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EXISTENCE OF SOLUTIONS FOR A NONLINEAR HYPERBOLIC-PARABOLIC EQUATION IN A NON-CYLINDER DOMAIN

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ABSTRACT. In this paper, we study the existence of global weak solutions for the equation

$$k_2(x)u'' + k_1(x)u' + A(t)u + |u|^{\rho}u = f \tag{I}$$

in the non-cylinder domain Q in R^{n+1}; k_1 and k_2 are bounded real functions, $A(t)$ is the symmetric operator

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}\right)$$

where a_{ij} and f are real functions given in Q. For the proof of existence of global weak solutions we use the Faedo-Galerkin method, compactness arguments and penalization.

KEY WORDS AND PHRASES. Existence of weak solutions, Faedo-Galerkin method, compactness arguments.
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INTRODUCTION AND TERMINOLOGY.

Let $T \geq 0$ be a positive real number, O a bounded open set of $\mathbb{R}^n$ and $Q \subset O \times [0,T)$ a non-cylindrical domain in $\mathbb{R}^{n+1}$.

In the cylinder $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, Bensoussan et al. [1] and Lions [7] have studied the homogenization for the following Cauchy problem:

$$\begin{aligned} &k_2(x)u'' + k_1(x)u' + \Delta u = f \text{ in } \Omega \\ &u(x,0) = u_0(x) \text{ and } k_2(x)u'(x,0) = k_2^{1/2}(x)u_1(x), x \in \Omega \end{aligned} \tag{II}$$

Many authors have been investigating the solvability of solution for the nonlinear equations associated with problem (I) see: Larkin [4], Lima [5], Medeiros [9], Medeiros [10], Medeiros [11], Melo [12], Maciel [13], Neves [14] and Vagrov [16].

In the non-cylindrical domain Q, Lions, J.L. [8] studied the existence and uniqueness of global weak solutions for nonlinear equations associated with problem (II) with nonlinearity of type $|u|^{\rho}u$.

Let $\Omega_t = Q \cap \{t = s\}$ be a plane in $\mathbb{R}^{n+1}$. Analogously $\Omega_0 = Q \cap \{t = 0\}$ and $\Omega_T = Q \cap \{t = T\}; \partial Q = \Gamma$ the boundary of $Q; \Gamma_s = \partial Q \cap \{t = s\}$ the boundary de Ω_s and $\Sigma = \cup_{0<s<T}\Gamma_s$ lateral boundary of Q. Therefore Q is a subset of $O \times (0,T)$ whose boundary is $\Omega_0 \cap \Sigma \cap \Omega_T$.

Let's denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm in $L^2(\Omega)$ and by $((\cdot,\cdot))$ and

$\|\cdot\|$ the inner product and norm in $H_0^1(\Omega)$. We identify $L^2(\Omega_t)$ and $H_0^1(\Omega_t)$ the sub-space of the $L^2(O)$ and $H_0^1(O)$ respectively, $\forall\ 0 \leq t \leq T$.

We define $L^p(0,T;L^2(\Omega_t))$ to be the space of functions v in $L^p(0,T;L^2(O))$ such that $v(t)$ in $L^2(\Omega_t)$ a.e. on t, for $1 \leq p \leq \infty$. By analogy we define $L^p(0,T;H_0^1(\Omega_t))$.

In this work we study the following problem: Let f, k_1, k_2, u_0 e u_1 be functions in appropriate spaces. We want to find the function $u: Q \to \mathbb{R}$ such that:

$$k_2(x)u'' + k_1(x)u' + A(t)u + |u|^\rho u = f \text{ in } Q, \text{ with } 0 < \rho \in \mathbb{R}, \text{ where}$$

$$u(x,0) = u_0(0), u_t(x,0) = u_1(x) \text{ in } \Omega_0 \text{ and}$$

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}\right) \text{ with } a_{ij} \text{ in } Q$$

We use Faedo-Galerkin's method and compactness arguments, see Lions, J.L. [7]

1. ASSUMPTIONS AND MAIN RESULT.

If we assume the following hypothesis:

(H.1) Let Ω_t^* be the projection of the Ω_t on the hyperplane $t = 0$. We may assume $\Omega_t^* \subset \Omega_s^*$ if $t \leq s$.

(H.2) For each $t \in [0,T], \Omega_t$ has the following regularity: If $u \in H_1(O)$ and $u = 0$ in $O - \Omega_t^*$ a.e., then the restriction of u to Ω_t belongs to $H_0^1(\Omega_t)$.

On the functions k_1, k_2 and a_{ij} we take:

(H.3) $k_1, k_2 \in L^\infty(\Omega_t); k_1(x) \geq \beta > 0, \beta \in \mathbb{R}; k_2(x) \geq 0$ for each $t \in [0,T]$.

(H.4) $a_{ij} = a_{ji} \in L^\infty(O \times (0,T))$ and $a'_{ij} = \frac{\partial}{\partial t}a_{ij} \in L^\infty(O \times (0,T))$.

