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LINEAR PROGRAMMING WITH INEQUALITY CONSTRAINTS VIA ENTROPIC PERTURBATION

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ABSTRACT. A dual convex programming approach to solving linear programs with inequality constraints through entropic perturbation is derived. The amount of perturbation required depends on the desired accuracy of the optimum. The dual program contains only non-positivity constraints. An ε-optimal solution to the linear program can be obtained effortlessly from the optimal solution of the dual program. Since cross-entropy minimization subject to linear inequality constraints is a special case of the perturbed linear program, the duality result becomes readily applicable. Many standard constrained optimization techniques can be specialized to solve the dual program. Such specializations, made possible by the simplicity of the constraints, significantly reduce the computational effort usually incurred by these methods. Immediate applications of the theory developed include an entropic path-following approach to solving linear semi-infinite programs with an infinite number of inequality constraints and the widely used entropy optimization models with linear inequality and/or equality constraints.

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1. INTRODUCTION.

Since Karmarkar's projective scaling algorithm was introduced in 1984 [1], various interior-point methods [2,3] have been proposed to compete with the classical simplex method [4] for linear programs. Among many new research directions, an unconstrained convex programming approach was proposed [5], in a framework of geometric programming [6], for solving linear programming problems in Karmarkar's form. The approach involves solving an unconstrained convex programming dual problem and converting the dual optimal solution to an ε-optimal solution for the linear program. The work was extended for linear programming problems in standard form [7] with a quadratically convergent global algorithm, based on the curved search methods [8]. This paper further extends the approach to solve linear programming problems with inequality constraints directly without a conversion to the standard form. In this way, no artificial variables are added and the dimensionality of the original problem is kept. In accordance with the earlier work, we derive the geometric dual, although the same dual program can be derived using the Lagrangian approach.

The motivation of this study is twofold. First, Fang and Wu [9] recently proposed an entropic path-following approach to solving linear semi-infinite programs with finitely many variables and infinitely many inequality constraints. Their algorithms require solving an entropically perturbed linear program with finitely many inequality constraints. After introducing artificial variables, the resulting equality-constrained convex program is no longer an entropically perturbed linear program due to the absence of the entropic terms for the artificial variables. Therefore, the algorithms proposed in [7] is no longer applicable and an algorithm for solving directly the entropically perturbed linear programs with inequality constraints is needed. Second, the widely applicable entropy optimization problem with linear inequality constraints turns out to be a special case of the perturbed linear program being treated. Although such minimization problems subject to *equality* constraints have been used widely and treated extensively in recent literature [e.g. 10-16], the inequality case has received little attention. Nevertheless, the inequality formulation is particularly appealing when point estimates for the linear moments of the underlying distribution, i.e. the right-hand sides of the equality formulation, cannot be accurately obtained but the interval (range) estimates for the moments are available.

In this paper, we extend the geometric programming approach to derive the dual program in Section 2, discuss other applications of the duality results in Section 3, and conclude the paper in Section 4.

2. A DUAL APPROACH WITH ENTROPIC PERTURBATION.

Consider the following (primal) linear program:

$$\text{Program P:}\quad \text{Minimize}\quad \mathbf{c}^T\mathbf{x}$$

$$\text{subject to}\quad \mathbf{Ax} \le \mathbf{b} \tag{2.1}$$

$$\mathbf{x} \ge 0\ . \tag{2.2}$$

where $\mathbf{c}$ and $\mathbf{x}$ are n-dimensional column vectors, $\mathbf{A}$ is an $m \times n$ $(m \le n)$ matrix, $\mathbf{b}$ is an m-dimensional column vector, and 0 is the n-dimensional zero column vector.

The linear dual of Program P is given as follows:

$$\text{Program D:}\quad \text{Maximize}\quad \mathbf{b}^T\mathbf{w}$$

$$\text{subject to}\quad \mathbf{A}^T\mathbf{w} \le \mathbf{c}$$

$$\mathbf{w} \le 0\ ,$$

where $\mathbf{w}$ is an m-dimensional column vector.

