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# (QUASI) - UNIFORMITIES ON THE SET OF BOUNDED MAPS

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**ABSTRACT.** From real analysis it is known that if a sequence $\{f_n, n\in\mathbb{N}\}$ of real-valued functions defined and bounded on $X\subset\mathbb{R}$ converges uniformly to f, then f is also bounded and the sequence $\{f_n, n\in\mathbb{N}\}$ is uniformly bounded on X. In the present paper we generalize results as the above using (quasi)-uniform structures.

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## 1. INTRODUCTION

Let $(Y,\mathcal{U})$ be a uniform space. A set $A\subset Y$ is said $\mathcal{U}$-bounded (see [1], [3] and [6]) (there, "$\mathcal{U}$-bounded" is called "bounded"), if given an entourage $V\in\mathcal{U}$, there exists a positive integer n and a finite set $F\subset Y$, such that $A\subset V^n(F)$. Also it is known that a set $A\subset Y$ is precompact (totally bounded) in $(Y,\mathcal{U})$ if given an entourage $V\in\mathcal{U}$, there exists a finite set $F\subset Y$, such that $V(F)\supset A$ (see [1]). Instead of the term "precompact" we will use in the following the term $\mathcal{U}^*$-bounded.

It is obvious that the class of $\mathcal{U}^*$-bounded subsets of a uniform space $(Y,\mathcal{U})$ is broader than the class of $\mathcal{U}$-bounded subsets. It is well known that $\mathcal{U}$-boundedness and $\mathcal{U}^*$-boundedness are also boundedness in the sense of Hu [4].

Given a uniform space $(Y,\mathcal{U})$ by $\mathcal{T}_{\mathcal{U}}$ we denote the topology of the uniform space.

If X is a set, if $(Y,\mathcal{U})$ is a uniform space and if $\alpha$ is a covering of X, then the uniformity $u_\alpha$ of uniform convergence on members of $\alpha$ on $\mathcal{F}(X,Y)$, (the set of all functions from X to Y) is generated by the subbasis $\mathcal{Y} = \{(A,V): A\in\alpha, V\in\mathcal{U}\}$, where $(A,V) = \{(f,g)\in\mathcal{F}(X,Y)\times\mathcal{F}(X,Y): (f(x), g(x))\in V, \text{ for each } x\in A\}$. The corresponding topology of $u_\alpha$, $\mathcal{T}_{u_\alpha}$, is called the topology of uniform convergence on the members of $\alpha$. The subbasic $\mathcal{T}_{u_\alpha}$-neighborhoods of an arbitrary $f\in\mathcal{F}(X,Y)$ are of the form $(A,V)(f) = \{g\in\mathcal{F}(X,Y): (f,g)\in(A,V)\}$, where $A\in\alpha$, $V\in\mathcal{U}$ (see [5]).

We will also use the following symbols:

$\mathcal{B}_\alpha(X,Y) = \{f\in\mathcal{F}(X,Y): f(A) \text{ is } \mathcal{U}\text{-bounded for every } A\in\alpha\}$

$\mathcal{B}^*_\alpha(X,Y) = \{f\in\mathcal{F}(X,Y): f(A) \text{ is } \mathcal{U}^*\text{-bounded for every } A\in\alpha\}$

$\mathcal{B}(X,Y) = \{f\in\mathcal{F}(X,Y): f(X) \text{ is } \mathcal{U}\text{-bounded}\}$

$\mathcal{B}^*(X,Y) = \{f\in\mathcal{F}(X,Y): f(X) \text{ is } \mathcal{U}^*\text{-bounded}\}$.

The uniformity of uniform convergence is denoted by $\mu$, the topology of uniform convergence by $\mathcal{T}_\mu$ (see [5]).

By $(\mathcal{F}(X,Y),\mathcal{T})$ we denote the set $\mathcal{F}(X,Y)$ equipped with the topology $\mathcal{T}$.