There is $0 < \delta \in \mathbb{R}$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \delta(|\xi_1|^2 + \cdots + |\xi_n|^2), (x,t) \in O \times (0,T), \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$$

Let $a(t,u,v)$ denote the bilinear form associated to the operator $A(t)$ From (H.4) and, using Cauchy-Schwartz, we obtain:

$$a(t,u,v) \leq C\|u\| \cdot \|v\|; \forall u,v \in H_0^1(O).$$

Also by Poincaré-Friedrichs inequality and of (H.4), there exists $\alpha > 0$, real, such that:

$$a(t,u,v) \geq \alpha \|u\|^2; \forall u \in H_0^1(O).$$

Therefore, from the above inequalities, we conclude that $a(t,\cdot,\cdot)$ is continuous and coercive in $H_0^1(O) \times H_0^1(O)$.

Now lets consider the main result.

THEOREM 1. Suppose the hypothesis (H.1)-(H.4) are satisfied and that

$$f \in L^2(Q) \tag{1.1}$$

$$u_0 \in H_0^1(\Omega_0) \tag{1.2}$$

$$u_1 \in L^2(\Omega_o) \text{ are given, with } 0 < \rho \leq \frac{4}{n-2} \tag{1.3}$$

Then there exists a function $u: Q \to \mathbb{R}$ such that

$$u \in L^\infty(0,T;H_0^1(\Omega_t)) \tag{1.4}$$

$$u' \in L^\infty(0,T;L^2(\Omega_t)),\ \sqrt{k_2(x)}u' \in L^\infty(0,T;L^2(\Omega_t)) \tag{1.5}$$

$$k_2(x)u'' \in L^{p'}(0,T;H^{-1}(\Omega_t)) \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1,\ p = \rho + 2 \text{ and } u \tag{1.6}$$

is a solution (1) in the weak sense in Q, i.e.,

$$\frac{d}{dt}(k_2(x)u_t(t),v) + (k_1(x)u_t(t),v) + a(t,u(t),v) + (\,|\,u(t)\,|^{\rho}u(t),v) = (f(t),v), \tag{1.7}$$

in $D'(0,T), \forall v \in H_0^1(\Omega_t)$.

$$u(x,0) = u_0(x); \qquad k_2(x)u_t(x,0) = \sqrt{k_2(x)}u_1 \text{ in } \Omega_0 \tag{1.8}$$

PROOF. The idea is to transform, the non-cylinder problem in the cylinder problem, through the penalization function, $M \in L^\infty(O\times(0,T))$, that was introduced by J.L. Lions [8], given by:

$$M(x,t) = \begin{cases} 0, & \text{in } Q \\ 1, & \text{in } O\times(0,T)\backslash Q. \end{cases}$$

For each $\epsilon > 0$, we will find U^ϵ in the cylinder $O\times(0,T)$, solution of the perturbed problem (P_ϵ) below

$$\tilde{k}_{2\epsilon}(x)U_{tt}^\epsilon + \tilde{k}_1(x)U_t^\epsilon + A(t)U^\epsilon + \tfrac{1}{\epsilon}MU_t^\epsilon + \|\,U^\epsilon\,\|^{\rho}U^\epsilon = \tilde{f} \tag{1.9}$$

$$U^\epsilon(0) = \tilde{u}_0 \tag{1.10}$$

$$\tilde{k}_{2\epsilon}U_t^\epsilon(0) = \sqrt{\tilde{k}_{2\epsilon}(x)}\tilde{u}_1 \tag{1.11}$$

$$U^\epsilon = 0 \text{in the } \partial(O\times(0,T)) = \tilde{\Sigma} \tag{1.12}$$

where $\tilde{k}_{2\epsilon}(x) = \tilde{k}_2(x) + \epsilon; U_t = \frac{\partial}{\partial t}U;\ U_{tt} = \frac{\partial}{\partial t^2}U;\ \tilde{u}_0 = \begin{cases} u_0 & \text{in } \Omega_0 \\ 0 & \text{in } O\backslash\Omega_0 \end{cases}$

Therefore, $\tilde{u}_0 \in H_0^1(O)$. Analogously $\tilde{u}_1 \in L^2(O)$;

$$\tilde{f} = \begin{cases} f, & \text{in } Q \\ 0, & \text{in } O\times(0,T)\backslash Q; \end{cases}$$

Therefore $\tilde{f} \in L^2(O\times(0,T))$;

$$\tilde{k}_1(x) = \begin{cases} k_1(x) & \text{in } Q \\ \beta & \text{in } O\times(0,T)\backslash Q \end{cases} \quad\text{and}\quad \tilde{k}_2(x) = \begin{cases} k_2(x) & \text{in } Q \\ 0 & \text{in } O\times(0,T)\backslash Q \end{cases}$$

So $\tilde{k}_1$ and $\tilde{k}_2 \in L^\infty(O\times(0,T))$.