Following the approach developed in [5], for any given scalar $\mu > 0$, instead of solving Program P directly, we tackle the following nonlinear program with an entropic perturbation:

$$\text{Program P}_\mu:\quad \text{Minimize}\quad f_\mu(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \mu\sum_{j=1}^{n} x_j \ln x_j$$

$$\text{subject to}\quad \mathbf{Ax} \le \mathbf{b} \tag{2.3}$$

$$\mathbf{x} \ge 0\ . \tag{2.4}$$

Note that the entropic function $x\ln x$ is a strictly convex function well-defined on $[0, \infty)$, with the convention that $0\ln 0 = 0$. It has a unique minimum value of $-1/e$ at $x = 1/e$, where $e = 2.718\ldots$

Like all interior-point methods, we make an Interior-Point Assumption, namely, Program P has an interior feasible solution $\mathbf{x} > 0$. Under this assumption, Program P_μ is feasible for any $\mu > 0$. Moreover, since $0\ln 0 = 0$, $c_j x_j + x_j \ln x_j \to +\infty$ as $x_j \to \infty$, and $x_j \ln x_j$ is strictly convex over its domain for each j, Program P_μ achieves a finite minimum at a unique point $\mathbf{x}^* \in R^n$, for each $\mu > 0$. More interestingly, as discussed in [7], if Program P has a bounded feasible domain (i.e., the Bounded Feasible Domain Assumption), then as $\mu \to 0$ the optimal solution of Program P_μ approaches an optimal solution of Program P. To derive the geometric dual of P_μ, consider the following simple inequality:

$$\ln z \le z - 1\,, \qquad \text{for } z > 0 \tag{2.5}$$

Note that this inequality becomes an equality if and only if $z=1$.

For any $\mu > 0$, $w_i \in R$ $(i = 1,\ldots,m)$, and $x_j > 0$ $(j = 1,\ldots,n)$, we define

$$z_j = \frac{e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1}}{x_j}\,, \qquad \text{for } j = 1,\ldots,n.$$

In this way, $x_j > 0$ implies $z_j > 0$ and, by inequality (2.5), we have

$$[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1 - \ln x_j \le \frac{e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/u] - 1}}{x_j} - 1\,. \tag{2.6}$$

Multiplying both sides by $x_j>0$ and rearranging terms lead to

$$x_j[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1} \le x_j \ln x_j\,.$$

Note that this inequality holds even if $x_j = 0$. Now, multiplying both sides by μ and summing over j, we obtain

$$\sum_{j=1}^{n} x_j \left[\sum_{i=1}^{m} a_{ij}w_i\right] - \mu \sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1} \le \sum_{j=1}^{n} c_j x_j + \mu \sum_{j=1}^{n} x_j \ln x_j\,.$$

If (i) $x_j \ge 0$ $(j = 1,\ldots,n)$ satisfies $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ and (ii) $w_i \le 0$, for $i = 1,2,\ldots,m$, then

$$\sum_{j=1}^{n} x_j \left[\sum_{i=1}^{m} a_{ij}w_i\right] = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_{ij}x_j\right] w_i \ge \sum_{i=1}^{m} b_i w_i\,. \tag{2.7}$$

Therefore, for any $\mathbf{x} \ge 0$ such that $\mathbf{Ax} \le \mathbf{b}$ and $\mathbf{w} \le 0$,

$$\sum_{i=1}^{m} b_i w_i - \mu \sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1} \le \sum_{j=1}^{n} c_j x_j + \mu \sum_{j=1}^{n} x_j \ln x_j\,. \tag{2.8}$$

Recall that the right-hand side of (2.8) is exactly the objective function of Program P_μ. We now define the following geometric dual program D_μ of P_μ:

Program D_μ: Maximize $d_\mu(\mathbf{w}) \equiv \sum_{i=1}^{m} b_i w_i - \mu \sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1}$ subject to $\mathbf{w} \le 0$.

