j,k}$ *is as defined in* (8.4a) *or* (8.4c) *and* q_j *denotes the density of the joint probability distribution function of the random variable* S_{ν_0} *and the first passage time* $\mathfrak{T}_1$ *of the first excess above level r by the queueing process* $\{Q(t)\}$.

Now we are ready to apply the Main Convergence Theorem to the semi-regenerative kernel in the form of corollary A.8.

8.6 THEOREM. *Given the equilibrium condition* $\rho < r$ *for the embedded process* $\{Q_n\}$, *the stationary distribution* $\boldsymbol{\pi} = (\pi_x; x \in \Psi)$ *of the queueing process* $\{Q(t)\}$ *exists; it is independent of any initial distribution and is expressed in terms of the generating function* $\pi(z)$ *of* $\boldsymbol{\pi}$ *by the following formulas:*

(8.6a) $$P\beta\pi(z) = d(z)P(z) + \sum_{i=0}^{N} p_i\Big[\tfrac{1}{\lambda_i}K_i(z)G_{ir}(z) + z^i(d_i(z) - d(z))\Big],\ \textit{with}$$

(8.6b) $$d_j(z) = \frac{1 - K_j(z)}{\lambda_j\big(1 - a_j(z)\big)},$$

where $P(z)$ *is the generating function of* $\boldsymbol{P}$, $P\beta$ *is determined in proposition* 8.2, $G_{ir}(z)$ *is determined in* (3.5a), *and* $d(z)$ *is defined as* $d_j(z)$ *with all subscripts dropped.*

PROOF. Recall that the Markov renewal process $\{Q_n, T_n\}$ is ergodic if $\rho < r$. By corollary A.8 the semi-regenerative process $\{Q(t)\}$ has a unique stationary distribution $\boldsymbol{\pi}$ provided that $\rho < r$. From (8.5a) we can see that the semi-regenerative kernel is Riemann integrable over $\mathbb{R}_+$. Thus, following corollary A.8 we need to find the integrated semi-regenerative kernel H (which is done with routine calculus) and then generating functions $h_j(z)$ of all rows of H. First we find that

(8.6c) $$\sum_{p=j}^{\infty} z^p \int_0^{\infty} \delta_{j,p-j}(u)[1 - B_j(u)]\,du = z^j d_j(z)\ .$$

Then it follows that

(8.6d) $$h_j(z) = z^j \tfrac{1}{\lambda_j}\mathfrak{D}_x^{r-j-1}\Big\{\frac{1}{(1-x)(1-a_j(xz))}\Big\} + d_j(z)G_{jr}^{(r)}(z),\ 0 \le j < r\ ,$$

where $G_{jr}^{(r)}(z)$ denotes the tail of the generating function $G_{ir}(z)$ summing its terms from r to ∞. However, it is easy to show that $G_{jr}^{(r)}(z)$ and $G_{ir}(z)$ coincide. Then it appears that

(8.6e) $$h_j(z) = z^j d_j(z)\,,\ j \ge r,$$

where the index j can be dropped for all j exceeding N, in accordance with assumption (AS2) made in section 4. Formula (8.6a) now follows from corollary A.8, equations (8.6c-8.6e), (3.5a), (3.6a), remark 3.10, and some algebraic transformations. □

The following corollary (which follows from (8.3a), (8.4a), (8.2a) and (8.5a) by means of routine calculus) gives an elegant formula for the service cycle $P\beta$ and the generating function $\pi(z)$ if we just drop the modulation of the input but retain bulks of the input, service control and state dependent service delay.

8.7 COROLLARY. *The service cycle* $P\beta$ *and the generating function* $\pi(z)$ *of* $\boldsymbol{\pi}$ *in the queueing system with no modulation of the input can be determined from the following formulas:*

(8.7a) $$P\beta = \frac{r}{\lambda\alpha}\ .$$

(8.7a) $$\pi(z) = \frac{\alpha(1 - z^r)P(z)}{r[1 - a(z)]}\ .$$

8.8 EXAMPLES.

(i) Observe that the same result as (8.7a) holds true by retaining a "weak modulation", i.e. assuming that $\lambda_i = \lambda$ and $\alpha_i = \alpha$ but having no further restriction to the generating functions $a_i(z)$.

