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BUNDLES OF BANACH ALGEBRAS

J. W. KITCHEN

Department of Mathematics
Duke University
Durham, NC 27706 USA

and

D. A. ROBBINS

Department of Mathematics
Trinity College
Hartford, CT 06106 USA

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ABSTRACT: We study bundles of Banach algebras $\pi : A \to X$, where each fiber $A_X = \pi^{-1}(\{x\})$ is a Banach algebra and X is a compact Hausdorff space. In the case where all fibers are commutative, we investigate how the Gelfand representation of the section space algebra $\Gamma(\pi)$ relates to the Gelfand representation of the fibers. In the general case, we investigate how adjoining an identity to the bundle $\pi : A \to X$ relates to the standard adjunction of identities to the fibers.

KEYWORDS AND PHRASES: Banach bundle, bundle of Banach algebras, Gelfand representation, fiber space.
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1. INTRODUCTION

This paper continues the study of Banach bundles which has been pursued by the authors in a number of papers. This time, the focus is on bundles of Banach algebras. The reader is referred to [1] or [2] for general information about Banach bundles, and to [3] for the precise definition of bundles of Banach algebras which we use here.

We make the blanket assumption for this paper that the base space X for any Banach bundle $\pi : E \to X$ which we deal with is both compact and Hausdorff. As a result, all of our bundles are full; that is, for each $x \in X$, the fiber $E_X = \pi^{-1}(\{x\}) = \{ \sigma(x) : \sigma \in \Gamma(\pi)\}$. (See [1, p. 26])

The paper is divided into two sections. The first section deals with bundles of commutative Banach algebras. If $\pi : A \to X$ is a bundle of commutative Banach algebras, then the section space $\Gamma(\pi)$ is also a commutative Banach algebra under the pointwise operations. We study how the (usual) Gelfand representation of $\Gamma(\pi)$ relates to the Gelfand representations of the fibers $A_X = \pi^{-1}(\{x\})$. We show, in particular, that $\Delta(\Gamma(\pi))$, the maximal ideal space of $\Gamma(\pi)$, can be identified with the disjoint union of the maximal ideal spaces of the fibers. This generalizes a theorem of Rickart.

The second section concerns the adjunction of identities. Suppose that $\pi : A \to X$ is a bundle of (possibly non-commutative) Banach algebras. We show that there is a bundle of Banach algebras $\pi' : A' \to X$ such that

1) for each $x \in X$, $(A')_X = (A_X)_1$, the algebra obtained by adjoining an identity to A_X ;

and

2) the section space $\Gamma(\pi')$ is an algebra with identity which contains (an isomorphic copy of) $\Gamma(\pi)$.

2. BUNDLES OF COMMUTATIVE BANACH ALGEBRAS

Consider a bundle $\pi : A \to X$ of commutative Banach algebras. We will show that there is a natural way of embedding the maximal ideal spaces of the fibers $A_X = \pi^{-1}(\{x\})$ in the maximal ideal space of $\mathcal{A} = \Gamma(\pi)$. Moreover, so embedded, the maximal ideal spaces $\Delta(A_X)$ provide a fibering of $\Delta(\mathcal{A})$. This result, Proposition 6, generalizes a theorem of Rickart [4].

To relate our result to that of Rickart, we consider first Banach algebras with identities and adopt the following definition.

DEFINITION 1: Let $\pi : A \to X$ be a bundle of commutative Banach algebras. We say that the bundle <u>has an identity</u> if

1) each fiber $A_X = \pi^{-1}(\{x\})$ has an identity e_X ; and

2) the section e, defined by $e(x) = e_X$ for each $x \in X$, belongs to the section space $\Gamma(\pi)$ (and hence is the

identity for $\Gamma(\pi)$).

We now show that if π: $A \to X$ is a bundle with identity of commutative Banach algebras then the section space $\Gamma(\pi)$ fulfills the hypotheses of Theorem 3.2.2 of Rickart [4].

PROPOSITION 2: Let $\pi : A \to X$ be a bundle with identity of commutative Banach algebras. Then the algebra $\mathcal{A} = \Gamma(\pi)$ is a subdirect sum sum algebra of the family of fibers $\{A_X : x \in X\}$ which satisfies conditions (i), (ii), and (iii) of Theorem 3.2.2 of [4], namely:

(i) $\mathcal{A}$ contains the identity selection e defined above;

(ii) $\mathcal{A}$ is closed under (pointwise) multiplication by elements of C(X); and

(iii) for each $\sigma \in \mathcal{A}$, the map $x \mapsto \|\sigma(x)\|$ is upper semicontinous on X.