Program D_μ is a convex program with only non-positivity constraints and the sum in each of the n exponents in the second term of its objective function is simply the amount of violation of the corresponding constraint in Program D. More importantly, if Program D_μ attains a finite optimum at $\mathbf{w}^*(\mu)$ for every $\mu>0$, then $\mathbf{w}^*(\mu)$ approaches a feasible solution of Program D as μ approaches 0. Program D_μ can also be derived via the Lagrangian approach. Note that this dual program differs from the one obtained for standard-form linear programs in [7] only in the extra non-positivity requirements. While it is usually the case and easy to see that, in the Lagrangian max-min derivation, a change of sign in a primal constraint results in a change of range of the corresponding dual variable, this causal relationship is not apparent in the geometric programming derivation. Our derivation, in contrast with its counterpart for the equality-constrained program, illustrates the difference in deriving the geometric dual program between the equality-constrained and the inequality-constrained cases.

We now turn to establishing the duality theory.

THEOREM 1. (Weak Duality Theorem) If P_μ is feasible, then $\mathrm{Min}(P_\mu) \geq \mathrm{Sup}(D_\mu)$.

PROOF. Inequality (2.8) implies that $f_\mu(\mathbf{x}) \leq d_\mu(\mathbf{w})$ as long as $\mathbf{x}$ is primal feasible and $\mathbf{w}$ is dual feasible. The weak duality follows consequently. □

THEOREM 2. Assume that (i) $\mathbf{x}^*$ is primal feasible and (ii) $\mathbf{w}^*$ is dual feasible. If

$$x_j^* = e^{[(\sum_{i=1}^{m} a_{ij}w_i^* - c_j)/\mu] - 1}, \quad \text{for } j = 1,\ldots,n, \quad \text{and} \tag{2.9}$$

$$w_i^* \left[\sum_{j=1}^{n} a_{ij}x_j^* - b_i\right] = 0, \quad \text{for } i=1,2,\ldots,m, \tag{2.10}$$

then $\mathbf{x}^*$ is an optimal solution to Program P_μ and $\mathbf{w}^*$ is an optimal solution to Program D_μ. Moreover, $\mathrm{Min}(P_\mu) = \mathrm{Max}(D_\mu)$.

PROOF. Inequality (2.8) becomes an equality if and only if both inequalities (2.6) and (2.7) become equalities, for each $j=1,2,\ldots,n$. But, inequality (2.7) becomes an equality if and only if

$$w_i \left[\sum_{j=1}^{n} a_{ij}x_j - b_i\right] = 0, \quad i=1,2,\ldots,m.$$

Recall that inequality (2.5) becomes an equality if and only if $z = 1$. Hence inequality (2.6) becomes an equality if and only if

$$z_j = \frac{e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1}}{x_j} = 1$$

or, equivalently,

$$x_j = e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1}.$$

By equations (2.9) and (2.10), inequality (2.7) becomes an equality. By Theorem 1, the feasibility of $\mathbf{x}^*$ and $\mathbf{w}^*$ implies their optimality. □

THEOREM 3. The objective function $d_\mu(\mathbf{w})$ of Program D_μ is concave. If the constraint matrix $\mathbf{A}$ in Program P has full row-rank, then $d_\mu(\mathbf{w})$ is strictly concave.

PROOF. The k_1-th element of the gradient vector of the dual objective function $d_\mu(\mathbf{w})$ is

$$\frac{\partial d_\mu(\mathbf{w})}{\partial w_{k_1}} = b_{k_1} - \sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij}w_i^* - c_j)/\mu] - 1} a_{k_1 j}. \tag{2.11}$$

Consequently, the (k_1,k_2)-th element of the Hessian matrix of function $d_\mu(\mathbf{w})$ is given by

$$\frac{\partial d_\mu(\mathbf{w})}{\partial w_{k_1}\partial w_{k_2}} = -\frac{1}{\mu}\sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij}w_i - c_j)/\mu] - 1} a_{k_1 j}a_{k_2 j}.$$

Therefore, the Hessian matrix can be written as $\mathbf{A}\mathbf{D}_r(\mathbf{w})\mathbf{A}^T$, where $\mathbf{D}_r(\mathbf{w})$ is an $n \times n$ diagonal matrix with $r_j(\mathbf{w})$ as its j-th diagonal element and

$$r_j(\mathbf{w}) = -\frac{1}{\mu} e^{[(\sum_{i=1}^{m} a_{ij} w_i - c_j)/\mu] - 1} < 0 .$$

