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P-SYSTEMS IN LOCAL NOETHER LATTICES

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ABSTRACT. In this paper we introduce the concept of a p-system in a local Noether lattice and obtain several characterizations of these elements. We first obtain a topological characterization and then a characterization in terms of the existence of a certain type of decreasing sequence of elements. In addition, p-systems are characterized in quotient lattices and completions.

KEY WORDS AND PHRASES. Local, Noether, multiplicative lattice.
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In 1962 R. P. Dilworth [7] introduced the concept of a Noether lattice as an abstraction of the lattice of ideals of a Noetherian ring. Many of his ideas have since been extended and have proved to be extremely useful (e.g., [1], [5], [6], [12], and [13]). In this paper we introduce the idea of a p-system in a local Noether lattice and obtain numerous characterizations of these elements. We begin with a topological characterization of p-systems (Theorem 1) and then obtain a characterization of p-systems in terms of the existence of a special type of decreasing sequence of elements in the lattice (Theorem 2). Next p-systems are investigated in quotient lattices (Theorem 3) and completions (Theorem 4). When the local Noether lattice under consideration is the lattice of ideals of a local Noetherian ring, a p-system is known as a principal system. The concept of a principal system was introduced by Northcott and Rees [15] and was motivated by Macaulay's well-known theory of inverse systems. As a consequence of our lattice results, we obtain several characterizations of principal systems in rings. Finally we end by giving an example of a local Noether lattice which has a variety of p-systems and which is not the lattice of ideals of any commutative ring.

In general we adopt the terminology of [2], [4], and [7]. Following [7], a local Noether lattice is a modular, principally generated multiplicative lattice $\mathcal{L}$ which satisfies the ascending chain condition, has a unique maximal element, and has the property that the greatest element I of $\mathcal{L}$ is a multiplicative identity. Let $\mathcal{L}$ be a local Noether lattice with maximal element M. An element Q of $\mathcal{L}$ is primary if for all elements A and B of $\mathcal{L}$, $AB \leq Q$ implies $A \leq Q$ or $B^k \leq Q$ for some integer k. The radical of an element A of $\mathcal{L}$, denoted by Rad(A), is defined by

$$\mathrm{Rad}(A) = \vee\{X \in \mathcal{L} \mid X^s \leq A \text{ for some integer } s\}. \tag{1}$$

Furthermore, an element Q of $\mathcal{L}$ is said to be M-primary if Rad(Q) = M. (If Rad(Q) = M, then Q is primary [11, Corollary 2.5, page 191].) For an element A of $\mathcal{L}$, we define $\wp_A$ to be the set $\{Q \in \mathcal{L} \mid Q$ is a meet-irreducible M-primary element of $\mathcal{L}$ such that $A \leq Q\}$. Also define a metric d (called the M-adic metric) on $\mathcal{L}$ as follows:

$$d(C,D) = 0 \text{ if } C \vee M^n = D \vee M^n \text{ for all nonnegative integers n;} \tag{2}$$

otherwise,

$$d(C,D) = 2^{-s(C,D)} \text{ where } s(C,D) = \sup\{n \mid C \vee M^n = D \vee M^n\}. \tag{3}$$

This metric gives rise to the M-adic completion of $\mathcal{L}$ [8]. Finally an element A of $\mathcal{L}$ is defined to be a p-system in $\mathcal{L}$ if $A \neq I$ and for every M-primary element Q' of $\mathcal{L}$ with $A \leq Q'$, there exists a meet-irreducible M-primary element Q of $\mathcal{L}$ such that $A \leq Q \leq Q'$. Note that all meet-irreducible M-primary elements of $\mathcal{L}$ are p-systems in $\mathcal{L}$.

We begin with the following characterizations of p-systems.

THEOREM 1. Let $\mathcal{L}$ be a local Noether lattice with maximal element M and let A be an element of $\mathcal{L}$ different from I. Then the following are equivalent:

(1.1) A is a p-system in $\mathcal{L}$

(1.2) for every positive integer n, there exists a meet-irreducible M-primary element Q of $\mathcal{L}$ such that $A \leq Q \leq A \vee M^n$

(1.3) A is a closure point of $\wp_A$ in the M-adic topology on $\mathcal{L}$.