PROOF: Since $\Gamma(\pi)$ is a Banach algebra under the sup norm, it is a closed subalgebra of the full direct sum $\sum \{A_X : x \in X\}$ as defined by Rickart [4, p. 77]. Because the base space X is compact and Hausdorff, the set of function values $\{\sigma(x): \sigma \in \Gamma(\pi)\}$ exhausts A_X for each $x \in X$. Hence $\mathcal{A} = \Gamma(\pi)$ is a subdirect algebra of the family $\{A_X : x \in X\}$. Condition (i) is simply our definition of the algebra bundle having an identity. Condition (ii) is part of what is meant by the assertion that $\Gamma(\pi)$ is a C(X)-module, and condition (iii) holds for the sections of any bundle of Banach spaces. □□□

The next result turns out to be a strengthened version of Rickart's theorem.

PROPOSITION 3: Let $\pi : A \to X$ be as above. Then $\Delta(\mathcal{A})$, the maximal ideal space of $\mathcal{A} = \Gamma(\pi)$, can be identified with

$$\Omega = \{(x , h) : x \in X, h \in \Delta(A_X)\} ,$$

the disjoint union of the maximal ideal spaces of the fibers, in such a way that the Gelfand representation of $\mathcal{A}$ is described by the equation

$$\hat{\sigma}(x, h) = [\sigma(x)]^{\wedge} (h)$$

for all $(x, h) \in \Omega$ and all $\sigma \in \mathcal{A}$. Moreover,

(1) for each $x \in X$, the topology on $\Delta(A_X)$ is the same as the topology which $\Delta(A_X)$ inherits from $\Delta(\mathcal{A})$

($= \Omega$); and

(2) the coordinate projection $p : \Omega \to X$ is continuous and closed with respect to the topology which Ω

inherits from $\Delta(\mathcal{A})$.

Thus, $\Delta(\mathcal{A}) = \Omega$ is a fibered space over X whose fibers are the maximal ideal spaces $\Delta(A_X)$.

PROOF: In [4, p. 129-130], Rickart shows that $\Delta(\mathcal{A})$ and Ω can be identified as point sets in such a way that the identity $\hat{\sigma}(x, h) = [\sigma(x)]\hat{}\ (h)$ holds.

We now prove assertions (1) and (2), which relate the topologies of the spaces $\Delta(\mathcal{A})$, X, and $\Delta(A_X)$.

To prove continuity of the map $p : \Omega \to X$, we consider a convergent net $\{ (x_\alpha , h_\alpha)\}$ in the compact space $\Delta(\mathcal{A}) = \Omega$, say $\lim (x_\alpha , h_\alpha) = (x, h)$. We must prove that $\lim x_\alpha = x$, or equivalently that $\lim \phi(x_\alpha) = \phi(x)$ for all $\phi \in C(X)$ (since the topology of X is the same as the weak topology generated by the functions in C(X).) But $\lim (\phi e)\hat{}(x_\alpha, h_\alpha) = (\phi e)\hat{}(x, h)$, and

$$(\phi e)\hat{}(x, h) = [(\phi e)(x)]\hat{}\ (h) = [\phi(x) e_X]\hat{}(h) = \phi(x)\, \hat{e}_X(h) = \phi(x)\, 1 = \phi(x),$$

and similarly $(\phi e)\hat{}(x_\alpha, h_\alpha) = \phi(x_\alpha)$. Thus, $\lim \phi(x_\alpha) = \phi(x)$, as we wished to show. (A slightly more complicated proof appears later, when we no longer assume the existence of identity elements.)

To prove (1), it suffices to show that a net $\{h_\alpha\}$ in $\Delta(A_X)$ converges to a point h iff the net $\{ (x, h_\alpha) \}$ converges to (x, h) in $\Delta(\mathcal{A})$. Assuming that $\lim h_\alpha = h$ in $\Delta(A_X)$, it follows that

$$\lim \hat{\sigma}(x, h_\alpha) = \lim [\sigma(x)]\hat{}\ (h_\alpha) = [\sigma(x)]\hat{}\ (h) = \hat{\sigma}(x, h)$$

for all $\sigma \in \mathcal{A}$. If a is any element in A_X , we can choose $\sigma \in \mathcal{A}$ such that $\sigma(x) = a$. Then, if $\lim (x, h_\alpha) = (x, h)$ in $\Delta(\mathcal{A})$,

$$\lim\ \hat{a}(h_\alpha) = \lim [\sigma(x)]\hat{}\ (h_\alpha) = \lim \hat{\sigma}(x, h_\alpha) = \hat{\sigma}(x, h) = [\sigma(x)]\hat{}\ (h) = \hat{a}(h),$$

which implies that $\lim h_\alpha = h$ in $\Delta(A_X)$. Because the algebra $\mathcal{A}$ has an identity, the maximal ideal space $\Omega = \Delta(\mathcal{A})$ is compact, and as a result the map $p : \Omega \to X$ is not only continuous but closed. □□□

Our next result shows that Proposition 3 is indeed a strengthened version of Rickart's theorem.

