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FIXED POINT THEOREMS FOR NON-SELF MAPS I

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ABSTRACT. Suppose $f: C \to X$ where C is a closed subset of X. Necessary and sufficient conditions are given for f to have a fixed point. All results hold when X is complete metric space. Several results hold in a much more general setting.

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1. INTRODUCTION.

Fixed point theorems for non-self maps are unusual. We surely require that $C \cap f(C)$ is non-empty. $f(x) = x + 1$ for X in $[0,1]$ is a linear isometry from the compact space $[0,1]$ into the compact space $[0,2]$ but f is fixed point free. The mapping $f(x) = x + \frac{1}{x}$ for x in $[1,\infty)$ is a continuous mapping from $[1,\infty)$ into $[0,\infty)$. It is fixed point free but $|f(x) - f(y)| < |x - y|$ for $x \neq y$.

THEOREM (Brouwer [1]). If E is a non-empty convex compact subset of E^n and $f: E \to E$ is continuous, then $f(x) = x$ for some x in E.

2. RESULTS.

THEOREM 1. Let C be a closed subset of a complete metric space X and suppose f maps C onto X or f maps C into X with $C \subset f(C)$. If for some $k > 1$, $d(f(x), f(y)) \geq k\ d(x,y)$ for every x, y in C, then f has a unique fixed point in C.

PROOF. Clearly, f is one-to-one. Let $g = f^{-1}$ restricted to C. Now g maps C into C. For x, y in C, $d(x,y) = d(f(gx), f(gy)) \geq k\ d(g(x), g(y))$ or $d(g(x), g(y)) \leq \frac{1}{k}\ d(x,y)$ and $0 < \frac{1}{k} < 1$. g has a unique fixed point from Banach's fixed point theorem. But $f(x_0) = f(g(x_0)) = x_0$. If $x_1 = f(x_1)$, then $g(x_1) = g(f(x_1)) = x_1$ and $c_1 = x_0$.

The above result suggests that one should consider non-self maps that satisfy $C \subset f(C)$. It is well known that a continuous function from an arc onto a containing arc must have a fixed point. $[0,1]$ or any homeomorphic image is called an arc. Thus Brouwer's theorem extends to the case $C \subset f(C)$ for $n = 1$. In [7], Sam Nadler showed that for $n \geq 2$ Brouwer's theorem does not extend. For $n \geq 2$, let A and B be closed balls in E^n with $A \subset B$ and $A \neq B$. He showed that there exists f and g such that:

(a) $f: A \to B$ where f is continuous, onto, $f(\partial A) = B$, and f is fixed point free,

(b) $g: A \to B$ where g is continuous, onto, $g^{-1}(\partial B) = \partial A$, and g if fixed point free.

THEOREM 2. Let C be a closed bounded, and convex subset of a uniformly convex Banach space and suppose f maps C onto X or f maps C into X with $C \subset f(C)$. If for every x,y in C $\| f(x) - f(y) \| \geq \| x - y \|$, then f has a fixed point in C.

PROOF. Clearly, f is one-to-one. Let $g = f^{-1}$ restricted to C and observe that $\| g(x) - g(y) \| \leq \| x - y \|$ where $g: C \to C$. From Kirk's theorem [6], g has a fixed point x_0 in C. Clearly, $f(x_0) = x_0$.

The following is an example of a mapping f that takes a closed, bounded, and convex subset C of a Banach space X into X where $C \subset f(C)$, $\| f(x) - f(y) \| = \| x - y \|$ for all $x, y \in C$, and f has no fixed points.

EXAMPLE 1. Let X be the space of sequences which converge to zero with $\| x \| = \sup_n | x_n |$ for x in X. Let $C = \{x \in X: \| x \| = 1 \text{ and } x_0 = 1\}$. C is closed, bounded, and convex. Define $f: C \to X$ by $f(x) = y$ where $y_n = x_{n+1}, n = 0, 1, 2, \cdots$. $\| f(x) - f(y) \| = \| x - y \|$ and f is linear. To see that $C \subset f(C)$ consider the following. For $z \in C$, define r to be the sequence where $r_0 = 1$ and $r_n = z_{n-1}, n = 1, 2, 3, \cdots$. Then $r \in C$, and $f(r) = z$ so $C \subset f(C)$. If $s = \{1, 0, 0, \cdots\}, s \in C$ but $f(s) = \{0, 0, 0, \cdots\} \notin C$. Hence $C \neq f(C)$. If $f(x) = x$ for some x in C, then $x_n = x_{n+1}$ for $n = 0, 1, 2, \cdots$. Since $x_0 = 1, x_n = 1$ for all n and $x \notin C$. Therefore, f does not have a fixed point in C.

The following example shows that Banach's fixed point theorem does not generalize to non-self maps.