PROOF. To show that (1.1) implies (1.2), suppose A is a p-system in $\mathcal{L}$ and n is a positive integer. Since $M \geq \mathrm{Rad}(A \vee M^n) \geq \mathrm{Rad}(M^n) = M$, it follows that $A \vee M^n$ is an M-primary element of $\mathcal{L}$. Thus, since A is a p-system in $\mathcal{L}$, there exists a meet-irreducible M-primary element Q of $\mathcal{L}$ such $A \leq Q \leq A \vee M^n$. We now show that (1.2) implies (1.3). Suppose (1.2) holds and $\epsilon > 0$. Let n be a positive integer such that $2^{-n} < \epsilon$. By (1.2), there exists a meet-irreducible M-primary element Q of $\mathcal{L}$ such that $A \leq Q \leq A \vee M^n$. It follows that $A \vee M^n = Q \vee M^n$, and so $Q \in \wp_A$ and $d(A,Q) < 2^{-n} < \epsilon$. Hence, A is a closure point of $\wp_A$ in the M-adic topology on $\mathcal{L}$. To complete the proof, we show that (1.3) implies (1.1). Suppose A is a closure point of $\wp_A$ in the M-adic topology on $\mathcal{L}$ and that Q' is an M-primary element of $\mathcal{L}$ such that $A \leq Q'$. Since Q' is M-primary, choose n to be a positive integer such that $M^n \leq Q'$. So by (1.3), there exists $Q \in \wp_A$ such that $d(A,Q) < 2^{-n}$. Hence, we have that Q is a meet-irreducible M-primary element of $\mathcal{L}$ such that $A \leq Q$ and $A \vee M^n = Q \vee M^n$. Thus, it follows that

$$A \leq Q \leq Q \vee M^n = A \vee M^n \leq Q'. \tag{4}$$

Hence, A is a p-system in $\mathcal{L}$. This completes the proof.

We now characterize principal systems in a local Noether lattice in terms of the existence of a certain type of decreasing sequence of special elements in the lattice.

THEOREM 2. Let $\mathcal{L}$ be a local Noether lattice with maximal element M and let A be an element of $\mathcal{L}$ different from I. Then A is a p-system in $\mathcal{L}$ if and only if there exists a decreasing sequence $\{Q_n\}$ of meet-irreducible M-primary elements of $\mathcal{L}$ such that

(i) $A = \bigwedge_{n=1}^{\infty} Q_n$, and

(ii) if Q is an M-primary element of $\mathcal{L}$ satisfying $A \le Q$, then there is a positive integer n such that $Q_n \le Q$.

PROOF. Assume there is a decreasing sequence $\{Q_n\}$ of elements of $\mathcal{L}$ satisfying conditions (i) and (ii) and suppose Q' is an M-primary element of $\mathcal{L}$ such that $A \le Q'$. From condition (ii), we get that there exists a positive integer n such that $Q_n \le Q'$. Also by condition (i), we have that $A \le Q_n$. Therefore, Q_n is a meet-irreducible M-primary element of $\mathcal{L}$ such that $A \le Q_n \le Q'$. Hence, A is a p-system in $\mathcal{L}$. Conversely, assume A is a p-system in $\mathcal{L}$. We recursively define a sequence $\{Q_n\}$ of elements of $\mathcal{L}$ as follows: Choose Q_1 to be M. For $n > 1$, choose Q_n to be a meet-irreducible M-primary element of $\mathcal{L}$ such that

$$A \le Q_n \le Q_{n-1} \wedge (A \vee M^n). \tag{5}$$

This is possible using (1.2) since $A \le Q_{n-1} \wedge (A \vee M^n)$ and

$$\mathrm{Rad}(Q_{n-1} \wedge (A \vee M^n)) = \mathrm{Rad}(Q_{n-1}) \wedge \mathrm{Rad}(A \vee M^n) = M \tag{6}$$