The proof of Theorem 1 will be a consequence of the following Theorem:

THEOREM 2. For each $\epsilon > 0$, there exists one function $U_\epsilon : O\times(0,T)\to\mathbb{R}$, solution of the problem (P_ϵ), such that:

$$U^\epsilon \in L^\infty(0,T;H_0^1(O)) \tag{1.13}$$

$$U^\epsilon \in L^\infty(0,T;L^2(O)), \sqrt{\tilde{k}_{2\epsilon}(x)}U_t^\epsilon \in L^\infty(0,T;L^2(O)) \tag{1.14}$$

$$\tilde{k}_{2\epsilon}(x)U_{tt}^\epsilon \in L^{p'}(0,T;H^{-1}(O)) \tag{1.15}$$

with $\frac{1}{p}+\frac{1}{p'} = 1$ and $p = \rho + 2$

$$\tilde{k}_{2\epsilon}(x)U_{tt}^\epsilon + \tilde{k}_1(x)U_t^\epsilon + A(t)U^\epsilon + \tfrac{1}{\epsilon}MU_t^\epsilon + |\,U^\epsilon\,|^{\rho}U^\epsilon = \tilde{f} \tag{1.16}$$

in the weak sense in $O\times(0,T)$.

$$U^\epsilon(x,0) = \tilde{u}_0(x) \tag{1.17}$$

$$\tilde{k}_{2\epsilon}(x)U_t^\epsilon(x,0) = \sqrt{\tilde{k}_{2\epsilon}(x)}\tilde{u}_1(x) \tag{1.18}$$

REMARK 1. The condition $U^\epsilon = 0$ in $\hat{\Sigma}$ is a consequence of the fact that U^ϵ in $L^2(0,T;H_0^1(O))$.

REMARK 2. For the proof of Theorem 1 it is sufficient to prove that the solution U^ϵ in Theorem 2 converges for U in the weak sense when $\epsilon \to 0$ and that the restriction of U to Q satisfies all the assertions of Theorem 1.

In this part, we use a result due to W.A. Strauss see [15].

PROOF OF THEOREM 2.

(i) Approximate Problem. It will be done by the Faedo-Galerkin method. Let $\{w_\nu\}_{\nu=1}^{\nu=\infty} \subset H^1(O)$ be a basis of $H^1(O)$ and V_m the subspace spanned by the m first vectors $w_1, w_2, \cdots, w_m$. Let U_m^ϵ be the function

$$U_m^\epsilon(x,t) = \sum_{j=1}^{m} g_{jm\epsilon}(t) w_j(x)$$

defined by the system

$$\begin{aligned}(\tilde{k}_{2\epsilon}(x)\frac{\partial^2}{\partial t^2}U_m^\epsilon(t), w_j) + (\tilde{k}_1(x)\frac{\partial}{\partial t}U_m^\epsilon(t), w_j) + a(t, U_m^\epsilon(t), w_j)\\ + \frac{1}{\epsilon} M\left(\frac{\partial}{\partial t}U_m^\epsilon(t), w_j\right) + (\,|\,U_m^\epsilon(t)\,|^\rho, w_j) = (f(t), w_j), \quad \forall j = 1, \cdots, m\end{aligned} \tag{1.19}$$

$$U_m^\epsilon(0) = U_{0m} = \sum_{j=1}^{m} \alpha_{jm} w_j \to \tilde{u}_0 \text{ strong in } H^1(O) \tag{1.20}$$

$$\frac{\partial}{\partial t}U_m^\epsilon(0) = U_{1m} = \sum_{j=1}^{m} \beta_{jm} w_j \to \frac{\tilde{u}_1}{\sqrt{\tilde{k}_{2\epsilon}}} \text{ strong in } L^2(O) \tag{1.21}$$

The system (1.19)-(1.21) satisfies the condition of Caractheodory's theorem see [2]. Therefore it has a solution U_m^ϵ defined in $[0, t_{\epsilon m})$, where $0 < t_{\epsilon m} \le T$. The a priori estimates to be obtained in the following step, show, in particular, that $t_{\epsilon m} = T$.

(ii) A Priori Estimates. By multiplying both sides of (1.19) by $2g'_{jm\epsilon}(t)$, and adding from $j = 1$ to $j = m$ we obtain:

$$\begin{aligned}\frac{d}{dt}\left|\sqrt{\tilde{k}_{2\epsilon}(x)}U'_m(t)\right|^2 + 2\left|\sqrt{\tilde{k}_1(x)}U_m(t)\right|^2 + 2a(t, U_m(t), u'_m(t)) + \frac{2}{\epsilon}\int_O M(U'_m)^2 dx\\ + 2\int_O \|\,U_m(s)\,\|^\rho U_m(s) U'_m(s) dx = 2(\tilde{f}(t), U'_m(t)),\end{aligned} \tag{1.22}$$

where we wrote U_m instead of U_m^ϵ and denoted by $U'_m = \frac{\partial}{\partial t}U_m$.