EXAMPLE 2. Let $X = C(\boldsymbol{R}, \boldsymbol{R})$ with $\| f \| = \sup_{t \in \boldsymbol{R}} | f(t) |$ for $f \in X$. Let $C = \{f \in X: f(t) = 0 \text{ for all } t \leq 0 \text{ and } \lim_{t \to \infty} f(t) \geq 1\}$. C is a closed and convex subset of X. Define $T: C \to X$ by $(Tf)(t) = \frac{1}{2} f(t+1)$. To see that $C \subset T(C)$ consider the following. For f in C set $g(t) = 2f(t-1)$. $g(t) = 0$ for $t \leq 0$ since $t - 1 < 0$ and $f(t) = 0$ for all $t \leq 0$. $\lim_{t \to \infty} g(t) = \lim_{t \to \infty} 2\, f(t-1) \geq 2$. Thus $g \in C$ and $(Tg)(t) = f(t)$. Hence $C \subset T(C)$. Let $f(t)$ be defined as 0 if $t \leq 0$, t if $0 < t < 1$, and 1 if $t \geq 1$. Then $f \in C$. Now $(Tf)(t)$ is 0 if $t \leq -1$, $\frac{1}{2}(t+1)$ if $-1 < t < 0$, and $\frac{1}{2}$ if $t \geq 0$. Therefore, $Tf \notin C$ and $C \neq T(C)$. For $f, g \in C$, $\| Tf - Tg \| = \frac{1}{2} \| f - g \|$. If $Tf = f$ for some $f \in C$, then $f(t) = \frac{1}{2} f(t+1)$ and it follows that $f(n) = 0$ for all integers n. Hence $\lim_{t \to \infty} f(t) \ngeq 1$ and $f \notin C$. Therefore T does not have a fixed point in C. Note that T is linear, one-to-one, and $T(C)$ is closed.

We now turn to finding necessary and sufficient conditions for a non-self map to have a fixed point. Then it becomes clear that $C \subset f(C)$ is a natural assumption.

Let (X, t) be a topological space and $d: X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}$ is convergent in (X, t). These spaces include complete (quasi) metric spaces and d-complete (symmetric) semi-metric spaces. In [2] and [3] several basic metric space fixed point theorems were extended to this setting. $f: X \to X$ is w-continuous at x if $x_n \to x$ as $n \to \infty$ implies $f(x_n) \to f(x)$ as $n \to \infty$.

The following definition was given by G. Jungck in [5].

DEFINITION 1. Two maps f and g are compatible if, for any sequence $\{x_n\}$ such that $\lim_n f(x_n) = \lim_n g(x_n) = t$ it follows that $\lim_n d(f(gx_n), g(fx_n)) = 0$. Commuting maps are compatible but the converse is false.

DEFINITION 2. Given a map f, a map g is compatible with f, if for any sequence $\{x_n\}$ such that $\lim_n f(x_n) = \lim_n g(x_n) = t$ it follows that $\lim_n f(g(x_n)) = g(t)$.

REMARK 1. If f and g are w-continuous and (X, d) is a metric space, then, using definition

2, f is compatible with g is equivalent to g is compatible with f. In this case, we say that f and g are compatible.

PROOF. Assume f and g are w-continuous and that $\lim_n f(x_n) = \lim_n g(x_n) = t$ implies $\lim_n f(g(x_n)) = g(t)$. If we are in a metric space,

$$d(f(gx_n), g(fx_n)) \leq d(f(gx_n), g(t)) + d(g(t), g(fx_n))$$

and

$$d(g(fx_n), f(t)) \leq d(g(fx_n), f(gx_n)) + d(f(gx_n), f(t)).$$

It follows that f is compatible with g implies that g is compatible with f. Interchanging g and f above gives the converse.

It also follows from the above argument that if f and g are w-continuous and (X,d) is a metric space, then the two definitions of compatibility are equivalent.

REMARK 2. If (X,t) is a d-complete topological space, g is w-continuous, and f and g commute, then g is compatible with f using definition 2. We use definition 2 for d-complete topological spaces.

Theorem 3 and its corollaries are generalizations of theorems due to Hicks and Rhoades [4] which are generalizations of theorems due to Jungck [5].

THEOREM 3. Let (X,t) be a Hausdorff d-complete topological space and suppose $f: C \to X$ where f is w-continuous and C is a closed subset of X. Then f has a fixed point in C if and only if there exists $\alpha \in (0,1)$ and a w-continuous function $g: C \to C$ such that $g(C) \subset f(C), g$ is compatible with f on $f^{-1}(C)$ and

(1) $d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ for all $x, y \in C$.

Indeed, if (1) holds, f and g have a unique common fixed point.

PROOF. If $f(a) = a$ for some $a \in C$, set $g(x) = a$ for every $x \in C$. If $x \in f^{-1}(C)$, $f(x) \in C$ and $g(f(x)) = a = f(a) = f(g(x))$. If $x \in C, g(x) = a = f(a)$ gives $g(C) \subset f(C)$. Also, $0 = d(a,a) = d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ for all $x, y \in C$.