By matrix theory, the Hessian matrix is nonsingular and negative definite as long as **A** has full row-rank. Therefore, $d_\mu(\mathbf{w})$ is strictly concave if **A** has full row-rank. □

THEOREM 4. (Strong Duality Theorem) If Program P has an interior feasible solution, then Program D_μ attains a finite maximum and $\mathrm{Min}(P_\mu) = \mathrm{Max}(D_\mu)$. If, in addition, the constraint matrix **A** has full row-rank, then Program D_μ has a unique optimal solution $\mathbf{w}^*(\mu) \le 0$. In either case, formula (2.9) provides a dual-to-primal conversion which defines the optimal solution $\mathbf{x}^*(\mu)$ of Program P_μ.

PROOF. Under the Interior-Point Assumption, Program P (hence P_μ) has an interior feasible solution. From convex analysis (Fenchel's Theorem [6,17]), we know that there is no duality gap between the Programs P_μ and D_μ. Recall that Program P_μ always achieves a finite optimum as long as $\mu > 0$. Therefore, if **A** has full row-rank, then $D_\mu(\mathbf{w})$ is strictly concave and Program D_μ must also achieve a finite optimum at a unique maximizer $\mathbf{w}^*(\mu) \le 0$. Since any $\mathbf{w} \le 0$ is a regular point for the non-positivity constraints and $d_\mu(\mathbf{w})$ is continuously differentiable, the Kuhn-Tucker Conditions hold at $\mathbf{w}^*(\mu)$. In other words, there exists a $\mathbf{u} \ge 0$ such that

$$-\nabla d_\mu(\mathbf{w}^*(\mu)) + \mathbf{u}^T = 0 , \text{ and} \tag{2.12}$$

$$\mathbf{u}^T \mathbf{w}^*(\mu) = 0 . \tag{2.13}$$

By equation (2.11), equation (2.12) becomes

$$-b_k + \sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij} w_i^*(\mu) - c_j)/\mu] - 1} a_{kj} + u_k = 0 , \ k=1,2,\dots,m . \tag{2.14}$$

If we further define $\mathbf{x}^*(\mu) > 0$ according to (2.9), then the above equation becomes

$$\mathbf{A}\mathbf{x}^*(\mu) \le \mathbf{b} ,$$

which is simply the primal feasibility. Furthermore, by this definition and equation (2.14), equation (2.13) becomes

$$w_i^*(\mu) \left(\sum_{j=1}^{n} a_{ij} x_j^*(\mu) - b_i \right) = 0 .$$

The desired conclusion follows from Theorem 2. □

So far, we have concentrated on solving Program P, which contains only inequality constraints. The theory can be easily extended for linear programs with both inequality and equality constraints in the following form:

$$\begin{aligned} \text{Program } P': \quad & \text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ & \text{subject to} \quad \mathbf{A}_1\mathbf{x} \le \mathbf{b}_1 \\ & \qquad\qquad\quad \mathbf{A}_2\mathbf{x} = \mathbf{b}_2 \\ & \qquad\qquad\quad \mathbf{x} \ge 0 , \end{aligned}$$

where **c** and **x** are n-dimensional column vectors, $\mathbf{A}_1$ is an $m_1 \times n$ ($m_1 \le n$) matrix, $\mathbf{A}_2$ is an $m_2 \times n$ ($m_2 \le n$) matrix, $\mathbf{b}_1$ is an m_1-dimensional column vector, $\mathbf{b}_2$ is an m_2-dimensional column vector, and 0 is the n-dimensional zero column vector.