(ii) Assuming that the input is an orderly modulated Poisson process, in other words if $a_j(z) = z$, but retaining all other assumptions we arrive at the result by Dshalalow and Russell [7]. Indeed, h_j is reduced to

$$h_j(z)\tfrac{1}{\lambda_j}(1 - z)z^{-j} = 1 - \hat{g}_j(z),$$

with $\hat{g}_j(z) = z^{(j-i)^+} g_j(z)$. Then we obtain from (8.6a) that

$$\pi(z) = \frac{\frac{1}{\lambda}[1 - g(z)]P(z) + \sum_{j=0}^{N} p_j\left[z^j\left(\frac{1}{\lambda_j} - \frac{1}{\lambda}\right) - \left(\frac{1}{\lambda_j} z^r \hat{g}_j(z) - \frac{1}{\lambda} z^j g(z)\right)\right]}{P\beta\,(1-z)},$$

$$P\beta = b + \sum_{j=0}^{N} p_j[(b_j - b) + \tfrac{1}{\lambda_i}(r - i)^+]\,,$$

where $P(z)$ and $p_0,\ldots,p_N$ satisfy equations (8.9a-8.9d).

By dropping the bulk of the input process we obtain from (8.7a) that $P\beta = \frac{r}{\lambda}$ and from (8.7a) that

$$\pi(z) = \frac{1 - z^r}{r(1-z)} P(z)\,.$$

(*iii*) Now we will obtain a few results in connection with special cases treated in section 7. By further dropping service control in the condition of corollary 8.7 and using (8.6a) we get from (7.1a)

$$r\pi(z) = \frac{1 - z^r}{1 - z}\delta(z)\,.$$

By retaining some vague service control in the light of corollary 8.7 formula (7.2b) then yields

$$r\,\pi(z) = \frac{1 - z^r}{1 - z} L(z)\delta(z)\,,$$

which reduces to

$$r(1-z)(\gamma + \lambda - \lambda a(z))\pi(z) = \gamma(1 - z^r)\delta(z)$$

when using (7.3b).

(*iv*) By virtue of obvious probability arguments we can derive the probability density function of an idle period in the steady state:

$$u \mapsto \frac{\sum_{i=0}^{r-1} p_i \gamma^{(i)}(\theta, 1)}{\sum_{i=0}^{r-1} p_i}\,,$$

The mean value of the idle period $\mathfrak{I}$ in the steady state is then

(8.8a)
$$\mathfrak{I} = \frac{\sum_{i=0}^{r-1} p_i \frac{1}{\lambda_i} \bar{\gamma}_r^{(i)}}{\sum_{i=0}^{r-1} p_i}\,.$$

(*v*) Formula (8.8a) and theorem 8.6 allow to derive the mean busy period $\mathfrak{B}$ in the equilibrium. Clearly $\sum_{i=0}^{r-1}\pi_i$ is the probability that the server idles. On the other hand, it also equals $\frac{\mathfrak{I}}{\mathfrak{I} + \mathfrak{B}}$. Thus we have

$$\mathfrak{B} = \frac{\mathfrak{I}\sum_{n=r}^{\infty}\pi_n}{\sum_{i=0}^{r-1}\pi_i}\,.$$

(*vi*) If the input is a stationary compound Poisson process (i.e. nonmodulated) then its intensity is $\alpha\lambda$, which is also the mean number of arriving units per unit time. In the case of a modulated input process its intensity is no longer a trivial fact. We define the intensity of any random measure Z by the formula $\kappa = lim_{t\to\infty}\frac{1}{t}\mu_t(x)$, where $\mu_t(x) = E^x[Z([0,t])]$. We will apply the formula from theorem A.9 (Dshalalow [6]) for more general Poisson process modulated by a semi-Markov process:

$$\kappa = \frac{P\rho}{P\beta}\,,$$

where by theorem 4.3 $P\rho = r$ and $P\beta$ satisfies (8.2a). Thus we have that:

(8.8b)
$$\kappa = \frac{r}{P\beta}\,.$$

A trivial special case appears when we assume the weak modulation of the input defined in (*i*) and therefore use formula (8.7a) combining it with formula (8.8b). Then $\kappa = \lambda\alpha$. Specifically $\kappa = \lambda\alpha$ for the input without modulation, as it should be.

APPENDIX

A.1 THEOREM (Abolnikov and Dukhovny [2]). *Let $\{Q_n\}$ be an irreducible aperiodic Markov chain with the transition probability matrix A in the form of a $\Delta_{r,N}$-matrix. $\{Q_n\}$ is recurrent-positive if and only if*

(A.1a) $$\frac{d}{dz} A_i(z)\Big|_{z=1} < \infty,\ i = 0,1,\ldots,N,$$

and

(A.1b) $$\frac{d}{dz} g(z)\Big|_{z=1} < r.$$

A.2 THEOREM (Abolnikov and Dukhovny [2]). *Under the condition of* (A.1b) *the function $z^r - g(z)$ has exactly* r *roots that belong to the closed unit ball $\bar{\Gamma}^+$. Those of the roots lying on the boundary Γ are simple.*

A.3 DEFINITION. Let T be a stopping time for a stochastic process $\{\Omega,\mathcal{F},(P^x)_{x\in\Psi},$ $Q(t);\ t \geq 0\} \rightarrow (\Psi,\ \mathfrak{B}(\Psi))$. $\{Q(t)\}$ is said to have the *locally strong Markov property at* T if for each bounded random variable $\zeta\colon \Omega \rightarrow \Psi^r$ and for each Baire function $f\colon \Psi^r \rightarrow \mathbb{R}$, $r = 1,2,\ldots$, it holds true that $E^x[f \circ \zeta \circ \theta_T \,|\, \mathcal{F}_T] = E^{Z_T}[f \circ \zeta]$ $\ P^x$-*a.s.* on $\{T < \infty\}$, where θ_y is the shift operator.