PROPOSITION 4: Let X be a compact Hausdorff space, and let $\{A_X : x \in X\}$ be a family of commutative Banach algebras indexed by X. Let $\mathcal{A}$ be a subdirect sum algebra of the family $\{A_X : x \in X\}$ which satisfies conditions (i), (ii), and (iii) of Proposition 2. Then the disjoint union

$$A = \{ (x, a) : x \in X, a \in A_X\}$$

can be uniquely topologized so that $\pi : A \to X$ is a bundle of Banach algebras having the elements of $\mathcal{A}$ as sections (where π is the obvious coordinate projection). Moreover, $\mathcal{A}$ is the entire section space $\Gamma(\pi)$.

PROOF: The existence of the desired topology on $\mathcal{A}$ follows from Proposition 1.3 in [2]. Because the norm on $\mathcal{A}$ is a sup norm, it is easily shown that $\mathcal{A}$ is C(X)-locally convex as a module over C(X), and from this it follows that $\mathcal{A} = \Gamma(\pi)$. □□□

COROLLARY 5: (Rickart's Theorem) Let $\{A_X : x \in X\}$ and $\mathcal{A}$ satisfy the conditions of Proposition 4. Then $\Delta(\mathcal{A})$ can be identified as a point set with the disjoint union

$$\Omega = \{(x, h) : x \in X, h \in \Delta(A_X)\}$$

of the maximal ideal spaces of the algebras A_X .

So, Proposition 3 is a strengthening of Rickart's result in this sense: not only can $\Delta(\mathcal{A})$ be identified as a point set with the disjoint union of the family $\{\Delta(A_x) : x \in X\}$, but under this identification $\Delta(A_x)$ is a closed subset of $\Delta(\mathcal{A})$ for each x, and $\{\Delta(A_x) : x \in X\}$ is an upper semicontinuous decomposition of $\Delta(\mathcal{A})$. We show next that much of this is true if we drop the assumption that the algebras in question have identities.

PROPOSITION 6: Let $\pi : A \to X$ be a bundle of commutative Banach algebras. Then $\Delta(\mathcal{A})$, the maximal ideal space of $\mathcal{A} = \Gamma(\pi)$, can be identified as a point set with the disjoint union

$$\Omega = \{(x, h) : x \in X, h \in \Delta(A_x)\}$$

in such a way that

$$\hat{\sigma}(x, h) = [\sigma(x)]\hat{\ }(h)$$

holds for all $(x, h) \in \Omega$ and all $\sigma \in \Gamma(\pi)$. Moreover, under this identification, $\Delta(A_x)$ is a closed subset of $\Delta(\mathcal{A})$ for each $x \in X$.

PROOF: If B is a commutative Banach algebra, we shall regard each point in $\Delta(B)$ as an algebra homomorphism from B onto $\mathbb{C}$.

Consider an element $(x, h) \in \Omega$. Since the evaluation map $ev_x : \Gamma(\pi) \to A_x$, defined by $ev_x(\sigma) = \sigma(x)$ is a surjective algebra homomorphism, its composition with $h : A_x \to \mathbb{C}$ gives us an algebra homomorphism $\phi_{x, h} = h \circ ev_x$ of $\mathcal{A} = \Gamma(\pi)$ onto $\mathbb{C}$. Thus, $\phi_{x, h}$ belongs to $\Delta(\mathcal{A})$, and for all $\sigma \in \mathcal{A}$,

$$\hat{\sigma}(\phi_{x, h}) = \phi_{x, h}(\sigma) = (h \circ ev_x)(\sigma) = h(\sigma(x)) = [\sigma(x)]\hat{\ }(h) .$$

Let $\phi : \Omega \to \Delta(\mathcal{A})$ assign to each point (x, h) in Ω the multiplicative linear functional $\phi_{x, h}$ We must show that the map is bijective.

To show injectivity, consider two distinct points (x_1, h_1) and $(x_2, h_2) \in \Omega$. We will show that $\phi_{x_1,h_1} \neq \phi_{x_2,h_2}$.