The perturbed problem, Program P'_μ, is defined by

$$\text{Program } P'_\mu: \quad \text{Minimize} \quad \mathbf{c}^T\mathbf{x} + \mu\sum_{j=1}^{n} x_j \ln x_j$$
$$\text{subject to} \quad \mathbf{A}_1\mathbf{x} \le \mathbf{b}_1$$
$$\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$$
$$\mathbf{x} \ge 0 .$$

With $m \equiv m_1+m_2$ and the notation $\mathbf{w}^T \equiv (\mathbf{w}_1^T, \mathbf{w}_2^T)$, where $\mathbf{w}_1$ is an m_1-dimensional column vector and $\mathbf{w}_2$ is an m_2-dimensional column vector, the geometric dual is defined as

$$\text{Program } D'_\mu: \quad \text{Maximize} \quad d'_\mu(\mathbf{w}) \equiv \sum_{i=1}^{m} b_i w_i - \mu \sum_{j=1}^{n} e^{[(\sum_{i=1}^{m} a_{ij} w_i - c_j)/\mu] - 1} \quad \text{subject to } \mathbf{w}_1 \le 0 .$$

With the notation $\mathbf{A}^T \equiv (\mathbf{A}_1^T, \mathbf{A}_2^T)$, we state the following theorem, whose proof is straightforward in light of the derivation provided above and treatment of the standard-form linear programs in [7].

THEOREM 5. If Program P' has an interior feasible solution, then Program D'_μ, for every $\mu>0$, attains a finite maximum and $\text{Min}(P'_\mu) = \text{Max}(D'_\mu)$. If, in addition, the constraint matrix $\mathbf{A}$ has full row-rank, then Program D'_μ, for every $\mu>0$, has a unique optimal solution $\mathbf{w}^*(\mu)$. In either case, equation (2.9) provides a dual-to-primal conversion which defines the optimal solution $\mathbf{x}^*(\mu)$ of Program P'_μ.

As we stated before, if the feasible domain of Program P is bounded, then the optimal solution of Program P_μ converges to an optimal solution of Program P, as μ reduces to zero. Actually, by simply modifying a parallel result in [7], we can easily construct an ε-optimal solution according to the following theorem without any difficulty:

THEOREM 6. If Program P' has an interior feasible solution $\mathbf{x} > 0$ and its feasible domain is contained in a spheroid centered at the origin with a radius of $M > 0$, then, for any $\mu > 0$ such that

$$\mu \le \varepsilon / 2n\tau , \tag{2.15}$$

where

$$\tau \equiv \max\{ 1/e, \ |M\ln M| \}, \tag{2.16}$$

the optimal solution of Program P'_μ is an ε-optimal solution of Program P'.

3. CROSS-ENTROPY MINIMIZATION SUBJECT TO INEQUALITY CONSTRAINTS

The cross-entropy minimization problem has received much attention in the recent literature [10-16]. However, most of the attention has focused on the case with equality constraints (in addition to the non-negativity constraints). In fact, a more general setting of linearly-constrained minimum cross-entropy problem can be described in the following form (assuming $p_j > 0$, $j=1,2,\ldots,n$):

$$\text{Program Q}: \quad \text{Minimize} \quad \sum_{j=1}^{n} x_j \ln\left(\frac{x_j}{p_j}\right)$$
$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}^1 \, x_j \le b_i^1 \quad , \quad i = 1, 2, \ldots, m_1 ,$$
$$\sum_{j=1}^{n} a_{ij}^2 \, x_j = b_i^2 \quad , \quad i = 1, 2, \ldots, m_2 ,$$
$$x_j \ge 0 \quad , \quad j = 1, 2, \ldots, n .$$

Although the inequality constraints can be converted into equality ones by adding slack variables, the resulting program is no longer a regular entropy optimization problem due to the absence of the entropic terms $x_j \ln x_j$ for the slack variables in the objective function. Therefore, the duality theory developed in [16] and the algorithms developed in [10] are not applicable. Also note that Program Q is a special case of Program P_μ' with $\mu = 1$ and $c_j = -\ln p_j$. Therefore, the theory developed in the previous section applies readily to Program Q. In particular, the geometric dual program of Program Q can be derived as follows:

$$\text{Program F}: \quad \max \quad f(\mathbf{w}) = \sum_{i=1}^{m} b_i w_i - \sum_{j=1}^{n} p_j e^{\sum_{i=1}^{m} a_{ij} w_i - 1} \qquad \text{subject to } \mathbf{w}_i \le 0 .$$

In light of Theorem 5, we have the following corollary for the strong duality:

COROLLARY 1. If Program Q has an interior feasible solution and a constraint matrix **A** of full rank, then Program F has a unique optimal solution $\mathbf{w}^*$ and equation (2.9), with $c_j \equiv -\ln p_j$, provides a dual-to-primal conversion which defines the optimal solution $\mathbf{x}^*$ of Program Q. Moreover, Min(Q) = Max(F).

