ADDENDUM TO A PAPER OF CRAIG AND GOODMAN

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ABSTRACT. In [1], Craig and Goodman develop the geometrical optics solution of the linearized Korteweg-deVries equation away from caustic, or turning, points. Here we develop an analogous solution valid at caustic points.

1. **INTRODUCTION.** In [1], Craig and Goodman consider the linearized Korteweg-deVries (KdV) equation

$$\partial_t u(x,t) = a(x,t)\partial_x^3 u(x,t). \tag{1.1}$$

where x represents space and t represents time. Their analysis begins with a geometrical optics (WKB) solution. They first rescale the independent variables to obtain

$$\partial_t u(x,t) = \epsilon^2 a(x,t)\partial_x^3 u(x,t) \tag{1.2}$$

where ϵ is a small parameter, then assume a solution of the form

$$u(x,t) = A(x,t;\epsilon)exp\left[\frac{i}{\epsilon}\,S(x,t)\right] \tag{1.3}$$

where $A(x,t;\epsilon) = \sum_{j=0}^{\infty} \epsilon^j A_j$ may be regarded as amplitudes and $S(x,t)$ as the phase. Then

substituting (1.3) into (1.2) and regrouping, they develop an eikonal equation for the phase

$$\partial_t S + a(x,t)(\partial_x S)^3 = 0 \tag{1.4}$$

a first order transport equation for A_0

$$\partial_t S + 3a(x,t)\partial_x\left[(\partial_x S)^2 A_0\right] = 0 \tag{1.5}$$

and the Hamiltonian, $H(x,t,k,w) = w + a(x,t)k^3$, flow

$$\dot{x} = \partial_k H = 3a(x,t)k^2 \qquad \dot{k} = -\partial_x H = -\partial_x(a(x,t)k^3)$$

$$\dot{t} = \partial_w H = 1 \qquad \dot{w} = -\partial_t H = -\partial_t(a(x,t)k^3),$$

where $\dot{w} = -\partial_t S$ and the dots indicate differentiation with respect to time. Near caustic curves this system is singular and the geometrical optics technique does not apply. An approach which does apply at caustics is the Lagrange manifold technique ([2], [3], [4]). Here we apply this technique at caustics associated with the linearized Korteweg-deVries equation.

2. FORMALISM.

Near caustics we assume an asymptotic solution of the form

$$u(x,t) = \int A(x,k,t;\epsilon) \, exp\left[\tfrac{i}{\epsilon}\,(xk - S(k,t))\right]\!dk, \tag{2.1}$$

where amplitude $A(x,k,t;\epsilon) = \sum_{j=0}^{\infty} \epsilon^j A_j$ and its derivatives are assumed bounded and

$xk - S(k,t) = \phi(x,k,t)$ may be regarded as a phase. Then carrying the differentiation in (1.2) across the integral in (2.2) leads to

$$\int \left[\tfrac{i}{\epsilon}\,A(-\partial_t S + a(x,t)k^3) + (\partial_t A + 3a(x,t)k^2\partial_x A)\right.$$

$$\left. - i\epsilon(3ka(x,t)\partial_x^2 A) - \epsilon^2 a(x,t)\partial_x^3 A\right] exp\left[\tfrac{i}{\epsilon}\,\phi\right]\!dk = 0(\epsilon). \tag{2.2}$$

Here, the coefficient of the $\frac{1}{\epsilon}$ term is the Hamiltonian. On the caustic, the integral is evaluated using the stationary phase condition $[\partial_k \phi = 0]$, which obtains the Lagrange manifold

$$x = \partial_k S(k,t) \tag{2.3}$$

and turns the Hamiltonian into an eikonal equation, cf. (1.4),

$$-\partial_t S + a(x,t)k^3 = 0. \tag{2.4}$$

We obtain the phase from the Hamiltonian flow, (1.5). First these equations are solved to obtain

$$x = x(\mu,\sigma) \qquad k = k(\mu,\sigma)$$

$$t = t(\mu,\sigma) \qquad w = w(\mu,\sigma),$$

using μ as an arbitrary ray-path parameter and σ as an initial condition. Then the inversion of the wave vector (k) and time transformations, followed by substitution into the coordinate space map, determines the Lagrange manifold explicitly,

$$x = x(\mu(t,k),\sigma(t,k)) = \partial_k S(k,t).$$

An integration along the trajectories obtains

$$S(k,t) = \int_{k_0}^{k} x \, dk$$

and hence the phase

$$\phi(x,k,t) = xk - S(k,t). \tag{2.5}$$

We obtain a transport equation for the amplitudes by first Taylor-expanding the Hamiltonian near the Lagrange manifold to obtain

$$A(x,k,t;\epsilon)(-\partial_t S + a(x,t)k^3) = A(\partial_k S,k,t;\epsilon)(-\partial_t S + a(\partial_k S,t)k^3)$$

$$+ (x - \partial_k S)D = (x - \partial_k S)D,$$

where

$$D = \int_0^1 \partial_x H(\gamma(x - \partial_k S) + \partial_k S, k, t) d\gamma$$

with H the Hamiltonian, i.e., $H = \partial_t S + a(x,t)k^3$. Next substituting into (2.2) leads to

$$\int \left[(-\partial_k A)D + (-\partial_k D)A + \partial_t A + 3a(x,t)k^2 \partial_x^2 A \right.$$
$$\left. - i\epsilon a(x,t)(3k\partial_x^2 A) - \epsilon^2 a(x,t)\partial_x^3 A \right] exp\left[\frac{i}{\epsilon} \phi\right] dk = 0(\epsilon).$$

Finally, introducing the non-Hamiltonian flow

$$\dot{x} = 3a(x,t)k^2\partial_x^2 A \qquad k = -D \quad \dot{t} = 1$$

and requiring that

$$(-\partial_k A)D + (-\partial_k D)a + \partial_t A + 3a(x,t)k^2\partial_x^2 A$$
$$- i\epsilon a(x,t)(3k\partial_x^2 A) - \epsilon^2 a(x,t)\partial_x^3 A = 0$$

in a neighborhood of the Lagrange manifold determines the transport equation

$$\frac{dA_j}{dt} - A_j\partial_k D - 3ka(x,t)\partial_x^2 A_{j-1} - a(x,t)\partial_x^3 A_{j-2} = 0 \tag{2.6}$$

for the evolution of the amplitudes A_j.

The asymptotic evaluation of the field integrals at the caustic points has been detailed elsewhere ([2], [3], [4]). For brevity, we do not repeat the procedure here.

REFERENCES

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