## 2. THE SET OF BOUNDED FUNCTIONS OF $\mathcal{F}(X,Y)$

PROPOSITION 2.1. Let $a$ be a collection of subsets covering the set X and $(Y,\mathcal{U})$ a uniform space. The collection $\mathcal{Y} = \{<A,V>: A\in a,\ V\in\mathcal{U}\}$, where $<A,V> = \{(f,g) \in \mathcal{F}(X,Y) \times \mathcal{F}(X,Y):\ f(A) \subset V(g(A))\}$, is a subbasis for a quasi uniformity $\omega_a$ on $\mathcal{F}(X,Y)$, which is contained in the uniformity $u_a$ of uniform convergence on the members of $a$.

PROOF. Let an arbitrary $<A,V>\in\mathcal{Y}$. Then $(f,f)\in<A,V>$, because $f(A)\subset V(f(A))$. Also given an arbitrary $<A,V>\in\mathcal{Y}$ we choose a $U\in\mathcal{U}$, $U o U\subset V$ and we observe that $<A,U> o <A,U> \subset <A,V>$.

Now we prove that $\omega_a \subset u_a$. Let $<A,V>\in\mathcal{Y}$. We choose a symmetric $U\in\mathcal{U}$, $U\subset V$ and we have that $(A,U)\subset<A,V>$. Indeed if $(f,g)\in(A,U)$, then $(f(x),g(x))\in U$, for each $x\in A$. This means that $f(x)\in U^{-1}(g(x)) = U(g(x))$ for each $x\in A$ and thus $f(A)\subset U(g(A))$. So $(A,U)\subset<A,V>$ and hence $\omega_a\subset u_a$.

REMARK 2.2. a) The quasi uniformity $\omega_a^{-1}$ is generated by the sets of the form $[A,V] = \{(f,g) \in \mathcal{F}(X,Y) \times \mathcal{F}(X,Y):\ g(A) \subset V(f(A)),$ where $A\in a$, $V\in\mathcal{U}$. It is obvious that the conjugate quasi uniformity $\omega_a^{-1}$ is also contained in $u_a$.

Finally the supremum uniformity $\omega_a \vee \omega_a^{-1}$ is also contained in $u_a$ and has a basis $\mathcal{B} = \{<A,V> \cap [A,V]:\ A\in a,\ V\in\mathcal{U}\}$, where $<A,V> \cap [A,V] = \{(f,g) \in \mathcal{F}(X,Y) \times \mathcal{F}(X,Y):\ f(A) \subset V(g(A))$ and $g(A)\subset V(f(A))\}$.

b) If we consider $a=\{X\}$, it is easily seen that $\mathcal{Y} = \{<X,V>:\ V\in\mathcal{U}\}$ is a basis for $\omega_a$.

PROPOSITION 2.3. Let X be a set, $a$ be a collection of subsets covering X and let $(Y,\mathcal{U})$ be a uniform space. Then the sets $\mathcal{B}_a(X,Y)$, $\mathcal{B}^*_a(X,Y)$ are closed in the topological space $(\mathcal{F}(X,Y),\mathcal{T}_{\omega_a})$.

PROOF. First we prove that $\mathcal{B}_a(X,Y)$ is closed in the topological space $(\mathcal{F}(X,Y),\ \mathcal{T}_{\omega_a})$. Let a net $\{f_\lambda,\ \lambda\in\Lambda\} \subset \mathcal{B}_a(X,Y)$, which converges to f with respect to $\omega_a$. We prove that $f\in\mathcal{B}_a(X,Y)$. Let an arbitrary $V\in\mathcal{U}$ and $A\in a$. Then $f\in<A,V>(f)$. So, there exists a $\lambda_0\in\Lambda$, such that for each $\lambda>\lambda_0$, $f_\lambda\in<A,V>(f)$. Let a $\lambda>\lambda_0$. Then $(f,f_\lambda)\in<A,V>$, which means that $f(A)\subset V(f_\lambda(A))$. But $f_\lambda\in\mathcal{B}_a(X,Y)$, so there exists $m\in\mathbb{N}$ and a finite set $F\subset Y$, such that $f_\lambda(A)\subset V^m(F)$. Thus $f(A)\subset V^{m+1}(F)$, which means that $f\in\mathcal{B}_a(X,Y)$ and hence $\mathcal{B}_a(X,Y)$ is closed in the topological space $(\mathcal{F}(X,Y),\mathcal{T}_{\omega_a})$.