A.4 DEFINITION. A stochastic process $\{\Omega,\mathcal{F},(P^x)_{x\in\Psi},\ Q(t);\ t \geq 0\} \rightarrow (\Psi,\ \mathfrak{B}(\Psi))$ with $\Psi \preceq \mathbb{N}$ is called *semi-regenerative* if

a) there is a point process $\{T_n\}$ on $\mathbb{R}_+$ such that $T_n \rightarrow \infty$ $(n \rightarrow \infty)$ and such that each T_n is a stopping time relative to the canonic filtering $\sigma(Q_y\,;y \leq t)$,

b) the process $(Q(t))$ has the locally strong Markov property at T_n, $n = 1,2,\ldots$,

c) $\{Q(T_n + 0), T_n\,;\ n = 0,1,\ldots\}$ is a Markov renewal process.

A.5 DEFINITION. Let (X_n, T_n) be an irreducible aperiodic Markov renewal process with a discrete state space Ψ. Denote $\beta_x = E^x[T_1]$ as the mean sojourn time of the Markov renewal process in state $\{x\}$ and let $\beta = (\beta_x\,; x \in \Psi)^{\mathrm{T}}$. Suppose that the embedded Markov chain (X_n) is ergodic and that P is its stationary distribution. We call $P\beta$ the *mean inter-renewal time*. Then we call the Markov renewal process *recurrent-positive* if its mean inter-renewal time is finite. An irreducible aperiodic and recurrent-positive Markov renewal process is called *ergodic*.

A.6 DEFINITION. Let $\{\Omega,\mathcal{F},(P^x)_{x\in\Psi},\ Q(t);\ t \geq 0\} \rightarrow (\Psi,\ \mathfrak{B}(\Psi))$ be a semi-regenerative process relative to the sequence $\{T_n\}$ of stopping times. Introduce the probability

$$K_{jk}(t) = P^j\{Q(t) = k\,, T_1 > t\},\ j,k \in \Psi.$$

We will call the functional matrix $K(t) = (K_{jk}(t)\,;\ j,k\in\Psi)$ the *semi-regenerative kernel.*

A.7 THEOREM (The Main Convergence Theorem, *cf.* Çinlar [5], p. 347). *Let* $\{\Omega,\mathcal{F},(P^x)_{x\in\Psi},\ Q(t);\ t \geq 0\} \rightarrow (\Psi,\ \mathfrak{B}(\Psi))$ *be a semi-regenerative stochastic process relative to the sequence $\{t_n\}$ of stopping times and let $K(t)$ be the corresponding semi-regenerative kernel. Suppose that the associated Markov renewal process is ergodic and that the semi-regenerative kernel is Riemann integrable over $\mathbb{R}_+$. Then the stationary distribution $\pi = (\pi_x\,; x \in \Psi)$ of the process $\{Q(t)\}$ exists and it is determined from the formula:*

(A.7a) $$\pi_k = \frac{1}{P\beta}\ \sum\nolimits_{j\in\Psi} p_j \int_0^\infty K_{jk}(t)dt,\ k \in \Psi.$$

A.8 COROLLARY. *Denote $H = (h_{jk}\,; j,k \in \Psi) = \int_0^\infty K(t)dt$ as the integrated semi-regenerative kernel, $h_j(z)$ the generating function of* j*th row of matrix H and $\pi(z)$ as the generating function of vector π. Then the following formula holds true.*

(A.8a) $$\pi(z) = \frac{1}{P\beta} \sum_{j \in \Psi} p_j h_j(z)$$

PROOF. From (A.7a) we get an equivalent formula in matrix form, $\boldsymbol{\pi} = \frac{PH}{P\beta}$. Finally, formula (A.8a) is the result of elementary algebraic transformations. □

A.9 THEOREM (Dshalalow [6]). *Let Z^{ξ} be a compound Poisson process modulated by a semi-Markov process ξ in accordance with the above notation and definition in section 5. Let $\rho = \alpha*\beta*\lambda$ denote the Hadamard product of vectors α, β and λ. If $\{Q_n, T_n\}$ is ergodic then the intensity κ of the process Z^{ξ} is given by the formula*

$$\kappa = \lim_{t\to\infty} E^{\pi}[\tfrac{1}{t} Z^{\xi}([0,t])] = P\rho/P\beta .$$

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