4. CONCLUSION

We have extended the unconstrained convex programming approach to solving linear programs with inequality constraints without adding artificial variables. By the duality theory, one can solve a given linear program by solving the geometric dual of a perturbed linear program. Many standard constrained optimization techniques [e.g. 18] can be specialized to solve the dual program D_μ'. Such specializations, made possible by the simplicity of the constraints, significantly reduce the computational effort usually incurred by these methods. For example, the projection operation required by the projective gradient method is trivial, which makes the method a good candidate solution algorithm.

Immediate applications of the theory developed include an entropic path-following approach to solving linear semi-infinite programs with an infinite number of inequality constraints and the widely used entropy optimization models with linear equality and/or inequality constraints.

REFERENCES

1. Karmarkar, N., A New Polynomial Time Algorithm for Linear Programming, Combinatorica 4 (1984), 373-395.

2. Goldfarb, D. and Todd, M. J., Linear Programming, Technical Report No. 777, School of Operations Research and Industrial Engineering, Cornell University, 1988.

3. Fang, S.-C. and Puthenpura, S. C., Linear Optimization and Extensions: Theory and Algorithms , Prentice Hall, Englewood Cliffs, New Jersey, 1993.

4. Dantzig, G.B., Linear Programming and Extensions , Princeton University Press, Princeton, New Jersey, 1963.

5. Fang, S.-C., A New Unconstrained Convex Programming View of Linear Programming, Zeitschrift fur Operations·Research 36 (1992), 149-161.

6. Peterson, E. L., Geometric Programming, SIAM Review 19 (1976), 1-45.

7. Fang, S.-C. and Tsao, H.-S. J., Linear Programming with Entropic Perturbation, Zeitschrift fur Operations Research 37 (1993), 171-186.

8. Ben-Tal, A, Melman, A. and Zowe, J., Curved Search Methods for Unconstrained Optimization, Optimization 21 (1990), 669-695.

9. Fang, S.-C. and Wu, S.-Y., An Inexact Approach to Solving Linear Semi-Infinite Programming Problems, Optimization 28 (1994), 291-299.

10. Fang, S.-C. and Tsao, H.-S.J, A Quadratically Convergent Global Algorithm for the Linearly-Constrained Minimum Cross-Entropy Problem, European Journal of Operational Research 79 (1994), 369-378.

11. Grandy, W.T. Jr. and Schick, L.H. (Ed.), Maximum-Entropy and Bayesian Methods: Proceedings of the 10th International Workshop (1990), Kluwer Academic Publishers.

12. Censor, Y., On Linearly Constrained Entropy Maximization, Linear Algebra and Its Applications 80 (1986), 191-195.

13. Erlander, S., Accessibility, Entropy and the Distribution and Assignment of Traffic, Transportation Research 11 (1977), 149-153.

14. Guiasu, S, Maximum Entropy Condition in Queueing Theory, Journal of Operational Research Society 37 (1986), 293-301.

15. Ben-Tal, A., Teboulle, M., and Charnes, A., The Role of Duality in Optimization Problems involving Entropy Functionals with Applications to Information Theory, Journal of Optimization Theory and Applications 58 (1988), 209-223.

16. Ben-Tal A. and Charnes, A., A Dual Optimization Framework for Some Problems of Information Theory and Statistics, Problems of Control and Information Theory 8 (1979), 387-401.

17. Rockaffelar, R. T., Convex Analysis , Princeton University Press, Princeton, New Jersey, 1970.

18. Bazaraa, M.S. and Shetty, C.M., Nonlinear Programming: Theory and Algorithms , John Wiley & Sons, New York, New York, 1979.