REMARK 3. We have that

$$\frac{d}{dt}a(t, U_m(t), U_m(t)) = a'(t, U_m(t), U_m(t)) + 2a(t, U_m(t), U'_m(t));$$

where

$$a'(t, U_m(t), U_m(t)) = a'(t, U_m(t)) = \sum_{i,j=1}^{n} \int_O \frac{\partial}{\partial t}a_{ij}(x,t)\frac{\partial}{\partial x_i}U_m(t)\frac{\partial}{\partial x_j}U_m(t) dx.$$

Therefore,

$$2a(t, U_m(t), U'_m(t)) = \frac{d}{dt}a(t, U_m(t)) - a'(t, U_m(t)).$$

REMARK 4. We have that $\frac{1}{p}\frac{d}{dt}\int_O |\,U_m(s)\,|^p dx = \int_O |\,U_m(s)\,|^{p-1} \cdot \frac{U_m(s)}{|\,U_m(s)\,|} \cdot U'_m(s) dx = \int_O |\,U_m(s)\,|^\rho U_m(s) \cdot U'_m(s) dx$.

Therefore, in the remarks (3 and 4) below, we have, integrating (1.22) from 0 to t, $0 < t \le t_m$, that:

$$\left|\sqrt{\tilde{k}_{2\epsilon}(x)}\,U'_m(t)\right|^2 + 2\int_0^t \left|\sqrt{\tilde{k}_1(x)}U'_m(s)\right|^2 ds + a(t, U_m(t)) + \frac{2}{p}\int_O |U_m(s)|^p\, dx$$

$$+\frac{2}{\epsilon}\int_0^t\int_O M(U'_m(s))^2dxds=\left|\sqrt{\tilde{k}_{2\epsilon}(x)}U_{1m}\right|^2+a(0,U_{0m})+ \tag{1.23}$$

$$\frac{2}{p}\int_O |U_{0m}|^p\,dx+\int_0^t a'(s,U_m(s))ds+2\int_0^t (\tilde{f}(s),U'_m(s))ds$$

REMARK 5. From (20), (21) and the Sobolev Immersion, $H^1(O)\to L^p(O), \forall\ \frac{1}{p}=\frac{1}{2}-\frac{1}{n}$, we obtain:

$$\| U_{0m}\|_{L^p(O)}\le C.$$

$$\left|\sqrt{\tilde{k}_{2\epsilon}(x)}\,U_{1m}\right|^2\le C;\ \ |a(0,U_{0m})|\le C.$$

Here, the letter C denotes different constants.

REMARK 6. By using (H.4), we obtain:

$$\int_0^t |a'(s,U_m(s))|\,ds\le C\int_0^t \| U_m(s)\|^2ds;$$

Therefore, from the remarks (5 and 6) below, we can write (1.23) like

$$\begin{gathered}\left|\sqrt{\tilde{k}_{2\epsilon}(x)}\,U'_m(t)\right|^2+2\int_0^t\left|\sqrt{\tilde{k}_1(x)}U'_m(s)\right|^2ds+a(t,U_m(t))+\frac{2}{p}\int_O |U_m(s)|^p\,ds\\ +\frac{2}{\epsilon}\int_0^t\int_O M(U'_m(s))^2dxds\le C+C\int_0^t\| U_m(s)\|^2ds+\lambda\int_0^t |U'_m(s)|^2ds\end{gathered} \tag{1.24}$$

From (1.24), if we choose $\lambda=\beta>0$ (the $\beta>0$ of H.3) we obtain:

$$\int_0^t |U'_m(s)|^2ds\le C+C\int_0^t \| U_m(s)\|^2\,ds, \tag{1.25}$$

and

$$a(t,U_m(t))\le C+C\int_0^t \| U_m(s)\|^2ds+\beta\int_0^t |U'_m(s)|^2ds \tag{1.26}$$

Being $a(t,u,v)$ coercive, we obtain from (1.25) and (1.26), that:

$$\| U_m(t)\|^2\le C+C\int_0^t \| u_m(s)\|^2ds,\ \ \forall t\in[0,t_{\epsilon m}). \tag{1.27}$$

Gronwall's inequality implies that

$$\| U^\epsilon_m\|\ \le C,\ \forall m\in\mathbb{N},\ \forall\epsilon>0,\ \forall t\in[0,t_{\epsilon m}). \tag{1.28}$$

Returning to (1.25) we obtain:

$$\int_0^t\left|\frac{\partial}{\partial s}U^\epsilon_m(s)\right|^2ds\le C, \tag{1.29}$$

$\forall m\in\mathbb{N},\ \forall\epsilon>0,\ \forall t\in[0,t_{\epsilon m})$.

The priori estimative (1.24) shows that $t_{\epsilon m}=T$. Therefore,

$$\begin{gathered}\left|\sqrt{\tilde{k}_{2\epsilon}(x)}\,\frac{\partial}{\partial s}U^\epsilon_m(t)\right|^2+2\int_0^t\left|\sqrt{\tilde{k}_1(x)}\,\frac{\partial}{\partial s}U^\epsilon_m(s)\right|^2ds+a(t,U^\epsilon_m(t))\\ +\frac{2}{p}\int_O |U^\epsilon_m(s)|^p dx+\frac{2}{\epsilon}\int_0^t\int_O M\left(\frac{\partial}{\partial s}U^\epsilon_m(s)\right)^2dxds\le C\end{gathered} \tag{1.30}$$

$\forall m\in\mathbb{N}, \forall\epsilon>0$ and $\forall t\in[0,T]$.