It can be also easily proved that $\mathcal{B}^*_a(X,Y)$ is closed in the topological space $(\mathcal{F}(X,Y),\ \mathcal{T}_{\omega_a})$. The proof is the same as the above if we observe that $f\in<A,U>(f)$, where $U\in\mathcal{U}$, $U o U\subset V$ and that $f(A)\subset (U o U)(F)\subset V(F)$.

COROLLARY 2.4. Let X be a set, $a$ be a collection of subsets covering X and let $(Y,\mathcal{U})$ be a uniform space. Then the sets $\mathcal{B}_a(X,Y)$, $\mathcal{B}^*_a(X,Y)$ are closed in $(\mathcal{F}(X,Y),\ \mathcal{T}_{u_a})$. Hence, $\mathcal{B}_a(X,Y)$, $\mathcal{B}^*_a(X,Y)$ are complete if $(Y,\mathcal{U})$ is complete.

COROLLARY 2.5. Let X be a set and let $(Y,\mathcal{U})$ be a uniform space. Then the sets $\mathcal{B}(X,Y)$, $\mathcal{B}^*(X,Y)$ are closed in $(\mathcal{F}(X,Y),\ \mathcal{T}_\mu)$. Hence, $\mathcal{B}(X,Y)$, $\mathcal{B}^*(X,Y)$ are complete if $(Y,\mathcal{U})$ is complete.

PROOF. We set in the previous corollary $a=\{X\}$.

REMARK 2.6. If $(X,d)$ is a metric space and $\mathcal{U}_d$ is its corresponding uniformity generated by d, it is known (see [3]) that $\mathcal{U}_d$- boundedness coincides with d-boundedness. So by the above corollary corresponding theorems of metric spaces (see [2]) are generalized.

It is also known (see [3]) that in uniform locally totally bounded spaces, $\mathcal{U}$-boundedness and $\mathcal{U}^*$-boundedness coincide.

## 3. UNIFORMLY BOUNDED NETS

Let us recall the definition of uniform boundedness of a real- valued sequence of functions:

A sequence $\{f_n,\ n\in\mathbb{N}\}$, where $f_n:X\to\mathbb{R}$ is said uniformly bounded iff there exists M>0, such that for each $n\in\mathbb{N}$, $|f_n(x)|\le M$ for each $x\in X$.

Motivated by this fact we give the following definition.

DEFINITION 3.1. Let X be a set, $\alpha$ be a covering of X and let be $(Y,\mathcal{U})$ a uniform space.

a) A net $\{f_\lambda,\ \lambda\in\Lambda\}\subset\mathcal{F}(X,Y)$ is said to be finally $\mathcal{U}$-uniformly bounded on the members of $\alpha$, if for each $V\in\mathcal{U}$, there exists a $\lambda_0$, a finite set $F\subset Y$ and a $m\in\mathbb{N}$, such that $f_\lambda(A)\subset V^m(F)$, for each $\lambda>\lambda_0$ and for each $A\in\alpha$. We also say that $\{f_\lambda,\ \lambda\in\Lambda\}$ is uniformly bounded on the members of $\alpha$ if the last inclusion holds for each $\lambda\in\Lambda$.

b) A net $\{f_\lambda,\ \lambda\in\Lambda\}\subset\mathcal{F}(X,Y)$ is said to be finally $\mathcal{U}^*$-uniformly bounded on the members of $\alpha$, if for each $V\in\mathcal{U}$, there exists a $\lambda_0\in\Lambda$ and a finite $F\subset Y$, such that $f_\lambda(A)\subset V(F)$ for each $\lambda>\lambda_0$ and for each $A\in\alpha$.

If the inclusion holds for every $\lambda\in\Lambda$, we say that $\{f_\lambda,\ \lambda\in\Lambda\}$ is $\mathcal{U}^*$-uniformly bounded on the members of $\alpha$.