Case 1: $x_1 = x_2$ and $h_1 \neq h_2$. Because $h_1 \neq h_2$, there is an $a \in A_x$ such that $h_1(a) \neq h_2(a)$. If we choose $\sigma \in \mathcal{A}$ such that $\sigma(x_1) = a$, then

$$\phi_{x_1,h_1}(\sigma) = h_1(\sigma(x_1)) = h_1(a) \neq h_2(a) = h_2(\sigma(x_2)) = \phi_{x_2,h_2}(\sigma) .$$

Case 2: $x_1 \neq x_2$. Choose $a \in A_{x_1}$ such that $h_1(a) = 1$ and choose $\sigma \in \mathcal{A}$ such that $\sigma(x_1) =$ a. Next choose $f \in C(X)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. Then

$$\phi_{x_1,h_1}(f\sigma) = h_1((f\sigma)(x_1)) = h_1(f(x_1)\ \sigma(x_1)) = h_1(a) = 1$$

and

$$\phi_{x_2,h_2}(f\sigma) = h_2((f\sigma)(x_2)) = h_2(f(x_2)\ \sigma(x_2)) = h_2(0) = 0.$$

Thus, $\phi_{x_1,h_1} \neq \phi_{x_2,h_2}$.

Having shown that the map $\phi : \Omega \to \Delta(\mathcal{A})$ is injective, we show next that the map is surjective. Suppose, now, that $H \in \Delta(\mathcal{A})$. We choose $\sigma \in \mathcal{A}$ such that $H(\sigma) = 1$ and we define a functional $\psi : C(X) \to \mathbb{C}$ by $\psi(f) = H(f\sigma)$. Then it is easily checked that ψ is bounded and linear. Moreover, ψ is multiplicative, since

$$\psi(fg) = H(fg\sigma) = H(fg\sigma)\ H(\sigma) = H(fg\sigma^2) = H(f\sigma)\ H(g\sigma) = \psi(f)\ \psi(g)$$

for all $f, g \in C(X)$. Thus, $\psi \in \Delta(C(X))$ and, consequently, there is a unique $x \in X$ such that ψ is evaluation at x. We have, then,

$$H(f\sigma) = \psi(f) = f(x)$$

for all $f \in C(X)$.

We will now show that there exists an $h \in \Delta(A_x)$ such that $H = \phi_{x, h}$.

First, we review some facts about the natural embedding of $(A_x)^*$ into $\mathcal{A}^* = \Gamma(\pi)^*$. The evaluation map $ev_x : \mathcal{A} \to A_x$ is a quotient map, and consequently the adjoint map

$(ev_X)^* : (A_X)^* \to \mathcal{A}^*$ is an isometry. If $k \in (A_X)^*$ and $K = (ev_X)^*(k)$, then

$$K(\tau) = \{ (ev_X{}^*(k)\}(\tau) = k(ev_X(\tau)) = k(\tau(x))$$

for all $\tau \in \mathcal{A}$. Thus, if $f \in C(X)$ and $\tau \in \mathcal{A}$,

$$K(f\tau) = k((f\tau)(x)) = k(f(x)\ \tau(x)) = f(x)\ k(\tau(x)) = f(x)\ K(\tau).$$

Conversely, one can show that if $K \in \mathcal{A}^*$ and if the equation

$$K(f\tau) = f(x)\ K(\tau)$$

holds for all $f \in C(X)$ and $\tau \in \mathcal{A}$, then there is a unique $k \in (A_X)^*$ such that $K = (ev_X)^*(k)$. (See Proposition 2.1 of [2].)

Now, consider H again. If $f \in C(X)$ and $\tau \in \mathcal{A}$, then

$$H(f\tau) = H(f\tau)\ H(\sigma) = H(f\tau\sigma) = H(f\sigma)\ H(\tau) = f(x)\ H(\tau).$$

By the preceding paragraph, there is a unique $h \in (A_X)^*$ such that $H = (ev_X)^*(h)$, and thus

$$H(\tau) = h(\tau(x))$$

for all $\tau \in \mathcal{A}$. Since H is multiplicative, it immediately follows that h is multiplicative. Hence $h \in \Delta(A_X)$ and

$$H(\tau) = h(\tau(x)) = \phi_{x,\,h}(\tau)\ .$$

We have now proved that Ω can be identified with $\Delta(\mathcal{A})$ in such a way that

$$\hat{\sigma}(x, h) = \hat{\sigma}(\phi_{x,\,h}) = h(\sigma(x)) = [\sigma(x)]^{\wedge}\ (h)$$

for all $(x, h) \in \Omega$ and $\sigma \in \mathcal{A}$. From this it follows that the topology on $\Delta(A_X)$ is the same as the topology which it inherits as a subspace of $\Delta(\mathcal{A})$. (See the proof of condition 1) in Proposition 3.)