so that $Q_{n-1} \wedge (A \vee M^n)$ is an M-primary element of $\mathcal{L}$. By our construction, $\{Q_n\}$ is a decreasing sequence of meet-irreducible M-primary elements of $\mathcal{L}$. Moreover,

$$A \le \bigwedge_{n=1}^{\infty} Q_n \le \bigwedge_{n=1}^{\infty} (A \vee M^n) = A, \tag{7}$$

so $\bigwedge_{n=1}^{\infty} Q_n = A$. Finally, if Q is an M-primary element of $\mathcal{L}$ such that $A \le Q$, then there exists a positive integer n such that $M^n \le Q$, and it follows that $Q_n \le A \vee M^n \le Q$. This completes the proof.

We now recall the definition of quotient lattice given in [7]. Let $\mathcal{L}$ be a Noether lattice and let D be an element of $\mathcal{L}$. Define $\mathcal{L}/D$ to be the sublattice $\{X \in \mathcal{L} \mid D \le X\}$ of $\mathcal{L}$. Then $\mathcal{L}/D$ is a multiplicative lattice with multiplication $\circ$ defined by

$$A \circ B = AB \vee D. \tag{8}$$

If $\mathcal{L}$ is local with maximal element M and $D \le M$, then $\mathcal{L}/D$ is local with maximal element M; furthermore, for an element A of $\mathcal{L}$ satisfying $D \le A$, A is an M-primary element of $\mathcal{L}$ if and only if A is an M-primary element of $\mathcal{L}/D$.

THEOREM 3. Let $\mathcal{L}$ be a local Noether lattice with maximal element M and let A be an element of $\mathcal{L}$ different from I. Then the following are equivalent:

(3.1) A is a p-system in $\mathcal{L}$

(3.2) for all elements B of $\mathcal{L}$ satisfying $B \le A$, A is a p-system in $\mathcal{L}/B$

(3.3) the zero element of $\mathcal{L}/A$ is a p-system in $\mathcal{L}/A$

(3.4) there exists an element B of $\mathcal{L}$ satisfying $B \le A$ such that A is a p-system in $\mathcal{L}/B$.

PROOF. To show that (3.4) implies (3.1), suppose there exists an element B of $\mathcal{L}$ such that $B \le A$ and A is a p-system in $\mathcal{L}/B$. In addition, suppose that n is a positive integer. Then using (1.2), there exists a meet-irreducible M-primary element Q of $\mathcal{L}/B$ such that $A \le Q \le A \vee M^n$. Hence, we get that Q is a meet-irreducible M-primary element of $\mathcal{L}$ such that $A \le Q \le A \vee M^n$. Therefore, A is a p-system in $\mathcal{L}$. The proofs that (3.1) implies (3.2), that (3.2) implies (3.3), and that (3.3) implies (3.4) are straightforward and we omit the details. This completes the proof.

For a local Noether lattice $\mathcal{L}$ with maximal element M, we now investigate p-systems in the completion of $\mathcal{L}$ with respect to the M-adic metric described earlier. Following [8], let $\mathcal{L}^*$ denote the set of all formal sums $\sum_{i=1}^{\infty} A_i$ of elements of $\mathcal{L}$ such that

$$A_i = A_{i+1} \vee M^i \tag{9}$$

for all positive integers i. On $\mathcal{L}^*$, define

$$\sum_{i=1}^{\infty} A_i \le \sum_{i=1}^{\infty} B_i \text{ if and only if } A_i \le B_i \text{ for all } i \tag{10}$$

and

$$\left(\sum_{i=1}^{\infty} A_i\right)\left(\sum_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} (A_i B_i \vee M^i). \tag{11}$$