If $A=\{X\}$ we use the notation "$\mathcal{U}$-uniformly bounded" instead of "$\mathcal{U}$-uniformly bounded on the members of $\alpha=\{X\}$".

PROPOSITION 3.2. Let X be a space and let $\alpha$ be a covering of X and $(Y,\mathcal{U})$ a uniform space. Let $\{f_\lambda,\ \lambda\in\Lambda\}\subset\mathcal{B}_\alpha(X,Y)$ (resp. $\{f_\lambda,\ \lambda\in\Lambda\}\subset\mathcal{B}^*_\alpha(X,Y)$) be a net converging to f with respect to the topology $\mathcal{T}_{\mathcal{W}_\alpha\vee\mathcal{W}_\alpha^{-1}}$. Then $\{f_\lambda,\ \lambda\in\Lambda\}$ is a finally $\mathcal{U}$-uniformly (resp. $\mathcal{U}^*$-uniformly) bounded net on the members of $\alpha$.

PROOF. First we suppose that $\{f_\lambda,\ \lambda\in\Lambda\}\subset\mathcal{B}_\alpha(X,Y)$ and we prove that $\{f_\lambda,\ \lambda\in\Lambda\}$ is finally $\mathcal{U}$-uniformly bounded on the members of $\alpha$. Let $V\in\mathcal{U}$ and $A\in\alpha$. Then $[A,V](f)$ is a $\mathcal{T}_{\mathcal{W}_\alpha^{-1}}$ neighborhood of f, so it is also a $\mathcal{T}_{\mathcal{W}_\alpha\vee\mathcal{W}_\alpha^{-1}}$ neighborhood of f. So, there exists a $\lambda_0\in\Lambda$, such that $f_\lambda\in[A,V](f)$, for each $\lambda>\lambda_0$, which means that $f_\lambda(A)\subset V(f(A))$ for each $\lambda>\lambda_0$. But since the net $\{f_\lambda,\ \lambda\in\Lambda\}$ converges to f with respect to the topology $\mathcal{T}_{\mathcal{W}_\alpha\vee\mathcal{W}_\alpha^{-1}}$, it also converges to f with respect to $\mathcal{T}_{\mathcal{W}_\alpha}$. So by Proposition 2.2, $f\in\mathcal{B}_\alpha(X,Y)$. Thus, there exists a $m\in\mathbb{N}$ and F finite, $F\subset Y$, such that $f(A)\subset V^m(F)$. So by (1) we have that $f_\lambda(A)\subset V^{m+1}(F)$ for each $\lambda>\lambda_0$. This means that $\{f_\lambda,\ \lambda\in\Lambda\}$ is finally $\mathcal{U}$-bounded on the members of $\alpha$.

Now we suppose that $\{f_\lambda,\ \lambda\in\Lambda\}\subset\mathcal{B}^*_\alpha(X,Y)$ and we prove that $\{f_\lambda,\ \lambda\in\Lambda\}$ is finally $\mathcal{U}^*$-uniformly bounded on the members of $\alpha$. Let $V\in\mathcal{U}$ and $A\in\alpha$. We choose a $U\in\mathcal{U}$, such that $U\circ U\subset V$. Then $f\in[A,U](f)$ and following the above process we prove that $f_\lambda(A)\subset(U\circ U)(F)\subset V(F)$ for each $\lambda>\lambda_0$.

PROPOSITION 3.3. Let X be a set and let $\alpha$ covering of X and let $(Y,\mathcal{U})$ be a uniform space. If $\{f_n,\ n\in\mathbb{N}\}$ is a sequence contained in $\mathcal{B}_\alpha(X,Y)$, (resp. in $\mathcal{B}^*_\alpha(X,Y)$) and converging to f with respect to the topology $\mathcal{T}_{\mathcal{W}_\alpha\vee\mathcal{W}_\alpha^{-1}}$, then $\{f_n,\ n\in\mathbb{N}\}$ is $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded on the members of $\alpha$.