We obtain from (1.28), (1.29) and (1.30) the estimates,

$$\| U^\epsilon_m\|_{L^\infty(0,T;H^1_0(O))}\le C,\quad \forall m\in\mathbb{N},\quad \epsilon>0. \tag{1.31}$$

$$\left\| \frac{\partial}{\partial t} U^{\epsilon}_{m} \right\|_{L^2(0,T;L^2(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0 \tag{1.32}$$

$$\left\| \sqrt{k_{2\epsilon}} \frac{\partial}{\partial t} U^{\epsilon}_{m} \right\|_{L^\infty(0,T;L^2(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0 \tag{1.33}$$

$$\left\| \frac{1}{\sqrt{\epsilon}} M \frac{\partial}{\partial t} U^{\epsilon}_{m} \right\|_{L^\infty(0,T;L^2(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0; \tag{1.34}$$

where C is a constant independent of $m \in \mathbb{N}$ and $\epsilon > 0$.

By the estimates (1.31)-(1.34), there exist a subsequence of (U^{ϵ}_{m}), still denoted by (U^{ϵ}_{m}), and a function U^{ϵ} such that

$$U^{\epsilon}_{m} \rightarrow U^{\epsilon} \text{ weak-star in } L^\infty(0,T;H^1_0(O)), \tag{1.35}$$

$$\frac{\partial}{\partial t} U^{\epsilon}_{m} \rightarrow \frac{\partial}{\partial t} U^{\epsilon} \text{ weak in } L^2(0,T;L^2(O)), \tag{1.36}$$

$$\frac{1}{\sqrt{\epsilon}} M \frac{\partial}{\partial t} U^{\epsilon}_{m} \rightarrow \frac{1}{\sqrt{\epsilon}} M \frac{\partial}{\partial t} U^{\epsilon} \text{ weak-star in } L^\infty(0,T;L^2(O)). \tag{1.37}$$

THE NONLINEAR TERM.

By (1.30) and noting that $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\| \, |U^{\epsilon}_{m}|^{\rho} U^{\epsilon}_{m} \|^{p'}_{L^{p'}} = \int_O |U^{\epsilon}_{m}|^{(\rho+1)p'} dx = \int_O |U^{\epsilon}_{m}|^{(p-1)p'} dx = \int_O |U^{\epsilon}_{m}|^{p} dx \leq C,$$

which implies:

$$\| \, |U^{\epsilon}_{m}|^{\rho} U^{\epsilon}_{m} \|_{L^\infty(0,T;L^{p'}(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0. \tag{1.38}$$

From (1.31), (1.32) and the Aubin-Lions Theorem (see [7]) we obtain:

$$|U^{\epsilon}_{m}|^{\rho} U^{\epsilon}_{m} \rightarrow |U^{\epsilon}|^{\rho} U^{\epsilon} \text{ a.e. in } O \times (0,T), \tag{1.39}$$

and

$$|U^{\epsilon}_{m}|^{\rho} U^{\epsilon}_{m} \rightarrow W \text{ weak-star in } L^\infty(0,T;L^{p'}(O)) \tag{1.40}$$

The difficulty is to prove that $W = |U^{\epsilon}|^{\rho} U^{\epsilon}$. This is a consequence of the following result due to W.A. Strauss (see [15]).

LEMMA 1. Let Ω be a bounded open set of $\mathbb{R}^n$. Lets g_m and $g \in L^p(\Omega)$, $1 < p < \infty$ satisfy the following conditions:

(i) $g_m \rightarrow g$ a.e. in Ω

(ii) $\| g_m \|_{L^p(\Omega)} \leq C$, $\forall m \in \mathbb{N}$.

Then

(iii) $g_m \rightarrow g$ strongly in $L^q(\Omega)$, $1 \leq q < p$

(iv) $g_m \rightarrow g$ weakly in $L^p(\Omega)$.

Lemma 1 with $q = \frac{\rho+2}{\rho+1} = p'; \Omega = O \times (0,T)$ and $g_m = |U_m|^{\rho} U_m$, we obtain from (1.38) and (1.39) that

$$|U^{\epsilon}_{m}|^{\rho} U^{\epsilon}_{m} \rightarrow |U^{\epsilon}|^{\rho} U^{\epsilon} \text{ weak-start in } L^\infty(0,T;L^{p'}(O)) \tag{1.41}$$

and consequently weak in $L^{p'}(0,T;L^{p'}(O))$.

By multiplying both sides of (1.19) by $\theta \in C^\infty_0(0,T)$, integrating from $t = 0$ to $t = T$, passing to the limit and using the convergences (1.35)-(1.37), (1.41) and noting that $\{w_\nu\}_{\nu=1}^{\infty}$ is a basis of $H^1_0(O)$, we obtain:

$$\int_0^T (\tilde{k}_{2\epsilon}(x)\frac{\partial^2}{\partial t^2}U^\epsilon(t), v\theta)dt + \int_0^T (\tilde{k}_1(x)\frac{\partial}{\partial t}U^\epsilon(t), v\theta)dt + \int_0^T a(t, U^\epsilon(t), v\theta)dt +$$
$$\int_0^T \left(\frac{1}{\epsilon}M\frac{\partial}{\partial t}U^\epsilon(t), v\theta\right)dt + + \int_0^T (\,|\,U^\epsilon(t)\,|\,^\rho U^\epsilon(t), v\theta)dt = \int_0^T (\tilde{f}(t), v\theta)dt, \tag{1.42}$$

$\forall v \in H_0^1(O)$, $\forall \theta \in C_0^\infty(0,T)$.