For an element A of $\mathcal{L}$, let A^* denote the element $\sum_{i=1}^{\infty} (A \vee M^i)$ of $\mathcal{L}^*$. Then $\mathcal{L}^*$ is a local Noether lattice with maximal element $M^* = \sum_{i=1}^{\infty} M$. It can be seen that $\mathcal{L}^*$ is a collection of representatives of equivalence classes of Cauchy sequences of $\mathcal{L}$ with the M-adic metric and in fact is the completion of $\mathcal{L}$ with this metric. If $\sum_{i=1}^{\infty} B_i$ is an element of $\mathcal{L}^*$, then $C(\sum_{i=1}^{\infty} B_i)$ is the element $\bigwedge_{i=1}^{\infty} B_i$ of $\mathcal{L}$. For each positive integer i, the map $A \to A^*$ from $\mathcal{L}/M^i$ to $\mathcal{L}^*/(M^*)^i$ and the map $B \to C(B)$ from $\mathcal{L}^*/(M^*)^i$ to $\mathcal{L}/M^i$ are multiplicative lattice isomorphisms. Additional properties can be found in [8]-[10].

THEOREM 4. Let $\mathcal{L}$ be a local Noether lattice with maximal element M and let A be an element of $\mathcal{L}$ different from I. Then A is a p-system in $\mathcal{L}$ if and only if A^* is a p-system in $\mathcal{L}^*$.

PROOF. Suppose A is a principal system in $\mathcal{L}$ and n is a positive integer. Then by (1.2) there exists a meet-irreducible M-primary element Q of $\mathcal{L}$ such that $A \le Q \le A \vee M^n$. Thus, it follows that $A^* \le Q^* \le A^* \vee (M^*)^n$. Since $M = \mathrm{Rad}(Q)$, pick a positive integer k such that $M^k \le Q$. Since $\mathcal{L}/M^k$ is isomorphic to $\mathcal{L}^*/(M^*)^k$, it follows that Q^* is a meet-irreducible M^*-primary element of $\mathcal{L}^*$. Thus, by (1.2), A^* is an p-system in $\mathcal{L}^*$.

Conversely, suppose A^* is a p-system in $\mathcal{L}^*$ and n is a positive integer. Then by (1.2) there exists a meet-irreducible M^*-primary element Q' of $\mathcal{L}^*$ such that $A^* \le Q' \le A^* \vee (M^*)^n$. Thus, it follows that

$$A = C(A^*) \le C(Q') \le C(A^* \vee (M^*)^n) = A \vee M^n. \tag{12}$$

Since $M^* = \mathrm{Rad}(Q')$, pick a positive integer r such that $(M^*)^r \le Q'$. Therefore, we have that $M^r = C((M^*)^r) \le C(Q')$ and since $\mathcal{L}^*/(M^*)^r$ is isomorphic to $\mathcal{L}/M^r$, we get that $C(Q')$ is a meet-irreducible M-primary element of $\mathcal{L}$. Hence, by (1.2), A is a principal system in $\mathcal{L}$. This completes the proof.

We now summarize the results of the previous theorems.

THEOREM 5. Let $\mathcal{L}$ be a local Noether lattice with maximal element M and let A be an element of $\mathcal{L}$ different from I. Then the following are equivalent:

(5.1) A is a p-system in $\mathcal{L}$

(5.2) for every positive integer n, there exists a meet-irreducible M-primary element Q of $\mathcal{L}$ such that $A \le Q \le A \vee M^n$

(5.3) A is a closure point of $\wp_A$ in the M-adic topology on $\mathcal{L}$

(5.4) there exists a decreasing sequence $\{Q_n\}$ of meet-irreducible M-primary elements of $\mathcal{L}$ such that

(i) $A = \bigwedge_{n=1}^{\infty} Q_n$, and

(ii) if Q is an M-primary element of $\mathcal{L}$ satisfying $A \le Q$, then there is a positive integer n such that $Q_n \le Q$

(5.5) for all elements B of $\mathcal{L}$ satisfying $B \le A$, A is a p-system in $\mathcal{L}/B$

(5.6) the zero element of $\mathcal{L}/A$ is a p-system in $\mathcal{L}/A$

(5.7) there exists an element B of $\mathcal{L}$ satisfying $B \le A$ such that A is a p-system in $\mathcal{L}/B$

(5.8) A^* is a p-system in $\mathcal{L}^*$.