PROOF. Let $\{f_n,\ n\in\mathbb{N}\}\subset\mathcal{B}_\alpha(X,Y)$. We prove that $\{f_n,\ n\in\mathbb{N}\}$ is $\mathcal{U}$-uniformly bounded on the members of $\alpha$. Let an arbitrary $V\in\mathcal{U}$ and let an arbitrary $A\in\alpha$. By the previous proposition there exists a $n_0 \in \mathbb{N}$, a finite subset $F \subset Y$ and a $m \in \mathbb{N}$, such that $f_n(A) \subset V^m(F)$ for each $n \in \mathbb{N}$, $n \geq n_0$. But the functions $f_n$, $1 \leq n \leq n_0$, are $\mathcal{U}$-bounded on the members of $\alpha$, so for the given $V \in \mathcal{U}$, there exists $m_n \in \mathbb{N}$ and finite subsets $F_n \subset Y$, $1 \leq n \leq n_0$, such that $f_n(A) \subset V^{m_n}(F_n)$, $1 \leq n \leq n_0$ Setting $m^* = \max\{m_1, m_2, \dots m_n, m\}$ and $F^* = F \cup (\bigcup_{n=1}^{no} F_n)$ we observe that $V^{m^*}(F^*) \supset f_n(A)$, for each $n \in \mathbb{N}$, which means that $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$-uniformly bounded on the members of $\alpha$.

For the other case we follow the same process as above, choosing $U \in \mathcal{U}$, $U o U \subset V$.

COROLLARY 3.4. Let X be a set and let $\alpha$ be a covering of X and $(Y, \mathcal{U})$ a uniform space. Let $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X,Y)$ (resp. $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}^*_\alpha(X,Y)$) be a net converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha}$. Then $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded net on the members of $\alpha$.

PROOF. It is an immediate consequence of Proposition 3.2 if we observe that $\omega_\alpha \vee \omega_\alpha^{-1} \subset \mathcal{U}_\alpha$.

COROLLARY 3.5. Let X be a set and $(Y, \mathcal{U})$ be a uniform space. Let $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X,Y)$, (resp. $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}^*_\alpha(X,Y)$) be a net converging to f with respect to the topology $\mathcal{T}_\mathcal{U}$. Then $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}$-uniformly (resp. $\mathcal{U}^*$-uniformly bounded.

PROOF. It is an immediate consequence of the above corollary if we set $\alpha = \{X\}$.

COROLLARY 3.6. Let X be a set and $\alpha$ be a covering of X and let $(Y, \mathcal{U})$ be a uniform space. If $\{f_n, n \in \mathbb{N}\}$ is a sequence contained in $\mathcal{B}_\alpha(X,Y)$ (resp. in $\mathcal{B}^*_\alpha(X,Y)$) and converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha}$, then $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded on the members or $\alpha$.

COROLLARY 3.7. Let X be a set and $(Y, \mathcal{U})$ be a uniform space. If $\{f_n, n \in \mathbb{N}\}$ is a sequence contained in $\mathcal{B}(X,Y)$, (resp. in $\mathcal{B}^*(X,Y)$) and converging to f with respect to the topology $\mathcal{T}_\mu$ then $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded.

Let us complete the above paragraph by giving a classical theorem of real analysis, as a corollary of the above results.

COROLLARY 3.8. If the sequence $\{f_n, n \in \mathbb{N}\}$ of real - valued functions defined and bounded on X converges uniformly to f, then f is also bounded and the sequence $\{f_n, n \in \mathbb{N}\}$ is uniformly bounded on X.

## REFERENCES


1. BURBAKI, N. General Topology, Part 1, 1966.
2. BURBAKI, N. General Topology, Part 2, 1966.
3. HEJCMAN, J. Czechoslovak. Math. J., 84 (1959), 544-562.
4. HU, S.T. Boundedness in a topological space, J. Math. Pure. Appl. 28 (1949), 287-320.
5. MURDESHWAR, M.G. and NAIMPALY S.A. Quasi-uniform topological spaces, Nootdhoff, Gronigen, 1966.
6. MURDESHWAR, M.G. THECKEDATH K.K. Boundedness in a quasi uniform space, Canad. Bull. 13 (1970) 367-370.