Then, from (1.35)-(1.37) and from (1.42), we obtain U^ϵ satisfying (1.9)-(1.10) and (1.12). Noting that

$$L^2(0,T;L^2(O)) \hookrightarrow L^2(0,T;H^{-1}(O)),$$

we obtain

$$-\tfrac{1}{\epsilon}MU_t^\epsilon - \tilde{k}_1(x)U_t^\epsilon \in L^2(0,T;H^{-1}(O)).$$

The fact that $a_{ij}(x,t)\dfrac{\partial}{\partial x_i}U^\epsilon(t) \in L^2(O)$ implies that

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}U^\epsilon\right) \in L^2(0,T;H^{-1}(O)),$$

(see [3]). Also from (1.16), (1.41) and $\tilde{f} \in L^2(0,T;L^2(O))$ we obtain

$$\tilde{k}_{2\epsilon}(x)\frac{\partial^2}{\partial t^2}U^\epsilon \in L^2(0,T;H^{-1}(O)),$$

which proves (1.15).

The estimates (1.31)-(1.34) and (1.38) are independent form $\epsilon > 0$, we obtain the same convergences (1.35)-(1.37) and (1.41) by changing U_m^ϵ by U^ϵ and U^ϵ by W. Therefore, we have

$$U^\epsilon \rightarrow W \text{ weak-star in } L^\infty(0,T;H_0^1(O)) \tag{1.43}$$

$$U_t^\epsilon \rightarrow W_t \text{ weak in } L^2(0,T;L^2(O)) \tag{1.44}$$

$$\sqrt{\tilde{k}_{2\epsilon}(x)}U_t^\epsilon \rightarrow \sqrt{\tilde{k}_2(x)}W_t \text{ weak-star in } L^\infty(0,T;L^2(O)). \tag{1.45}$$

Note that $\sqrt{k_{2\epsilon}(x)} = \sqrt{\tilde{k}_2(x)+\epsilon} \rightarrow \sqrt{\tilde{k}_2(x)}$ strong in $L^2(0,T;L^2(\Omega))$.

$$|\,U^\epsilon\,|^\rho U^\epsilon \rightarrow |\,W\,|^\rho W \text{ weak-star in } L^\infty(0,T;L^{\rho'}(O)) \tag{1.46}$$

Also, we obtain the essential estimates:
$$\int_{O\times(0,T)} M(U_t^\epsilon)dxdt \le C\epsilon. \tag{1.47}$$

From (1.44) we have: $M(U_t^\epsilon)^2 \rightarrow M(W_t)^2$ weak in $L^2(0,T;L^2(O))$.

Therefore, from (1.47) we obtain

$$\int_{O\times(0,T)} M(W_t)^2 dxdt = 0.$$

From this and the definition of M, we deduce: $W_t = 0$ a.e. in $O\times(0,T)\supset Q$. Consequently $W(x,t)$ is constant in the variable t in $O\times(0,T)\supset Q$. Being $W(x,0) = \tilde{u}_0(x)$ in O, we conclude that $W(x,0) = 0$ in $O\backslash\Omega_0$. From this and from (H-1), we get:

$$W(x,t) = 0 \text{ a.e. in } O\times(0,T)\supset Q. \tag{1.48}$$

We conclude from (1.43) and (1.44) that $W(t) \in H^1(O)$. Let u be the restriction of W to Q.

Then from (1.48) and (H-2), we obtain that $u \in L^\infty(0,T;H_0^1(\Omega_t))$; which proves (1.4) in Theorem 1. Moreover, from (1.44) and (1.45), we conclude that u satisfies (1.5).

Let $\widehat{U}$ be the restriction of U to Q. Then, restricting the equation of Theorem 2 to the domain Q, we obtain:

$$\begin{aligned}&(k_{2\epsilon}(x)\widehat{U}^\epsilon_{tt}(t),v)+(k_1(x)\widehat{U}^\epsilon_t(t),v)+a(t,\widehat{U}^\epsilon(t),v)+\tfrac{1}{\epsilon}(M\widehat{U}^\epsilon_t(t),v)+\\&(\,|\,\widehat{U}^\epsilon(t)\,|\,^\rho\widehat{U}^\epsilon(t),v)=(\tilde{f}(t),v),\end{aligned}\tag{1.49}$$

$\forall v \in H_0^1(O)$, in the sense of the $D'(0,T)$.

By taking the limit when $\epsilon\to 0$ in (1.49), and using the convergences (1.43)-(1.46) we get:

$$\frac{d}{dt}(k_2(x)u_t(t),v)+(k_1(x)u_t(t),v)+a(t,u(t),v)+(\,|\,u(t)\,|\,^\rho u(t),v)=(f(t),v),$$

in $D'(0,T), \forall v \in H_0^1(\Omega_t)$, which proves (1.7).