We show finally that the map $p : \Delta(\mathcal{A}) \to X$ is continuous. Suppose that $\{(x_\alpha , h_\alpha)\}$ is a net in $\Omega = \Delta(\mathcal{A})$ which converges to a point (x, h). Thus

$$\lim \hat{\sigma}(x_\alpha, h_\alpha) = \lim h_\alpha(\sigma(x_\alpha)) = \hat{\sigma}(x, h) = h(\sigma(x))$$

for all $\sigma \in \mathcal{A}$. Choose $a \in A_X$ such that $h(a) = 1$ and choose $\tau \in \mathcal{A}$ such that $\tau(x) = a$. Then

$$(*)\ \ \lim h_\alpha(\tau(x_\alpha)) = h(\tau(x)) = h(a) = 1.$$

Let $\phi \in C(X)$. Then

$$\lim h_\alpha((\phi\ \tau)(x_\alpha)) = h((\phi\ \tau)(x)).$$

But $h((\phi\ \tau)(x)) = h(\phi(x)\ \tau(x)) = \phi(x)\ h(\tau(x)) = \phi(x)$, and similarly $h_\alpha((\phi\ \tau)(x_\alpha)) = \phi(x_\alpha)\ h_\alpha(\tau(x_\alpha))$. Thus

$$(**)\ \ \lim \phi(x_\alpha)\ h_\alpha(\tau(x_\alpha)) = \phi(x)\ .$$

From $(*)$ and $(**)$ it follows that

$$\lim \phi(x_\alpha) = \lim \frac{\phi(x_\alpha)\ h_\alpha(\tau(x_\alpha))}{h_\alpha(\tau(x_\alpha))} = \frac{\phi(x)}{1} = \phi(x).$$

Hence $\lim x_\alpha = x$ in X. □□□

The reader will note that there is a result analogous to Proposition 6 to be found in the theory of bundles of C^*-algebras; see, for example [5, p. 582]. Namely, if $\pi : A \to X$ is a bundle of C^*-algebras (with the total space A having continuous norm), then for every irreducible *-representation T of the C^*-algebra $\Gamma(\pi)$, there exists $x \in X$ and an irreducible *-representation S of A_X such that $T = S \circ ev_X$. The proof of Proposition 6 (which has an evident corollary in common with the C^*-algebra result) requires neither the *-machinery nor the existence of approximate identities which are used in [5], but, of course, does not deal with non-commutative algebras .

We consider next two examples. In the first, we show that the map $p : \Delta(\mathcal{A}) \to X$ need not be closed, even if all the fibers A_X have identities. In the second, we show that the map $p : \Delta(\mathcal{A}) \to X$ need not be open, even if $\pi : A \to X$ is a bundle with identity.

EXAMPLE 7: Let $X = [0, 1]$ and for each $x \in X$ let A_x be the one-dimensional Banach algebra $\mathbb{C}$ normed by absolute value. Then the full direct sum of the family $\{A_x : x \in X\}$ is the space $\ell^\infty(X)$ of all bounded complex-valued functions on X. We let $\mathcal{A}$ be the subalgebra consisting of all functions $f : [0, 1] \to \mathbb{C}$ such that f is continuous on [0, 1) and $\lim_{x\to 1^-} f(x) = 0$ (and f(1) is arbitrary). Then it is easily verified that $\mathcal{A}$ is a subdirect sum algebra of $\ell^\infty(X)$ which satisfies conditions (ii) and (iii) of Proposition 2, but not condition (i). (Note that if $f \in \mathcal{A}$, then the function $x \mapsto \|f(x)\| = |f(x)|$ is upper semicontinuous on [0, 1]. On the other hand, the identity selection $e(x) \equiv 1$ does not belong to $\mathcal{A}$.)

It is easy to show that $\Delta(\mathcal{A})$ consists of the evaluation functionals ev_x for $x \in [0, 1]$. Thus, as a point set, $\Delta(\mathcal{A})$ can be identified with [0, 1]. The topologies, however, do not match. In [0, 1] the sequence { (n-1)/n } converges to 1, whereas in $\Delta(\mathcal{A})$ the corresponding sequence $\{ev_{(n-1)/n}\}$ has no limit. (In $\mathcal{A}^*$ the sequence converges weak-* to the zero functional.) In $\Delta(\mathcal{A})$ the point ev_1 is isolated. The natural surjection $p : \Delta(\mathcal{A}) \to X$ maps ev_x onto x for each $x \in [0, 1]$. However, p is not closed: the set $\{ev_x : 0 \le x < 1\}$ is the complement of the open set $\{ev_1\}$ and hence is closed. Its image under p is [0, 1), and the latter is not closed in [0, 1].