Suppose there exists $\alpha \in (0,1)$ and a w-continuous function $g: C \to C$ such that $g(C) \subset f(C)$, g is compatible with f on $f^{-1}(C)$ and $d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ for all $x, y \in C$. Let $x_0 \in C$. $g(x_0) = f(x_1)$ for some $x_1 \in C$ since $g(C) \subset f(C)$. Construct a sequence $\{x_n\}$ with $\{x_n\} \subset C$ and $f(x_n) = g(x_{n-1})$ for $n = 1, 2, 3, \cdots$. Since

$$d(f(x_n), f(x_{n+1})) = d(g(x_{n-1}), g(x_n)) \leq \alpha d(f(x_{n-1}), f(x_n)),$$

it follows that $d(f(x_n), f(x_{n+1})) \leq \alpha^{n-1} d(f(x_1), f(x_2))$. Hence $\sum_{n=1}^{\infty} d(f(x_n), f(x_{n+1})) < \infty$. The space is d-complete so there exists $p \in X$ with $\lim_n f(x_n) = p$. $f(x_n) = g(x_{n-1}) \in C$ gives $p \in cl(C) = C$. Now f is w-continuous gives $f(g(x_{n-1})) \to f(p)$ as $n \to \infty$. Since g is compatible with f on $f^{-1}(C)$ and $p \in f^{-1}(C)$ we get $\lim_n f(g(x_n)) = g(p)$. The space is Hausdorff so $f(p) = g(p)$ and $p \in f^{-1}(C)$. Consider the sequence $y_n = p$ for all n. Then $f(y_n) \to f(p)$ as $n \to \infty$, $g(y_n) \to g(p)$ as $n \to \infty$, and compatibility give $f(g(p)) = f(g(y_n)) \to g(f(p))$ as $n \to \infty$. Thus, $f(g(p)) = g(f(p))$. Therefore, $f(f(p)) = f(g(p)) = g(f(p)) = g(g(p))$ together with $d(g(p), g(gp)) \leq \alpha d(f(p), f(gp)) = \alpha d(g(p), g(gp))$ implies $g(p) = g(g(p))$. Hence $g(p) = g(g(p)) = f(g(p))$ and $g(p)$ is a common fixed point of f and g.

If $x = f(x) = g(x)$, then $d(x, g(p)) = d(g(x), g(gp)) \leq \alpha d(f(x), f(gp)) = \alpha d(x, g(p))$ gives $x = g(p)$.

COROLLARY 1. Let (X,t) be a Hausdorff d-complete topological space and C be a closed subset of X. Suppose $f{:}C \to X$ and $g{:}C \to C$, where f and g are w-continuous, commute on $f^{-1}(C)$, and $g(C) \subset f(C)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that $d(g^k(x), g^k(y)) \leq \alpha d(f(x), f(y))$ for all $x, y \in C$, then f and g have a unique common fixed point.

PROOF. Clearly, g^k commutes with f on $f^{-1}(C)$ and $g^k(C) \subset g(C) \subset f(C)$. Applying the theorem to g^k and f gives a unique $p \in C$ such that $p = g^k(p) = f(p)$. Since f and g commute on $f^{-1}(C)$ and $p \in f^{-1}(C), g(p) = g(f(p)) = f(g(p)) = g^k(g(p))$ or $g(p)$ is a common fixed point of f and g^k. Uniqueness of the common fixed point of f and g^k gives $g(p) = p = f(p)$. If $q = g(q) = f(q)$ then $g^k(q) = f(q)$ and $q = p$.

COROLLARY 2. Let n be a positive integer and let $\alpha > 1$. Suppose C is a closed subset of a Hausdorff d-complete topological space and $f{:}C \to X$ with $C \subset f(C)$. If $d(f^n(x), f^n(y)) \geq \alpha d(x,y)$ for all x, y in $(f^{n-1})^{-1}(C)$, then f has a fixed point in C.

PROOF. For $n = 1$, this follows from corollary 1 by letting $g = I$. f^n is one-to-one. $C \subset f(C)$ implies $C \subset f^n(C)$. Let h be the restriction of $(f^n)^{-1}$ to C. $h{:}C \to C$ and $d(h(x), h(y)) \leq \frac{1}{\alpha}\, d(x,y)$ for all $x, y \in C$. From corollary 1 with $k = 1, h = g^k = g$ and $f = I$, there exists a unique x_0 such that $h(x_0) = x_0$. Hence $f(x_0) = f^{n+1}(x_0) = f^n(f(x_0))$ or $h(f(x_0)) = (f^n)^{-1}(f(x_0)) = f(x_0)$. Uniqueness of the fixed point for h gives $x_0 = f(x_0)$. If $f(y) = y$ then $f^n(y) = y$ and $y = h(y)$. Again, uniqueness of the fixed point for h gives $x_0 = y$.

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