We now turn our attention to rings where in general we adopt the terminology of [14]. Let R be a local Noetherian ring with maximal ideal M. Then $\mathcal{L}(R)$, the lattice of ideals of R, is a local Noether lattice. We say that an ideal A of R is irreducible if A is a meet-irreducible element of $\mathcal{L}(R)$. In addition, an ideal A of R is a principal system in R if and only if A is a p-system in $\mathcal{L}(R)$. Let R^* denote the ring M-adic completion of R and let AR^* denote the (ring) extension of an ideal A of R to R^*. Thus, we obtain the following result.

THEOREM 6. Let R be a local Noetherian ring with maximal ideal M and let A be a proper ideal of R. Then the following are equivalent:

(6.1) A is a principal system in R

(6.2) for every positive integer n, there exists an irreducible M-primary ideal Q of R such that $A \subseteq Q \subseteq A + M^n$

(6.3) A is a closure point of $\wp_A$ in the M-adic topology on $\mathcal{L}(R)$

(6.4) there exists a decreasing sequence $\{Q_n\}$ of irreducible M-primary ideals of R such that

(i) $A = \bigcap_{n=1}^{\infty} Q_n$, and

(ii) if Q is an M-primary ideal of R satisfying $A \subseteq Q$, then there is a positive integer n such that $Q_n \subseteq Q$

(6.5) for all ideals B of R satisfying $B \subseteq A$, A is a principal system in R/B

(6.6) the zero ideal of R/A is a principal system in R/A

(6.7) there exists an ideal B of R satisfying $B \subseteq A$ such that A is a principal system in R/B

(6.8) AR^* is a principal system in R^*.

PROOF. The equivalences of (6.1)-(6.7) follows from (5.1)-(5.7) and the fact that Q is an irreducible M-primary ideal of R if and only if Q is a meet-irreducible M-primary element of $\mathcal{L}(R)$. To show that (6.1) and (6.8) are equivalent, we have the following chain of equivalences:

A is a principal system in R

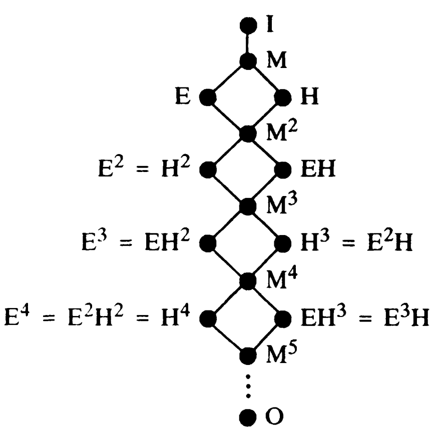
$\Leftrightarrow$ A is a p-system in $\mathcal{L}(R)$

$\Leftrightarrow$ A^* is a p-system in $\mathcal{L}(R)^*$

$\Leftrightarrow$ AR^* is a p-system in $\mathcal{L}(R^*)$

$\Leftrightarrow$ AR^* is a principal system in R^*.

The second equivalence follows from the equivalence of (5.1) and (5.8), whereas the third equivalence follows from the fact that there exists an isomorphism $\varphi : \mathcal{L}(R)^* \to \mathcal{L}(R^*)$ with the property that $\varphi(A^*) = AR^*$ (see [10, Theorem 3, p. 158] and its proof). This completes the proof.



We conclude this paper by giving an example of a local Noether lattice which is not the lattice of ideals of any commutative ring and which has a variety of principal systems. Let $\mathcal{L}$ be the sublattice of RL_2 [3] pictured above. It is easily seen that all elements of $\mathcal{L}$ except for I and O are M-primary elements of $\mathcal{L}$. In addition, the maximal element M as well as all elements of the form E^m (where $m > 0$) or E^nH (where $n \geq 0$) are p-systems since these elements are also meet-irreducible. However, the least element O is another p-system in $\mathcal{L}$ since the sequence $\{E^n\}$ satisfies the conditions of Theorem 2 and O is the meet of the elements of this sequence. Finally, since $\mathcal{L}$ is distributive and is not a chain, it follows [3, Theorem 3, p. 222] that $\mathcal{L}$ is not the lattice of ideals of any commutative ring.

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