EXAMPLE 8: Let $q : Y \to X$ be a continuous surjective map, where X and Y are compact Hausdorff. For each $x \in X$, set $Y_x = q^{-1}(\{x\})$ and $A_x = C(Y_x)$. Given $f \in C(Y)$ and $x \in X$, we define $f^*(x)$ to be the restriction of f to Y_x. In this way, $\mathcal{A} = C(Y)$ can be viewed as a subdirect sum of the family $\{A_x : x \in X\} = \{ C(Y_x) : x \in X\}$. (See [6] or [7].) Moreover, $\mathcal{A}$ satisfies conditions (i), (ii), and (iii) of Proposition 2. The maximal ideal space of $\mathcal{A} = C(Y)$ can be identified with Y, which in turn can be identified with

$$\Omega = \{ (x, y) : x \in X, y \in Y_x\}.$$

Now, for each $(x, y) \in \Omega$,

$$p((x, y)) = x = q(y).$$

That is, under the identification of $\Delta(\mathcal{A})$ with Y, the map $p : \Delta(\mathcal{A}) \to X$ is the same as the given map $q : Y \to X$. Since q may not be open, the map $p : \Delta(\mathcal{A}) \to X$ need not be open. (For example, let Y = [-1, 2], X= [0, 4], and $q(x) = x^2$. Then the set [-1, 1/2) is open in Y, but its image under q is [0, 1] , which is not open in [0, 4].)

We close this section with some simple consequences of our main result.

PROPOSITION 9: Let $\pi : A \to X$ be a bundle of commutative Banach algebras.

(1) If A_x is semi-simple for each $x \in X$, then $\Gamma(\pi)$ is semi-simple.

(2) Under our identification of $\Delta(\Gamma(\pi))$ with $\Omega = \{(x, h) : x \in X, h \in \Delta(A_x)\}$, the Silov boundary of $\Gamma(\pi)$ is the disjoint union of the Silov boundaries of the fibers A_x.

(3) If $\pi : A \to X$ is a bundle with identity, then for $\sigma \in \Gamma(\pi)$, the spectrum of σ is the union of the spectra of the section values $\sigma(x)$; if $\pi : A \to X$ is a bundle without identity, then for $\sigma \in \Gamma(\pi)$, the spectrum of σ is the union of the spectra of the section values $\sigma(x)$ and {0}.

PROOF: All three assertions follow easily from the identity $\hat{\sigma}(x, h) = [\sigma(x)]\hat{}\,(h)$. We omit the details. ▫▫▫

3. THE ADJUNCTION OF IDENTITIES TO BUNDLES OF BANACH ALGEBRAS

If B is a (complex) Banach algebra, then there is a standard way of embedding B into a Banach algebra with identity. We let B_1 be the vector space $B \times \mathbb{C}$ with norm and multiplicaton defined by

$$|(a, \lambda)| = \|a\| + |\lambda| \qquad \text{and} \qquad (a,\lambda)(b, \mu) = (ab + \lambda b + \mu a,\ \lambda\mu).$$

Then B_1 is a Banach algebra with identity (the pair (0, 1)) which contains a subalgebra, namely $B \times \{0\}$, which is isometrically isomorphic to B.

PROPOSITION 10: Let $\pi : A \to X$ be a bundle of Banach algebras. Then there exists a bundle $\pi' : A' \to X$ of Banach algebras such that

(1) for each $x \in X$, $(A')_x = (A_x)_1$;

(2) for all $\sigma \in \Gamma(\pi)$ and $f \in C(X)$, the selection $(\sigma, f)^{\sim} : X \to A'$ defined by $(\sigma,f)^{\sim}(x) = (\sigma(x), f(x))$

belongs to the section space $\Gamma(\pi')$; and

(3) $\Gamma(\pi')$ is a Banach algebra with identity which contains subalgebras isometrically isomorphic to $\Gamma(\pi)$,

$\{\Gamma(\pi)\}_1$, and C(X).