The proof of (1.6) is analogous to (1.15) of the cylinder problem.

(iii) The Initial Conditions.

Let $\sigma \in C^1([0,T];\mathbb{R})$ be such that $\sigma(0)=1$ and $\sigma(T)=0$. We have

$$\int_0^T\Big(\frac{\partial}{\partial t}U^\epsilon_m(t),v\Big)\sigma(t)dt=-(U^\epsilon_m(0),v)-\int_0^T(U^\epsilon_m(t),v)\sigma'(t)dt,\qquad \forall v\in L^2(O).$$

By passing to the limit in the above equality and using the convergences (1.20), (1.35) and (1.36) we obtain:

$$\int_0^T\Big(\frac{\partial}{\partial t}U^\epsilon(t),v\Big)\sigma(t)dt=-(\tilde{u}_0,v)-\int_0^T(U^\epsilon(t),v)\sigma'(t)dt,\quad \forall v\in L^2(O).$$

Integrating by parts the last integral above, we conclude that

$$(U^\epsilon(0),v)=(\tilde{u}_0,v),\forall v\in L^2(O).$$

From this it follows (1.17). The initial condition $u(x,0)=u_0(x)$ of Theorem 1 is done analogously.

Finally, we will verify condition (1.18). Initially we verify that $[(k_2(x)+\epsilon)U_t](0)$ does make sense.

Let U^ϵ be a solution of the perturbated problem. Then

$$\begin{aligned}&-\int_0^T\Big\langle\tilde{k}_{2\epsilon}(x)U^\epsilon_t(t),\theta'(t)v\Big\rangle dt+\int_0^T\Big\langle\tilde{k}_1(x)U^\epsilon_t(t),\theta(t)v\Big\rangle dt+\int_0^T\Big\langle A(t)U^\epsilon(t),\theta(t)v\Big\rangle dt+\\&\int_0^T\Big\langle\frac{1}{\epsilon}MU^\epsilon_t(t),\theta(t)v\Big\rangle dt+\int_0^T\Big\langle\,|\,U^\epsilon(t)\,|\,^\rho U^\epsilon(t),\theta(t)v\Big\rangle dt=\int_0^T\Big\langle\tilde{f}(t),\theta(t)v\Big\rangle dt\end{aligned}$$

$\forall v \in H_0^1(O)$ and $\forall\theta\in C_0^\infty(0,T)$; where $<\cdot,\cdot>$ is the duality between $H_0^1(O)$ and $H^{-1}(O)$. So

$$\begin{aligned}&\Big\langle-\int_0^T\tilde{k}_{2\epsilon}(x)U^\epsilon_t(t)\theta'(t)dt+\int_0^T\tilde{k}_1(x)U^\epsilon_t(t)\theta(t)dt+\\&\int_0^T A(t)U^\epsilon(t)\theta(t)dt+\int_0^T\frac{1}{\epsilon}MU^\epsilon_t(t)\theta(t)dt+\\&\int_0^T|\,U^\epsilon(t)\,|\,^\rho U^\epsilon(t)\theta(t)dt,v\Big\rangle=\Big\langle\int_0^T\tilde{f}(t)\theta(t)dt,v\Big\rangle\end{aligned}$$

$\forall v \in H_0^1(O)$ and $\forall\theta\in C_0^\infty(0,T)$.

Therefore, we have

$$< -\tilde{k}_{2\epsilon}(x)U_t^\epsilon(t),\theta'(t) > + < \tilde{k}_1(x)U_t^\epsilon(t),\theta(t) > + < A(t)U^\epsilon(t),\theta(t) > +$$
$$< \tfrac{1}{\epsilon} MU_t^\epsilon(t),\theta(t) > + < |U^\epsilon(t)|^\rho U^\epsilon(t),\theta(t) > = < \tilde{f}(t),\theta(t) > ,$$

$\forall\theta \in C_0^\infty(0,T)$; where, here $< \cdot,\cdot >$ denotes the vectorial distribution of $(0,T)$ in $H^{-1}(O)$ evaluated in scalar test application of $(0,T)$. Being $\tilde{k}_{2\epsilon} \in L^\infty(O \times (0,T))$ and $U_t^\epsilon \in L^2(0,T;L^2(O))$, we have $-\tilde{k}_{2\epsilon}U_t^\epsilon \in L^2(0,T;L^2(O))$.