PROOF: We let $M = \Gamma(\pi) \times C(X)$. Then it is easily verified that M becomes a Banach algebra if the norm and multiplication are defined as follows:

$$\|(\sigma,f)\| = \|\sigma\| + \|f\|, \quad \text{and} \quad (\sigma,f)\,(\tau,g) = (\sigma\tau + f\tau + g\sigma,\ fg).$$

(We have, for instance,

$$\begin{aligned} \|(\sigma,\ f)\ (\tau,\ g)\| &= \|\sigma\tau + f\tau + g\sigma\| + \|fg\| \\ &\le \|\sigma\tau\| + \|f\tau\| + \|g\sigma\| + \|fg\| \\ &\le \|\sigma\|\,\|\tau\| + \|f\|\,\|\tau\| + \|g\|\,\|\sigma\| + \|f\|\,\|g\| \\ &= (\|\sigma\| + \|f\|)\,(\|\tau\| + \|g\|) \\ &= \|(\sigma,f)\|\ \|(\tau,g)\| .\ ) \end{aligned}$$

Moreover, if we define $f\,(\tau, g) = (f\tau, fg)$ for all $f \in C(X)$ and $(\tau, g) \in M$, then M becomes a C(X)-module. We let $\mu : B \to X$ be the canonical bundle of M as a C(X)-module. We will show that $\mu : B \to X$ is an isomorphic copy of the bundle $\pi' : A' \to X$ which we seek.

Recall that for each $x \in X$, the fiber $B_x = \mu^{-1}(\{x\})$ is the quotient space $\frac{M}{I_xM}$, where $I_xM = \{\, f(\sigma, g) : f \in I_x ,\ (\sigma, g) \in M\}$ and I_x is the maximal ideal $\{f \in C(X) : f(x) = 0\}$ in C(X). (The set I_xM can also be characterized as the smallest closed subspace J_x of M which contains all products of the form $f\,(\sigma, g)$, where $f \in I_x$ and $(\sigma, g) \in M$. For J_x is clearly an essential I_x module, and the Banach algebra I_x has an approximate identity, so the Cohen factorization theorem implies that $J_x = I_xM$.) Moreover, the Gelfand sectional representation $\wedge : M \to \Gamma(\mu)$ is described by

$$(\sigma,\ f)\hat{\ }(x) = \Pi_x((\sigma,\ f))$$

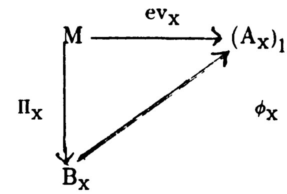
for all $(\sigma, f) \in M$, where $\Pi_x : M \to \frac{M}{I_xM}$ is the natural surjection. In this case, the set I_xM is a closed two-sided ideal in M, so that $B_x = \frac{M}{I_xM}$ is a Banach algebra and Π_x is an algebra homomorphism.

We will now show that for each $x \in X$ there is a unique isometric isomorphism $\phi_x : B_x \to (A_x)_1$ such that

$$\phi_x((\sigma,\ f)\hat{\ }(x)) = (\sigma(x),\ f(x))$$

for all $(\sigma, f) \in M$. To that end, consider the map $ev_x : M \to (A_x)_1$ defined by $ev_x((\sigma, f)) = (\sigma(x), f(x))$. This map is evidently a norm-decreasing algebra homomorphism whose kernel includes I_xM. On the other hand, if $(\sigma, f) \in \ker(ev_x)$, then $\sigma(x) = 0$, so $\sigma \in I_x\Gamma(\pi)$; we also have $f(x) = 0$, so $f \in I_x = I_x\, C(X)$. Hence f and σ have factorizations $f = hk$ and $\sigma = g\tau$, where $g, h \in I_x$, $\tau \in \Gamma(\pi)$, and $k \in C(X)$. It follows that $(\sigma, f) = g\,(\tau, 0) + h\,(0, k) \in I_xM$, since it is the sum of two elements of the subspace I_xM. Hence $\ker(ev_x) = I_xM$.

Because ker $(ev_X) = I_X M$, there is a unique injective algebra homomorphism $\phi_X : B_X \to (A_X)_1$
which makes the diagram



commute. Thus,

$$\phi_X((\sigma, f)\hat{}(x)) = (\phi_X \circ \Pi_X)(\sigma, f) = ev_X((\sigma, f)) = (\sigma(x), f(x))$$

for all $(\sigma, f) \in M$. To show that ϕ_X is an isometric isomorphism, it suffices to show that $ev_X : M \to (A_X)_1$ is a quotient map. If (z, λ) is any element in $(A_X)_1$, we can choose $\sigma \in \Gamma(\pi)$ and $f \in C(X)$ such that $\sigma(x) = z$, $f(x) = \lambda$, $\|\sigma\| = \|\sigma(x)\| = \|x\|$, and $\|f\| = |f(x)| = |\lambda|$. (We could, for instance, let f have the constant
value λ.) Then

$$ev_X((\sigma,f)) = (\sigma(x), f(x)) = (z, \lambda)$$

and

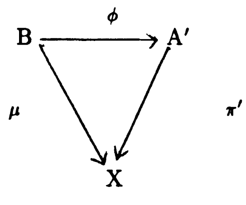
$$\|(\sigma,f)\| = \|\sigma\| + \|f\| = \|z\| + |\lambda| = \|(z, \lambda)\|.$$

This proves that $ev_X : M \to (A_X)_1$ is a quotient map.

We now let A' be the disjoint union

$$A' = \{ (x, z) : x \in X, z \in (A_X)_1 \}$$

and let $\pi' : A' \to X$ be the coordinate projection. Then our family of isometric isomorphisms $\phi_X : B_X \to (A_X)_1$ gives us a bijective map $\phi : B \to A'$ which makes the diagram



commute. (In other words, ϕ restricted to the fiber B_X is just ϕ_X.) We use the map ϕ to transfer the topology on the fiber space B to A'. Then $\pi' : A' \to X$ becomes a bundle of Banach algebras which is bundle isomorphic to $\mu : B \to X$. If $(\sigma, f) \in M$, then the selection $(\sigma, f)^{\sim} : X \to A'$ defined by $(\sigma, f)^{\sim}(x) = (\sigma(x), f(x))$ is just $\phi \circ (\sigma, f)\hat{}$ and hence is in the section space $\Gamma(\pi')$.

The maps

$$\alpha : \Gamma(\pi) \to \Gamma(\pi'),\ \alpha(\sigma) = (\sigma, 0);$$

$$\beta : \{\Gamma(\pi)\}_1 \to \Gamma(\pi'),\ \beta((\sigma,\lambda)) = (\sigma,\lambda)^{\sim};\ \text{and}$$

$$\gamma : C(X) \to \Gamma(\pi'),\ \gamma(f) = (0, f)^{\sim}$$

are clearly isometric algebra homomorphisms. □□□

We conclude with an unrelated result.

If $\pi : A \to X$ is a bundle of Banach spaces, and if $\sigma \in \Gamma(\pi)$, then the numerical function $f(x) = \|\sigma(x)\|$ is upper semicontinuous on X. If $\pi : A \to X$ is a bundle of Banach algebras we will show that the same is true of the function $g(x) = \|\sigma(x)\|_{sp}$, where $\|a\|_{sp}$ denotes the spectral radius of a.

PROPOSITION 11: Let $\pi : A \to X$ be a bundle of Banach algebras, and let $\sigma \in \Gamma(\pi)$. Then the function $g(x) = \|\sigma(x)\|_{sp}$ is upper semicontinuous on X. Moreover, if the algebras $\{A_x : x \in X\}$ are all commutative (so that $\Gamma(\pi)$ is commutative), then $\|\sigma\|_{sp} = \sup\{\|\sigma(x)\|_{sp} : x \in X\}$.

PROOF: Let $x_0 \in X$, and let $\epsilon > 0$ be given. Since

$$\|\sigma(x_0)\|_{sp} = \inf \|\{\sigma(x_0)\}^n\|^{1/n} = \lim \|\{\sigma(x_0)\}^n\|^{1/n},$$

we can choose a positive integer n such that $\|\{\sigma(x_0)\}^n\|^{1/n} < \|\sigma(x_0)\|_{sp} + \epsilon/2$. Now, $\|\{\sigma(x)\}^n\| = \|\sigma^n(x)\|$ is an upper semicontinuous function of x, and hence the same is true of $\|\{\sigma(x)\}^n\|^{1/n}$. Thus, there is a neighborhood V of x_0 such that

$$\|\{\sigma(x)\}^n\|^{1/n} < \|\{\sigma(x_0)\}^n\|^{1/n} + \epsilon/2$$

for all $x \in V$. Consequently,

$$\|\sigma(x)\|_{sp} \le \|\{\sigma(x)\}^n\|^{1/n} < \|\sigma(x_0)\|_{sp} + \epsilon$$

for all $x \in V$, thus proving that the function $g(x) = \|\sigma(x)\|_{sp}$ is upper semicontinuous at x_0.

The assertion in the case when all algebras A_x are commutative follows easily from 1) Proposition 6;

2) the fact that, if B is any Banach algebra, then for $b \in B$, we have $\|b\|_{sp} = \max\{|\lambda| : \lambda \in Sp(b)\}$ (where Sp(b) denotes the spectrum of b) (see e.g. [8, p. 23]); and

3) the fact that, if B is a commutative Banach algebra, and if $b \in B$, then the range of its Gelfand transform $\hat{b}$ is either Sp(b) or $Sp(b)\backslash\{0\}$. □□□

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