So $-\tilde{k}_{2\epsilon}U_t^\epsilon$ defines a vectorial distribution of $(0,T)$ in $L^2(O)$, whose derivative is:

$$< -\tilde{k}_{2\epsilon}U_t^\epsilon,\theta' > = < (\tilde{k}_{2\epsilon}U_t^\epsilon)_t,\theta > , \quad \forall\theta \in C_0^\infty(0,T).$$

Therefore,

$$< (\tilde{k}_{2\epsilon}U_t^\epsilon)_t,\theta > + < \tilde{k}_1U_t^\epsilon,\theta > + < A(t)U^\epsilon,\theta > +$$
$$< \tfrac{1}{\epsilon}MU_t^\epsilon,\theta > + < |U^\epsilon|^\rho U^\epsilon,\theta > = < \tilde{f},\theta > , \forall\theta \in C_0^\infty(0,T).$$

Or,

$$(\tilde{k}_{2\epsilon}U_t^\epsilon)_t + \tilde{k}_1U_t^\epsilon + A(t)U^\epsilon + \tfrac{1}{\epsilon}MU_t^\epsilon + |U^\epsilon|^\rho U^\epsilon = \tilde{f},$$

in $L^2(0,T;H^{-1}(O))$. As $\tilde{f},\tilde{k}_1U_t^\epsilon,\frac{1}{\epsilon}MU_t^\epsilon, |U^\epsilon|^\rho U^\epsilon \in L^2(0,T;L^2(O))$ and $A(t)U^\epsilon \in L^2(0,T;H^{-1}(O))$, we obtain, from the last equality above that: $(\tilde{k}_{2\epsilon}U_t^\epsilon)_t \in L^2(0,T;H^{-1}(O)) \hookrightarrow L^{p'}(0,T;H^{-1}(O))$, which proves (1.15). It is easy to see that $\tilde{k}_{2\epsilon}U_t^\epsilon \in C^0([0,T];H^{-1}(O))$. Therefore, $[\tilde{k}_{2\epsilon}U_t^\epsilon](0)$ makes sense. Let now $\theta \in C^1([0,t]);\mathbb{R})$ be such that $\theta(0) = 1$ and $\theta(T) = 0$. Then,

$$\int_0^T(\tilde{k}_{2\epsilon}\frac{\partial^2}{\partial t^2}U_m^\epsilon(t),v)\theta(t)dt = -\Big(\tilde{k}_{2\epsilon}\frac{\partial}{\partial t}U_m^\epsilon(0),v\Big)$$
$$-\int_0^T\Big(\tilde{k}_{2\epsilon}\frac{\partial}{\partial t}U_m^\epsilon(t),v\Big)\theta'(t)dt, \forall v \in V_m.$$

From this and taking $v = w_j$ in the approximate equation, we obtain:

$$-\Big(\tilde{k}_{2\epsilon}\frac{\partial}{\partial t}U_m^\epsilon(0),v\Big) - \int_0^T\Big(\tilde{k}_{2\epsilon}\frac{\partial}{\partial t}U_m^\epsilon(t),v\Big)\theta'(t)dt + \int_0^T\Big(\tilde{k}_1\frac{\partial}{\partial t}U_m^\epsilon(t),v\Big)\theta(t)dt +$$
$$\int_0^T a(t,U_m^\epsilon(t),v)\theta'(t)dt + \int_0^T\Big(\tfrac{1}{\epsilon}M\frac{\partial}{\partial t}U_m^\epsilon(t),v\Big)\theta(t)dt + \int_0^T(|U_m^\epsilon(t)|^\rho U_m^\epsilon(t),v)\theta(t)dt =$$
$$\int_0^T(\tilde{f}(t),v)\theta(t)dt, \quad \forall v \in V_m.$$

By passing to the limit in the above equality and using the convergences (1.21), (1.35)-(1.37) and (1.41) we obtain:

$$-\Big(\sqrt{\tilde{k}_{2\epsilon}}\tilde{u}_1,v\Big) - \int_0^T(\tilde{k}_{2\epsilon}U_t^\epsilon(t),v)\theta'(t)dt + \int_0^T(\tilde{k}_1(x)U_t^\epsilon(t),v)\theta(t)dt +$$
$$\int_0^T a(t,U^\epsilon(t),v)\theta(t)dt + \int_0^T\Big(\tfrac{1}{\epsilon}MU_t^\epsilon(t),v\Big)\theta(t)dt +$$
$$\int_0^T(|U^\epsilon(t)|^\rho U^\epsilon(t),v)\theta(t)dt = \int_0^T(\tilde{f}(t),v)\theta(t)dt,$$

As $-\int_0^T(\tilde{k}_{2\epsilon}U_t^\epsilon(t),v)\theta'(t)dt = <(\tilde{k}_{2\epsilon}U_t^\epsilon(t))_t, v> \theta(t)\forall v \in V_m$ and $\theta \in C^1([0,T];\mathbb{R})$ such that $\theta(0) = 1$ and $\theta(T) = 0$, we have, using the fact that U^ϵ is solution of the perturbed equation, that:

$$- < \sqrt{\tilde{k}_{2\epsilon}(x)}\tilde{u}_1, v> + < \tilde{k}_{2\epsilon}(x)U_t^\epsilon(0), v> = 0, \forall v \in V_m.$$

Or,

$$< \tilde{k}_{2\epsilon}(x)U_t^\epsilon(0) - \sqrt{\tilde{k}_{2\epsilon}(x)}\tilde{u}_1, v> = 0,$$

$\forall v \in H_0^1(\Omega)$. This proves (1.18) and, therefore, the proof of Theorem 2 is complete.

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